**MATH 610-600 Numerical Methods for PDEs**
**Programming assignments #4 & 5**
**FEM for elliptic problems on general meshes**

Consider the following elliptic boundary value problem: find $u(x,y)$ such that

$$
\begin{aligned}
-\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} + qu &= f(x,y) && \text{in} \quad \Omega,\\
u &= g_D && \text{on} \quad \Gamma_D,\\
\frac{\partial u}{\partial \nu} &= g_N && \text{on} \quad \Gamma_N,
\end{aligned}
$$

where $\Gamma$ is the boundary of $\Omega$, $\Gamma = \Gamma_D \cup \Gamma_N$ and $\nu$ is the outer unit normal vector to $\Gamma$.

Solve the given below problems by approximating the corresponding boundary value problem using linear triangular finite elements on a partition of the domain generated by TRIANGLE. Consider meshes with $|\tau| \le 1/n^2$, $n = 10, 20, 40$, where $|\tau|$ is the maximal area of the triangular elements. Submit a report according to the given below specification.

**Specifications**

Use **double precision** arithmetics. Use the preconditioned conjugate gradient method (PCG method) with preconditioner $B = D$, where $D$ is the diagonal of the matrix of the Ritz-Galerkin system $AU = b$. The evaluation of $AU$ for a given vector $U$ should be done element by element in a separate subroutine. **DO NOT ASSEMBLE** the global matrix $A$ of the Ritz-Galerkin system. But you may need to assemble the vector of the right hand side $b$. Choose as initial guess $U_0 = b$ and use stopping criterion $(r_m, r_m) \le \epsilon(r_0, r_0)$, where $r_m = b - AU_m$ is the residual of the iterate $U_m$ and $\epsilon$ is the iteration tolerance. Use $\epsilon = 10^{-22}$. Report the number of iterations.

**Computational examples**

Problem 1. Take $\Omega = (0,1) \times (0,1)$, $\Gamma = \Gamma_D \cup \Gamma_N$, $\Gamma_D = \{x = 1,\ 0 \le y \le 1\} \cup \{y = 1,\ 0 \le x \le 1\}$, $q = 5,\ g = 0$ and $f(x,y) = (5+8.5\pi^2) cos\frac{5\pi x}{2} cos\frac{3\pi y}{2}$ with exact solution $u(x,y) = cos\frac{5\pi x}{2} cos\frac{3\pi y}{2}$. Present in a table the $L^2$- and the $H^1$-norms of the error $u - u_h$.

Problem 2. Solve the problem with $q = 0,\ g_D = 1,\ f(x) = 1$, $\Omega$ is a polygon with vertices $(0,0),\ (0.5,0),\ (1,1),\ (0,2)$ and $\Gamma_D = \Gamma$. Plot the solution.

Problem 3. The domain $\Omega = (0,1)^2 \setminus \bar{\Omega}_1$, where $\Omega_1 = \{|x - 0.5| < 0.25, \quad \{|y - 0.5| < 0.25\}$ and $\Gamma_N$ is the boundary of $\Omega_1$ while $\Gamma_D$ is the rest of the boundary. Take $q = 1,\ g_D = 0,\ g_N = 1$, $f(x,y) = xy$. Plot the solution.

Problem 4. To approximately impose the Dirichlet B.C. in the previous Problem 3 use the penalty method, namely, instead $u = g_D$ on $\Gamma_D$ one uses the natural boundary condition

$$\frac{\partial u}{\partial \nu} + \delta(u - g_D) = 0.$$

Here $\delta > 0$ is a penalty (large) parameter. Experiment with $\delta = n$, $\delta = n^2$, $\delta = n^3$, and $\delta = 10^6$ to see how accurately the Dirichlet boundary conditions are imposed. Plot the difference between this solution and the one computed in the previous problem. In a table compare the number of iterations of this problem with those of Problem 3.

There is penalty of 5 points/day (out of 200) for delaying the programming assignment.