MATH 609-602
Homework #5
Polynomial interpolation of functions of one variable

(1) Let $f(x) = x^n$ and $f[x_0, x_1, ..., x_n]$ be the divided difference of order n using the points $x_0 < x_1 < ... < x_n$. Prove that:
  (a) (10 pts) $f[x_0, x_1, ..., x_n] = 1$;
  (b) (10 pts) $f[x_0, x_1, ..., x_{n-1}] = x_0 + x_1 + ... + x_{n-1}$.

(2) (10 pts) If $f[x_0, x_1, \ldots, x_n]$ denotes the divided difference of order n prove the **Leibnitz formula**

$$(fg)[x_0, x_1, \ldots, x_n] = \sum_{k=0}^{n} f[x_0, x_1, \ldots, x_k] g[x_k, x_{k+1}, \ldots, x_n].$$

(3) (10 pts) Find the Lagrange and backward Newton divided difference interpolating polynomials for the data $(0, 1)$, $(0.5, 2)$, $(1, 3)$, $(1.5, 4)$.

(4) (20 pts) Let

$$L_{n,k}(x) = \frac{(x - x_0)...(x - x_{k-1})(x - x_{k+1})...(x - x_n)}{(x_k - x_0)...(x_k - x_{k-1})(x_k - x_{k+1})...(x_k - x_n)}.$$

Show that for any x the following relations are valid:
  (a) $\sum_{k=0}^{n} L_{n,k}(x) = 1$;
  (b) $\sum_{k=0}^{n} x_k^m L_{n,k}(x) = x^m$ for $m = 1, ..., n$.

(5) (20 pts) Estimate the interpolation error of $\cos x$ in the interval $(0, 0.4)$ by a polynomial of degree 2 using the interpolation nodes $x_0 = 0$, $x_1 = 0.2$, $x_2 = 0.4$.

(6) (20 pts) $\sum_{k=0}^{n} (x - x_k)^m L_{n,k}(x) = 0$ for $m = 1, ..., n$.

In your free time and for your amusement:

Show that if $\omega(x) = (x - x_0)(x - x_1)...(x - x_n)$ then:
(1) $\sum_{k=0}^{n} (x - x_k)^{n+1} L_{n,k}(x) = (-1)^n \omega(x)$
(2) $\sum_{k=0}^{n} (x - x_k)^{n+2} L_{n,k}(x) = (-1)^n \omega(x) \sum_{k=0}^{n} (x - x_k)$
(3) $\sum_{k=0}^{n} L_{n,k}(0) x_k^{n+1} = (-1)^n x_0.x_1...x_n$.