MATH 609-602
Homework #6
Hermite and Spline Interpolation

Solve any set of problems for 100 points. The homework should be presented at the beginning of the class. There will be penalty for delay of the homework, 5 pts per day.

(1) (20 pts) The polynomial $p(x) = 2-(x+1)+x(x+1)-2x(x+1)(x-1)$ interpolates the first four points in the table $(-1, 2), (0, 1), (1, 2), (2, -7), (3, 10)$. By adding one additional term to p, find polynomial that interpolates the whole table.

(2) (20 pts) Find the interpolating polynomial using divided differences that interpolates the data: $(x = 0, f(0) = 0, f'(0) = 0, f''(0) = 1), (x = 1, f(1) = 1, f'(1) = 0, f''(1) = 1)$.

(3) (20 pts) Let $H_n(x)$ be the Hermite interpolating polynomial for the data:

$$(x_0, f_0, f_0'), (x_1, f_1, f_1'), ..., (x_m, f_m, f_m'),$$

where $f_i = f(x_i), f'(x_i) = f_i', k = 0, 1, ..., m$, and $x_i \in [a, b]$. Prove that if $f(x)$ has continuous $(2m+2)$-st derivative in $[a, b]$ (usually we denote that by $f \in C^{(2m+2)}[a, b]$), then for any $x \in [a, b]$ there is $\xi_x \in [a, b]$ such that

$$f(x) - H_n(x) = \frac{(x-x_0)^2(x-x_1)^2...(x-x_m)^2}{(2m+2)!} f^{(2m+2)}(\xi_x).$$

(4) (20 pts) Find the periodic cubic spline that interpolates the data: $(0, 0)$, $(0.5, 1)$, $(1, 0)$.

(5) (20 pts) Let $y(x) \in C^{(4)}[t_0, t_n]$ and $y(x)$ satisfies $y''(t_0) = 0$ and $y''(t_n) = 0$. Let $S(x)$ be the natural cubic spline interpolating the data $(t_0, y_0), \ldots, (t_n, y_n)$. Show that there is a constant $C > 0$, independent of h such that

$$|S''(x) - y''(x)| \le Ch^2 \max_{x \in [t_0, t_n]} |y^{(4)}(x)|, \ \forall \ x \in [t_0, t_n].$$

Remark. Use the fact that in class we have proved this inequality for the knots, i.e. for $x = t_i, \ i = 0, \ldots, n$.

(6) (40 pts) The natural **tension spline** that interpolates the data $(t_0, y_0), \ldots, (t_n, y_n)$ is a function $f(x)$ having the following properties:
(a) $f \in C^{(2)}[t_0, t_n]$;
(b) $f(t_i) = y_i, \ i = 0, \ldots, n$;
(c) On each open interval (t_{i-1}, t_i), f satisfies $f^{(4)} - \tau^2 f'' = 0$;
(d) $f''(t_0) = f''(t_n) = 0$.

Adopting the approach of cubic spline interpolation find the system of linear algebraic equations for the unknown parameters $z_i = f''(t_i)$. Remark. Consult with your text.