

MATH 661-600 Mathematical Theory of Finite Element Methods
Programming assignment #1
FEM for 1-D Boundary Value Problems
Due September 26, 01

Write a program for solving the boundary value problems for second order ordinary differential equations by Ritz-Galerkin finite element method using finite elements of order $r = 2, 3$. Submit a report with the graphs of the results and tables for the following norms (or semi-norms) of the error $e(x)$:

$$\|e\|_{L^2(I)}, \quad \|e'\|_{L^2(I)}, \quad \|e\|_{L^\infty(I)}, \quad \|e'\|_{L^\infty(I)}, \quad \|e\|_{L^\infty(\omega)}, \quad \|e'\|_{L^2(\omega)}.$$

Here we have used the following definitions for the last two norms:

$$\|e'\|_{L^2(\omega)} := \left\{ \sum_j |e'(\bar{x}_j)|^2 h_j \right\}^{1/2}, \quad \|e\|_{L^\infty(\omega)} := \max_j |e(x_j)|,$$

where the summation is over all super-convergence points \bar{x}_j (you have found in homework #2) and x_j are the nodes of the mesh. Use standard nodal basis functions. If you want to investigate other basis functions you have to revisit the theory of the error analysis.

Specifications

1. Run your program on a sequence on meshes ω_s with uniform step-sizes $h = 2^{-s}$ with $s = 4, 5, 6, 7, 8$. Plot the ratios

$$\frac{\|e_{\omega_s}\|_*}{\|e_{\omega_{s+1}}\|_*},$$

where $\|\cdot\|_*$ denotes one of the norms given above and e_{ω_s} denotes the error for the mesh ω_s .

2. Compare the normalized errors for all problems given below for the finest mesh (normalized means that you have to divide by the norm of the error by, say $\max |u|$).
3. How to compute the corresponding norms? I assume that you can figure this out. In any case one should use elementwise computations with sampling at enough number of points or Gaussian quadratures of higher order. Consult regarding this matter with your available text on Numerical Analysis (for example, by D. Kincaid and W. Cheney, Brooks and Cole Publ. Co., 1996).

Computational examples – solve at least four of the following problems:

1. $-u'' = -1, \quad x \in (0, 1), \quad u(0) = u'(1) = 0$. This problem has exact solution $u(x) = 0.5x^2$.
2. $-u'' = -1, \quad x \in (0, 1), \quad u(0) = u(1) = 0$. This problem has exact solution $u(x) = x(x - 1)$.
3. The deflection of a uniformly loaded, long rectangular plate under axial tension force and fixed ends, for small deflections, is governed by the second order differential equation. Let F represent the axial force and q the intensity of the uniform load. The deflection W along the elemental length is given by:

$$W''(x) - \frac{F}{D}W(x) = -\frac{qx}{2D}(l - x), \quad 0 < x < l, \quad W(0) = W(l) = 0,$$

where l is the length of the plate, and D is the flexural rigidity of the plate. Let $q = 200 \text{ lb/in}^2$, $F = 100 \text{ lb/in}$, $D = 8.8 \cdot 10^7 \text{ lb in}$, and $l = 50 \text{ in}$.

The exact solution is given by: $a = \frac{Fl^2}{D}$, $b = \frac{ql^4}{2D}$, $t = x/l$ and

$$W(t) = \frac{b}{a} \left\{ -t^2 + t - \frac{2}{a} + \frac{2}{a \sinh(\sqrt{a})} [\sinh(\sqrt{at}) + \sinh(\sqrt{a}(1-t))] \right\}.$$

4. Consider the plate deformation discussed in the previous problem when the right end of the plate is elastically supported, i.e. instead of the boundary condition $W(l) = 0$ now we have the condition $W'(l) + \frac{\beta}{D}W(l) = 0$, where β is the characteristic of the elastic support. Solve the problem in the following two cases: (a) $\beta = 0$, i.e. the end is free; (b) $\beta = 2000000 \text{ lb}$. The exact solution for the case of a plate with free end (i.e. $\beta = 0$) is:

$$W(t) = \frac{b}{a} \left\{ -t^2 + t - \frac{2}{a} + \frac{1}{a \cosh(\sqrt{a})} [\sqrt{a} \sinh(\sqrt{at}) + 2 \cosh(\sqrt{a}(1-t))] \right\}.$$

5. Consider the following convection-diffusion problem:

$$(-k(x)u' + u)' = 1, \quad x \in (0, 1), \quad u(0) = 0, \quad u'(1) = 1.$$

Here the coefficient $k(x)$ and the solution $u(x)$ are piece-wise smooth, namely:

$$k(x) = \begin{cases} 1 & x < 0.5, \\ 0.5 & x \geq 0.5, \end{cases} \quad u(x) = x + \begin{cases} (1 - e^x)/(2\sqrt{e}) & x < 0.5, \\ (1 - \sqrt{e})/(2\sqrt{e}) & x \geq 0.5. \end{cases}$$

6. Consider the following convection-diffusion problem (boundary layer):

$$(-u' + bu)' = 0, \quad x \in (0, 1), \quad u(0) = 0, \quad u(1) = 1, \quad b = 10, \quad \text{and} \quad b = 200.$$

The solution of this problem is $u(x) = (e^{b(x-1)} - e^{-b})/(1 - e^{-b})$.

Remark. For $b = 200$ you may need to go use step-size $h = 2^{-7}$ (and smaller) in order to get a computationally stable algebraic problem.