MATH 610-600
Homework #1
Some fundamental inequalities related of two-point boundary value problem and their approximations

Problem 1. (40 pts) Consider the following tridiagonal $n \times n$ matrices:

$$(1) \qquad A = \begin{bmatrix} 2+a & -1 & & & \\ -1 & 2+a & -1 & & \\ & \cdots & \cdots & \cdots & \\ & & & & -1 \\ & & & -1 & 2+a \end{bmatrix}, \quad A_0 = \begin{bmatrix} 1 & -1 & & & \\ -1 & 2 & -1 & & \\ & \cdots & \cdots & \cdots & \\ & & & & -1 \\ & & & -1 & 1 \end{bmatrix},$$

where $a > 0$ is a given constant. Find the eigenvalues and the eigenvectors of A and A_0.

Problem 2. Let $C^1[0,1]$ be the set of function $u(x)$ that are continuous and have continuous first derivative on $[0,1]$. Consider the following subsets of $C^1[0,1]$:

$$\mathcal{A} = \{u \in C^1[0,1] : \; u(0) = u(1) = 0\}, \; \mathcal{B} = \{u \in C^1[0,1] : \; u(0) = 0\}, \; \mathcal{C} = C^1[0,1].$$

(1) (20 pts) Derive the following inequalities:

$$\int_0^1 u^2 dx \le \frac{1}{2} \int_0^1 u'^2 dx, \; \forall u \in \mathcal{B},$$

$$\int_0^1 u^2 dx \le \frac{1}{4} \int_0^1 u'^2 dx, \; \forall u \in \mathcal{A}.$$

Can you improve the constants in these inequalities ? Make a short discussion or just derive the improved estimates.

(2) (20 pts) Show that for $u \in \mathcal{C}$ the following inequalities are valid:

$$\int_0^1 u^2 dx \le \frac{1}{6} \int_0^1 u'^2 dx + \Big(\int_0^1 u dx \Big)^2;$$

$$\max_{x \in [0,1]} |u(x)|^2 \le 2u^2(1) + 2 \int_0^1 u'^2 dx;$$

$$\int_0^1 u^2(x) dx \le 2u^2(1) + 2 \int_0^1 u'^2(x) dx;$$

$$\max_{x \in [0,1]} |u(x)|^2 \le 2 \int_0^1 (u^2 + u'^2) dx.$$

Problem 3. (20 pts) Consider the function $u(x) = |x|^{\frac{2}{3}} - 1$ defined on the interval $(-1,1)$. Show that the generalized derivative of this function is

$$u'(x) = \begin{cases} \frac{2}{3}(-x)^{-\frac{1}{3}}, & -1 \le x < 0, \\ \frac{2}{3}x^{-\frac{1}{3}}, & 0 < x \le 1. \end{cases}$$

Hint. Find a sequence $u_n(x) \in C_0^1(-1,1)$, such that $\|u - u_n\|_{H^1} \to 0$, when $n \to \infty$.