MATH 661-600
Homework #1
Finite element method for 1-D problems
Due, September 12, 2003

Below we use the notations from class and from the textbook: S.C. Brenner and L.R. Scott, *The Mathematical Theory of Finite Element Methods*, Springer-Verlag, 1994. Namely, $V = \{v,\ v' \in L^2(0,1):\quad v(0) = 0\}$, and $0 = x_0 < x_1 < \ldots < x_n = 1$ is a mesh in the interval $[0,1]$. Further, $S \subset V$ is the space of piece-wise linear functions over the mesh.

1. (20 pts) (problem 0.x.7, p. 18 of your textbook) Let $h = \max_{1 \le i \le n}(x_i - x_{i-1})$. Then

$$\inf_{v \in S} \|u - v\| \le Ch^2 \|u''\|.$$

2. (20 pts) (problem 0.x.8, p. 19 of your textbook) Prove that the problem $-u'' = f(x),\ x \in (0,1),\ u(0) = 0,\ u'(1) = 0$ has a solution $u \in C^2([0,1])$ provided $f \in C^0([0,1])$.

3. (20 pts) Let $a(u,v) = (u', v')$. Prove:

   (a) (problem 0.x.9, p. 19 of your textbook) the following *coercivity* result:

$$\|v\|^2 + \|v'\|^2 \le C_1 a(v,v), \quad \forall v \in V;$$

   (b) (problem 0.x.10, p. 19 of your textbook) the following version of *Sobolev's inequality*:

$$\|v\|^2_{\max} \le C_2 a(v,v), \quad \forall v \in V;$$

   Give the value for the constants C_1 and C_2. For simplicity, restrict the result to $v \in V \cap C^1([0,1])$.

4. (40 pts) Consider the b.v.p.:

$$-u'' + bu' = f(x),\ x \in (0,1),\ u(0) = 0,\ u'(1) = 0.$$

   Here b is a positive constant. Consider the finite element approximation u_S for this problem with linear elements. Construct the corresponding Green's function and provide an estimate for the error in maximum-norm:

   (1) First estimate $|u(x_i) - u_S(x_i)|$ for $i = 1, \ldots, n$;

   (2) Next, bound $\max_{x \in [0,1]} |u(x) - u_S(x)|$ (the error in max-norm).