MATH 609-601 Numerical Analysis
Homework #8
Runge-Kutta Methods for ODE's

(1) (20 pts) Consider the initial value problem $y' = t - t^3$, $y(0) = 0$. Suppose we use Euler's method with step-size h to compute approximate values $\eta(t_j; h)$ for $y(t_j)$ at $t_j = jh$. Find an explicit formula for $\eta(x_j; h)$ and for $e(x_j; h) = \eta(t_j; h) - y(t_j)$ and show that $e(t; h)$ for t fixed, goes to zero as $h = t/n$ tends to 0.

In all problems below we consider the Cauchy problem $y' = f(t, y), y(t_0) = y_0$ and its approximation by Runge-Kutta method

$$\eta_0 = y_0,\ \eta_{i+1} = \eta_i + h\ \Phi(t_i, \eta_i; h),\ i = 0, 1, ... \tag{1}$$

Here η_i is the approximation of $y(t_i)$.

(2) (20 pts) Show that if

$$\Phi(t, x; h) = \frac{1}{4}(k_1 + 3k_2),\ \text{where } k_1 = f(t, x),\ k_2 = f(t + \frac{2}{3}h, x + \frac{2}{3}hk_1)$$

then the method is of second order.

(3) (20 pts) Show that if

$$\Phi(t, x; h) = \frac{1}{6}(k_1 + 4k_2 + k_3),$$

where $k_1 = f(t, x),\ k_2 = f(t + 0.5h, x + 0.5hk_1),\ k_3 = f(t + h, x - hk_1 + 2hk_2)$ then the method is of third order.

(4) (20 pts) Let

$$\begin{aligned} \Phi(t, x; h) &= f(t, x) + 0.5hg(t + \tfrac{1}{3}h, x + \tfrac{1}{3}f(t, x)), \\ g(t, x) &= \frac{\partial f(t, x)}{\partial t} + \frac{\partial f(t, x)}{\partial x}.f(t, x). \end{aligned} \tag{2}$$

Show that this method is of order 3.

(5) (20 pts) For the above initial value problem consider the following Runge-Kutta method (1), where:

$$\begin{aligned} \Phi(t, x; h) &= h(a_1 k_1 + a_2 k_2) \\ k_1 &= f(t + \alpha_1 h, x + \beta_{11} hk_1) \\ k_2 &= f(t + \alpha_2 h, x + \beta_{21} hk_1 + \beta_{22} hk_2). \end{aligned} \tag{3}$$

(a) Find the conditions that the coefficients a, α, β need to satisfy so that the **explicit** method (i.e. $\beta_{11} = \beta_{22} = 0$) is of second order. Give at least one set of coefficients that satisfy these conditions.

(b) Find the conditions that the coefficients a, α, β need to satisfy so that the **implicit** method (i.e. $\beta_{11} \neq 0$ and $\beta_{22} \neq 0$) is of second order.