MATH 609-602 Numerical Analysis
Program assignment #6
Runge-Kutta Method with Automatic Step-Size Control

Write a program for solving Cauchy problem for a single differential equation of first order applying one of the methods that automatically controls the error less than ϵ by selecting appropriate step-size. You should program both adaptive methods discussed in class, the one based on Richardson extrapolation technique (using halfing of the stepsize) and the other embedded Runge-Kutta methods. For the embedded methods one possibility is to use the pair fourth order R-K and fifth order Runge-Kutta-Fehlberg (see, Kincaid and Cheney, p. 586,587). Another possibility is to use as two embedded methods the Euler method and any second order Runge-Kutta method (e.g. modified Euler method).

Specifications

(1) Run the program for two values of ϵ: 10^{-2} and 10^{-4}.
(2) Report the number of steps and the obtained solution at the end point. Plot the obtained approximate solution in the whole interval.

Computational examples - solve the following problems:

(1) The initial value problem

$$y' = \frac{2}{t}y + t^2 e^t, \ 1 \le t \le 2, \ y(1) = 0,$$

with exact solution $y(t) = t^2(e^t - e)$. Comapre with the exact solution.

(2) In a circuit impressed, voltage E, and resistance R, inductance L, capacitance C in parallel, the current satisfies the differential equation

$$i'(t) = CE''(t) + \frac{1}{R}E'(t) + \frac{1}{L}E.$$

Suppose $C = 0.3$ farad, $R = 1.4$ ohms, $L = 1.7$ henries, and the voltage is given by $E(t) = e^{-0.06\pi t} sin(2t - \pi)$. If $i(0) = 0$, find the current $i(t)$ for $0 < t < 5$.

(3) A liquid of low viscosity, such as water, flows from an inverted conical tank with circular orifice at the rate

$$y'(t) = -0.6\pi r^2 \sqrt{2g}\frac{\sqrt{y}}{A(y)},$$

where r is the radius of the orifice, y is the height of the liquid level from the vertex of the cone, and $A(y)$ is the area of the cross section of the tank y units above the orifice. Suppose $r = 0.1$ feet, $g = 32 feet/sec^2$, and the tank has an initial water level of 8 feet and initial volume of $512\pi/3$ cubic feet. Find the time when the tank is emptied.

(4) The irreversible chemical reaction in which two molecules of solid potassium dichomate ($K_2Cr_2O_7$), two molecules of water (H_2O), and three atoms of solid sulfur (S) combine to yield three molecules of the gas sulfur dioxide (SO_2), four molecules of solid potassium hydroxide (KOH), and two molecules of solid chrome oxide (Cr_2O_3) can be represented symbolically by the stoichiometric equation:
$2K_2Cr_2O_7 + 2H_2O + 3S - - - -4KOH + 2Cr_2O_3 + 3SO_2$. If n_1 molecules of $K_2Cr_2O_7$, n_2 molecules of H_2O and n_3 molecules of S are originally available, the following differential equation describes the amount $x(t)$ of KOH after time t:

$$x'(t) = k(n_1 - 0.5x)^2(n_2 - 0.5x)^2(n_3 - 0.75x)^3,$$

where k is the velocity constant of the reaction. If $k = 6.22 * 10^{-19}, n_1 = n_2 = 1000, n_3 = 1500$, how many units of potassium hydroxide will have been formed after two seconds ?