**MATH 609-601**
**Homework #7**
**Numerical Differentiation and Integration**

Solve a set for 100 points. Penalty for delay of the homework, 5 pts per day applies.

(1) (10 pts) Show $O(h^4)$ error of the following formula for approximating the second derivative:

$$f''(x) \approx \frac{1}{12h^2}(-f(x+2h) + 16f(x+h) - 30f(x) + 16f(x-h) - f(x-2h))$$

(2) (10 pts) Derive a five-point formula for the first derivative at the point $x_1$ using the values of the function $f(x)$ at the points $x_i = x_0 + ih$, $i = 0, 1, 2, 3, 4$ and $h$ is a positive number. Derive your formula using interpolation.

(3) (10 pts) Estimate the error of the composite Simpson's rule for numerical integration, i.e. estimate

$$error = \int_a^b f(x)dx - \frac{h}{3}\sum_{i=0}^{n-1}\{f(x_{2i}) + 4f(x_{2i+1}) + f(x_{2i+2})\}$$

where $h = \dfrac{b-a}{2n}$ and $x_i = a + ih$.

(4) (10 pts) Show that the quadrature

$$\int_0^\infty e^{-x} f(x)dx \approx \frac{2+\sqrt{2}}{4} f(2-\sqrt{2}) + \frac{2-\sqrt{2}}{4} f(2+\sqrt{2})$$

has algebraic degree of accuracy 3.

(5) (20 pts) Prove that if the interval is symmetric with respect to the origin and if $w(x)$ is an even function, then the Gaussian nodes will be symmetric respect to the origin. So if the roots are ordered $x_0 < x_1 < \cdots < x_n$, then $x_i = -x_{n-i}$ and $A_i = A_{n-i}$ for $i = 0, 1, \ldots, n$.

(6) (20 pts) Prove that no Gaussian quadrature formula with $n$ nodes can be exact on $\Pi_{2n}$.

(7) (20 pts) Define the Legendre polynomial $P_n(x)$ of degree $n$ by

$$P_n(x) = \frac{1}{2^n\ n!}\frac{d^n(x^2-1)^n}{dx^n},\ n = 0, 1, \ldots.$$

Show that $P_n(x)$ has $n$ distinct zeros in the interval $(-1, 1)$ which are symmetric with respect to the origin.

(8) (20 pts) Consider the Gaussian quadrature

$$\int_{-1}^1 f(x)dx \approx \sum_{i=1}^n A_k f(x_k).$$

Show that $A_k = 2/[(1-x_k^2)[P_n'(x_k)]^2]$, $k = 1, 2, \ldots, n$,where $P_n$ is the defined above Legendre polynomial.
Hint: Use the fact that $P_n(1) = 1$, $P_n(-1) = (-1)^n$ and the equality

$$\int_{-1}^1 P_n(x)P_n'(x)/(x-x_k)dx = A_k[P_n'(x_k)]^2.$$