MATH 661-600
Homework #2
Convergnece analysis of the FEM for 1-D problems
Due, September 22, 2003

Below we use the notations from class and from the textbook: S.C. Brenner and L.R. Scott, *The Mathematical Theory of Finite Element Methods*, Springer-Verlag, 1994. Namely, $V = \{v,\ v' \in L^2(0,1):\quad v(0) = v(1) = 0\}$, and $0 = x_0 < x_1 < \ldots < x_n = 1$ is a mesh in the interval $[0,1]$ (denoted sometimes as ω). Further, $S \subset V$ is the space of continuous piece-wise linear functions over the mesh. Denote by $h = \max_{1\le j\le n} h_j$, $I_j = (x_{j-1}, x_j)$, and $I = (0,1)$. The problems below are for the following b.v.p.

$$(-ku' + bu)' + cu = f, \quad x \in (0,1), \quad u(0) = u(1) = 0.$$

Further, $u_S \in S$ is the Ritz-Galerkin approximation to u. I allow you to assume that the coefficients of the DE and the solution u are as smooth as needed for your analysis.

1. (40 pts) Let S be the space of continuous piece-wise linear functions over the mesh ω and $x_{j-1/2} = x_j - h_j/2$, where $h_j = x_j - x_{j-1}$. Using scaling argument and Peano kernel theorem show that

$$|u'(x_{j-1/2}) - u_I'(x_{j-1/2})| \le C h_j^{2-1/p} \|u'''\|_{L^p(I_j)}, \quad 1 \le p \le \infty.$$

Here $\|v\|_{L^p(I_j)}$ is the L^p-norm of v on the interval I_j. Use this inequality to prove the following "super-convergence" result:

$$\max_{1\le j\le n} |u'(x_{j-1/2}) - u_S'(x_{j-1/2})| \le Ch^2 \|u'''\|_{L^\infty(I)};$$

$$\left\{ \sum_{j=1}^{n} |u'(x_{j-1/2}) - u_S'(x_{j-1/2})|^p h_j \right\}^{1/p} \le Ch^2 \|u'''\|_{L^p(I)}, \quad 2 \le p < \infty.$$

Remark. You may use the already established inequality $\|u' - u_S'\|_{L^\infty(I)} \le Ch\|u''\|_{L^\infty(I)}$.

2. (60 pts) Now consider the same problem as above but this time $S = \{v \in C^0([0,1]) :$ v is piecewise **quadratic** over the partiton $\omega\}$.

   (a) Find an estimate for $\|u' - u_S'\|$.

   (b) Using Green's function technique prove that:

$$|u(x_j) - u_S(x_j)| \le Ch^4 \|u^{(3)}\|_{L^2(I)}, \quad j = 1, 2, \ldots, n-1.$$

   (c) Find the "superconvergence points" $\bar{x}_j$ for the derivative and prove the corresponding estimate.