MATH 661-600
Homework #3
Sobolev spaces and related topics
Due, October 7, 2003

Below we use the notations from class and from the textbook: S.C. Brenner and L.R. Scott, *The Mathematical Theory of Finite Element Methods*, Springer-Verlag, 1994.

Solve any five of the following problems.

1. Prove Hölder's inequality: i.e. for $\frac{1}{p} + \frac{1}{q} = 1, \quad \|fg\|_{L^1(\Omega)} \leq \|f\|_{L^p(\Omega)} \|g\|_{L^q(\Omega)}$.
   Hint: Prove that for $0 < m < 1$ and $y > 0$, $y^m \leq 1 + m(y-1)$. Conclude from this that $A^{1/p} B^{1/q} \leq \frac{1}{p} A + \frac{1}{q} B$, for $A, B \geq 0$ so that Hölder's inequality follows from this.

2. Let $u \in W_p^1(\Omega)$, where $1 < p < \infty$ and Ω is a domain in $\mathbf{R}^n$ with Lipschitz boundary. Show that $f = |u|^p$ has weak derivative $D^\alpha f$ for $|\alpha| = 1$.

3. Using the result of the previous problem show that there a constant $C > 0$ such that

$$\int_{\partial\Omega} |u|^p \leq C \left(\epsilon^{1-1/p} \int_\Omega |\nabla u|^p dx + \epsilon^{-1/p} \int_\Omega |u|^p dx \right), \quad \forall \epsilon > 0.$$

4. (Exercise 1.x.13): Let $f(x) = |x|^r, \quad x \in \mathbf{R}^n$ for a given real number r. Prove that f has first order weak derivatives on the unit ball provided that $r > 1 - n$. For what values of r does f also have second order weak derivatives?

5. Let $f(x) = |x| \log |x|, \quad x \in \Omega = \{x \in \mathbf{R}^n : |x| < 1\}$. Prove that f has first order weak derivatives in Ω. Find the values of p so that $f \in W_p^1(\Omega)$.

6. Let $\xi \in \Omega \subset \mathbf{R}^n$. Suppose that Ω is a polygonal domain. Define $L(u) = \frac{\partial u}{\partial x_1}(\xi)$ for a continuously differentiable function u on Ω. For what values of n, k and p is L a continuous linear functional on $W_p^k(\Omega)$?

7. (This is Remark 1.4.8 of page 32) Consider the domain

$$\Omega = \{(x,y) \in \mathbf{R}^2 : \quad 0 < x < 1, \quad |y| < x^r\},$$

   where $r > 1$ and let $u(x,y) = x^{-\epsilon/p}$, with $0 < \epsilon < r$. Show that if $p < 1 + r - \epsilon$, then $u \in W_p^1(\Omega)$. Choosing ϵ small and r large one can satisfy the conditions for the Sobolev inequality (saying that $W_p^1(\Omega) \subset L^\infty(\Omega)$). However, this function is not in $L^\infty(\Omega)$. Explain.

8. (Problem 1.x.42) Let $v = r^\beta sin\beta\theta$ on $\Omega_\beta = \{(r,\theta) : \quad r < 1, \ 0 < \theta < \pi/\beta\}$ where $1/2 \leq \beta < \infty$. Determine the optimal values of k, p (in terms of β) so that $v \in W_p^k(\Omega)$. The case $\beta = 1/2$ is called *slit domain* and $\beta = 2/3$ a *re-entrant corner.*