MATH 661-600
Homework #4
Finite Elements
Due, October 20, 2003

Below we use the notations from class and from the textbook: S.C. Brenner and L.R. Scott, *The Mathematical Theory of Finite Element Methods*, Springer-Verlag, 1994. In particular we use the following notation for finite elements $(\tau, \mathcal{P}, \mathcal{N})$, where τ is the finite element domain, $\mathcal{P}$ is the space of shape functions (usually a polynomial space), and $\mathcal{N}$ is the set of liner functionals on $\mathcal{P}$ called degrees of freedom.

Problem 1. (40 pts) Define the homogeneous coordinates for τ being a simplex in $\mathcal{R}^d$, $d \geq 2$ and find the nodal basis functions for:

(a) $\mathcal{P} = \mathcal{P}_1$ and $\mathcal{N}$ are the values at the: (1) vertices (this is the conforming case), (2) the centers of the gravity of the faces (edges for $d = 2$);

(b) $\mathcal{P} = \mathcal{P}_2$ and $\mathcal{N}$ are the values at the vertices and the midpoints of the edges;

(c) For $d = 3$ what would be an appropriate set of liner functionals $\mathcal{N}$ for a Hermitian cubic finite element $(\tau, \mathcal{P}_3, \mathcal{N})$? Show that your choice of $\mathcal{N}$ determines $\mathcal{P}_3$;

(d) Determine the dimension of $\mathcal{P}_k$ in $\mathcal{R}^d$ and suggest a set of points that could be used to construct nodal basis functions for Lagrangian elements.

Problem 2. (30 pts) In $\mathcal{R}^2$ we consider $(\tau, \mathcal{P}_1, \mathcal{N})$ where τ is a non-degenerate triangle and $\mathcal{N}$ is the set of values at the vertices. For a given constant vector $\underline{b}$ compute the element "convection" matrix K_τ^{01}, i.e. the matrix produced by $\int_\tau \underline{b} \cdot \nabla u \; v \; dx$.

Can you suggest a quadrature formula (or a particular way of computing approximately K_τ^{01}) so that the element "convection" matrix is an M-matrix ?

Problem 3. (30 pts) In $\mathcal{R}^2$ we consider $(\tau, \mathcal{Q}_k, \mathcal{N})$ where $\tau = (0,1) \times (0,1)$ is a rectangle, $\mathcal{Q}_k = \sum_{i,j=0}^k a_{ij} x^i y^j$, and

$$\mathcal{N} = \{N_{ij}(v) = v(x_i, y_j), \quad (x_i, y_j) = (i/k, j/k), \quad i, j = 0, \ldots, k\}.$$

(This is example 3.5.4, p. 83 of your textbook). Show that $(\tau, \mathcal{Q}_k, \mathcal{N})$ is a finite element.