MATH 609-602
Homework #4
Newton's method for nonlinear systems

Problem 1. (30 pts) Let A and B be $n \times n$ real matrices and assume that $\|I - BA\| < 1$ for some matrix norm $\|\cdot\|$ subordinated to a vector norm $\|\cdot\|$ in $\mathcal{R}^n$. Prove the following inequalities (referred often as Banach Lemma)

$$\|A^{-1}\| \leq \frac{\|B\|}{1 - \|I - BA\|}$$

and

$$\|A^{-1} - B\| \leq \frac{\|B\|\|I - BA\|}{1 - \|I - BA\|}.$$

The second inequality indicates that if $\|I - BA\| \ll 1$ (i.e. very small) then B is close to A^{-1}, so B is an approximation to A^{-1}.

Problem 2. (70 pts) Consider the solution to the system of nonlinear equations:

$$F(x) = 0, \quad F : \mathcal{R}^n \mapsto \mathcal{R}^n, \tag{1}$$

where F is differentiable in an open set $\Omega \subset \mathcal{R}^n$. Make the following **$\alpha$-standard assumptions**:

(a) The equation (1) has a solution $x^* \in \Omega$.

(b) $F' : \Omega \mapsto \mathcal{R}^n \times \mathcal{R}^n$ is Hölder continuous in Ω with exponent $\alpha \in (0, 1]$, that is

$$\|F'(x) - F'(y)\| \leq K\|x - y\|^\alpha, \quad \forall\ x, y \in \Omega$$

with a constant $K > 0$. Here $\|x\|$ is a norm of $x \in \mathcal{R}^n$ and $\|F'\|$ is the corresponding subordinate matrix norm.

(c) $F'(x^*)$ is nonsingular.

Prove that Newton's method converges superlinaerly with order $1+\alpha$. More precisely, there is $\delta > 0$ such that, if $x_0 \in B(\delta) = \{x : \ \|x - x^*\| < \delta\}$ then the Newton iterates

$$x^{m+1} = x^m - F'(x^m)^{-1}F(x^m),\ m = 0, 1, \ldots$$

converge and

$$\|x^{m+1} - x^*\| \leq K\|x^m - x^*\|^{1+\alpha}$$

for some $K > 0$.