MATH 609-602
Homework #3
Fundamentals in iterative methods for linear systems

Solve any set of problems for 100 points.

1. (10 pts) Prove that if A is nonsingular matrix with real elements, then $A^t A$ is positive definite.

2. (10 pts) Prove that if A is a real matrix and such that (Ax, x) is real and positive for any nonzero vector $x \in C^n$, then its eigenvalues are positive.

3. (10 pts) Prove that if A is a real matrix and such that that (Ax, x) is real and positive for any nonzero vector $x \in C^n$, then so are A^2, A^3, ..., as well as A^{-1}, A^{-2}, ...

4. (10 pts) Prove that if A is unit column diagonally dominant, that is

$$a_{jj} = 1 > \sum_{i \neq j} |a_{ij}|, \ 1 \leq j \leq n,$$

   then Richardson iteration is convergent.

5. (10 pts) Prove that if A is nonsingular and if $|\lambda| < ||A^{-1}||^{-1}$, then λ is not an eigenvalue of A. Here the norm can be any subordinate matrix norm.

6. (10 pts) Let $|| \cdot ||$ be a subordinate matrix norm, and let S be a nonsingular matrix. Define $||A||_* = ||SAS^{-1}||$, and show that $|| \cdot ||_*$ is a subordinate norm.

7. (10 pts) Prove that if $\rho(A) < 1$, then the matrix $I - A$ is invertible and

$$(I - A)^{-1} = \sum_{k=0}^{\infty} A^k.$$

8. (20 pts) Prove that $\rho(A) < 1$ if and only if $\lim_{k \to \infty} A^k x = 0$ for every $x \in R^n$.

9. (20 pts) Assume that the matrix A is is split into $A = D - L - U$, where D is a nonsingular diagonal matrix and L and U are strictly lower and upper triangular matrices, respectively. Consider the iteration method in the form $x^{(k+1)} = Gx^{(k)} + c$, where G is the iteration matrix. Find the iteration matrices G of the SSOR method. If the matrix G is written in the form $G = I - BA$ show that if A is symmetric, them B is also symmetric, i.e. $B = B^T$.

10. (20 pts) Let $A \in C^{n \times n}$ and let $\| \cdot \|$ be any subordinate matrix norm. If $\rho(A)$ is the spectral radius of A, show that

$$\lim_{m \to \infty} \|A^m\|^{1/m} = \rho(A).$$

    Hint. Use Jordan decomposition or simply assume that A consists of Jordan blocks.