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ON L_p-APPROXIMATION OF FUNCTIONS WHOSE m^{th} DERIVATIVE IS OF BOUNDED VARIATION

by

RONALD DEVORE

1. Introduction. Let f be an element of $L_p[-1, 1]$, $1 \leqq p \leqq \infty$, and S and T be subsets of $L_p[-1, 1]$. Then $E_p(f, T) = \inf_{g \in T} \|f-g\|_p$ is the error in approximating f by elements of T in the $L_p[-1, 1]$ norm and $E_p(S, T) = \sup_{f \in S} E_p(f, T)$ is the error in approximating elements of S by elements of T in the $L_p[-1, 1]$ norm.

A problem of particular interest is the determination of $E_p(S, T)$ when S is characterized by some structural property of its elements and T is one of the classes $\mathbf{P}_n$ of algebraic polynomials of degree $\leqq n$ or $\mathbf{T}_n$ of trigonometric polynomials of degree $\leqq n$. The first result of this type is the classical result of J. FAVARD [1] which for the interval $[-\pi, \pi]$ instead of $[-1, 1]$ can be stated as follows:

If $\mathbf{W}_m$ is the class of those 2π periodic functions f for which $f^{(m-1)}$ is absolutely continuous and $|f^{(m)}(x)| \leqq 1$ a.e., then

$$E_\infty(\mathbf{W}_m, \mathbf{T}_n) = \frac{K_m}{n^m}, \quad \text{where} \quad K_m = \frac{4}{\pi} \sum_{j=0}^{\infty} \frac{(-1)^{(m+1)j}}{(2j+1)^{m+1}}.$$

Another result in this direction, for L_1 approximation, was obtained by S. NIKOLSKI [2]. Let $\mathbf{A}_m$ be the class of those functions f for which $f^{(m-1)}$ is absolutely continuous on $[-1, 1]$ and $f^{(m)}$ is equivalent to a function g whose total variation on $[-1, 1]$ is $\leqq 1$. Then for $n \geqq m$, $m = 1, 2, \ldots$

$$E_1(\mathbf{A}_m, \mathbf{P}_n) = \frac{M_1(m, \mathbf{P}_n)}{m!} E_1(x_+^m, \mathbf{P}_n) \tag{1.1}$$

where

$$(x-a)_+^m = \begin{cases} (x-a)^m & \text{for} \quad x > a \\ 0 & \text{for} \quad x \leqq a \end{cases} \tag{1.2}$$

and

$$M_p(m, \mathbf{P}_n) = \sup_{|a| \leqq 1} \frac{E_p((x-a)_+^m, \mathbf{P}_n)}{E_p(x_+^m, \mathbf{P}_n)} \tag{1.3}$$

Nikolski [3] has also studied the functions $M_1(m, \mathbf{P}_n)$ and $E_1(x_+^m, \mathbf{P}_n)$. He has determined $E_1(x_+^m, \mathbf{P}_n)$ explicitly when m is an odd positive integer and has proved the following assymptotic formulae.

$$E_1(x_+^m, \mathbf{P}_n) = \frac{c_m}{n^{m+2}} + O\left(\frac{\log n}{n^{m+2}}\right) \qquad (n \to \infty) \tag{1.4}$$

$$M_1(m, \mathbf{P}_n) = 1 + O\left(\frac{\log n}{n}\right) \qquad (n \to \infty). \tag{1.5}$$

In this paper, we shall study the L_p approximation ($1 \leqq p \leqq \infty$) of the class $\mathbf{A}_m$ by $\mathbf{P}_n$. First of all, we shall show that the result of NIKOLSKI (1.1) is valid for $1 \leqq p \leqq \infty$. More precisely, in Theorem 1 we show that for $1 \leqq p \leqq \infty$ and $n \geqq m$, $m = 1, 2, \ldots$

$$E_p(\mathbf{A}_m, \mathbf{P}_n) = \frac{M_p(m, \mathbf{P}_n)}{m!} E_p(x_+^m, \mathbf{P}_n).^*$$

Next, we shall solve explicitly the problem of best L_1 approximation to the functions $x_+^m, m = 1, 2, \ldots$, on the interval $[-1, 1]$ by algebraic polynomials by means of more general results which are interesting in themselves. In Theorem 2, we shall determine the polynomials of best L_1 approximation on $[-1, 1]$ of degree $\leqq 2n-2$, $2n-1$ to even functions of the form $h(x^2)$ with $h^{(n)}$ of constant sign on $(0, 1)$.

A similar result for odd functions is Theorem 3 which determines the polynomials of best L_1 approximation on $[-1, 1]$ of degree $\leqq 2n-1$, $2n$ to functions of the form $xh(x^2)$ with $h^{(n)}$ of constant sign on $(0, 1)$.

Finally, using the constructive method employed in the proof of Theorem 1 and the explicit determination of polynomials of best L_1 approximation to the functions x_+^m given by Theorems 2 and 3, we shall show that for each function f in $\mathbf{A}_m$ there is a polynomial

$$P_n(x) = A_0(n) + A_1(n)x + \ldots + A_n(n)x^n$$

satisfying:

$$1^\circ \quad \|f - P_n\|_1 \leqq \frac{C_1}{n^{m+1}}$$

$$2^\circ \quad |A_k(n)| \leqq C_2 3^n \qquad k = 0, 1, \ldots, n$$

where C_1 and C_2 depend only on m.

The same result was obtained by J. KOREVAAR [4] for the class $\mathbf{K}_m$ of those functions f for which $f^{(m-1)}$ is absolutely continuous on $[-1, 1]$ and $f^{(m)}$ is continuous except for a finite number of jump discontinuities on $[-1, 1]$.

2. 1. Approximation of $\mathbf{A}_m$ by $\mathbf{P}_n$ in the $\mathbf{L}_p$ norm.

THEOREM 1. *If* $1 \leqq p \leqq \infty$, *then for* $n \geqq m$, $m = 1, 2, \ldots$

$$E_p(\mathbf{A}_m, \mathbf{P}_n) = \frac{M_p(m, \mathbf{P}_n)}{m!} E_p(x_+^m, \mathbf{P}_n).$$

PROOF. Let $f \in \mathbf{A}_m$ and $g = f^{(m)}$ a.e. where the variation of g on $[-1, 1]$ is $\leqq 1$. If $\varepsilon > 0$, there is a partition $-1 = x_0 < x_1 < \ldots < x_r = 1$ for which

$$\int_{-1}^{1} |g(x) - l_0(x)|\, dx \leqq \frac{\varepsilon}{2^{\frac{1}{p}+m-1}}$$

* The assymptotic behaviour of $M_p(m, \mathbf{P}_n)$ and $E_p(x_+^m, \mathbf{P}_n)$ has been given by RAICIN in *Dokl. Trans. A. M. S.* 6 (1965) 1171—1174.

where l_0 is the step function which has the value $g(x_k)$ on $(x_k, x_{k+1}]$. We have

$$l_0(x) = g(x_0)+\sum_{k=1}^{r}(g(x_k)-g(x_{k-1}))(x-x_k)_+^0.$$

We recursively define for $j=1, 2, \ldots, m$

$$l_j(x) = \int_{-1}^{x} l_{j-1}(t)\,dt+f^{(m-j)}(-1).$$

Then for $x\in[-1, 1]$, we have

$$|f^{(m-j)}(x)-l_j(x)| = \left|\int_{-1}^{x}(f^{(m-j+1)}(t)-l_{j-1}(t))\,dt\right| \leqq$$

$$\leqq \int_{-1}^{1}|f^{(m-j+1)}(t)-l_{j-1}(t)|\,dt \leqq \frac{\varepsilon}{2^{\frac{1}{p}+m-j}}$$

for $j=1, 2, \ldots, m$. Thus, $\|f-l_m\|_p\leqq\varepsilon$.

We have

$$l_m(x) = P(x)+\frac{1}{m!}\sum_{k=1}^{r}(g(x_k)-g(x_{k-1}))(x-x_k)_+^m \tag{2.1}$$

where P is a polynomial of degree $\leqq m$. Let $P_k\in\mathbf{P}_n$ satisfy

$$\|(x-x_k)_+^m-P_k(x)\|_p = E_p((x-x_k)_+^m, \mathbf{P}_n).$$

Then for $Q = P+\frac{1}{m!}\sum_{k=1}^{r}(g(x_k)-g(x_{k-1}))\,P_k$, we have $Q\in\mathbf{P}_n$ and

$$\|l_m-Q\|_p \leqq \frac{1}{m!}\sum_{k=1}^{r}|g(x_k)-g(x_{k-1})|\quad\|(x-x_k)_+^m-\mathbf{P}_k(x)\|_p$$

and so

$$\|l_m-Q\|_p \leqq \frac{M_p(m, \mathbf{P}_n)}{m!}E_p(x_+^m, \mathbf{P}_n).$$

Thus,

$$\|f-Q\|_p \leqq \|f-l_m\|_p+\|l_m-Q\|_p \leqq \varepsilon+\frac{M_p(m, \mathbf{P}_n)}{m!}E_p(x_+^m, \mathbf{P}_n).$$

Since ε is arbitrarily small and f is any function in $\mathbf{A}_m$, we have

$$E_p(\mathbf{A}_m, \mathbf{P}_n) \leqq \frac{M_p(m, \mathbf{P}_n)}{m!}E_p(x_+^m, \mathbf{P}_n).$$

The functions $\frac{(x-a)_+^m}{m!}$ are in $\mathbf{A}_m$ and

$$E_p\left(\frac{(x-a)_+^m}{m!}, \mathbf{P}_n\right) = \frac{E_p((x-a)_+^m, \mathbf{P}_n)}{m!}.$$

Therefore,

$$E_p(\mathbf{A}_m, \mathbf{P}_n) = \frac{M_p(m, \mathbf{P}_n)}{m!} E_p(x_+^m, \mathbf{P}_n)$$

and the theorem is proved.

2. 2. L_1 **approximation.** We now establish two theorems on L_1 approximation which are of interest in themselves and give in particular the value of $E_1(x_+^m, \mathbf{P}_n)$. We denote by $L(f, x_1, x_2, \ldots, x_n, x)$ the Lagrange interpolation polynomial which interpolates the function f at the points $x_1, x_2, \ldots, x_n$.

THEOREM 2. *Let $f(x) = h(x^2)$ where $h^{(n)}(x) > 0$ $(h^{(n)}(x) < 0)$ on $(0, 1)$. Then the polynomial of best L_1 approximation to f on $[-1, 1]$ of degree $\leqq 2n-2, 2n-1$ is $L(f, t_1, t_2, \ldots, t_{2n}, x)$ where*

$$t_k = -\cos\left(\frac{k\pi}{2n+1}\right) \qquad k = 1, 2, \ldots, 2n.$$

Also,

$$E_1(f, \mathbf{P}_{2n-2}) = E_1(f, \mathbf{P}_{2n-1}) = \left| \int_{-1}^{1} f(x) \operatorname{sgn} U_{2n}\, dx \right|$$

where U_{2n} is the Čebyšev polynomial of the second kind.

THEOREM 3. *Let $f(x) = xh(x^2)$ where $h^{(n)}(x) > 0$ $(h^{(n)}(x) < 0)$ on $(0, 1)$. Then the polynomial of best L_1 approximation to f on $[-1, 1]$ of degree $\leqq 2n-1, 2n$ is $L(f, t_1, t_2, \ldots, t_{2n+1}, x)$ where*

$$t_k = -\cos\left(\frac{k\pi}{2n+1}\right) \qquad k = 1, 2, \ldots, 2n+1.$$

Also,

$$E_1(f, \mathbf{P}_{2n-1}) = E_1(f, \mathbf{P}_{2n}) = \left| \int_{-1}^{1} f(x) \operatorname{sgn} U_{2n+1}\, dx \right|.$$

PROOFS. The proofs are similar and only that of Theorem 2 will be given. From the theorem of S. N. BERNSTEIN [5, p.p. 330—332], it is sufficient to show that $f(x) - L(f, t_1, t_2, \ldots, t_{2n}, x)$ changes sign at $t_1, t_2, \ldots, t_{2n}$ and only these points on $[-1, 1]$. Let $Q(x) = L(h, t_1^2, t_2^2, \ldots, t_n^2, x)$. Then the degree of Q is $\leqq n-1$ and from Cauchy's remainder formula for Lagrange interpolation we have that for each $x \in (0, 1)$ there is a $\xi_x \in (0, 1)$ such that

$$h(x) - Q(x) = \frac{h^{(n)}(\xi_x)}{n!}(x - t_1^2) \ldots (x - t_n^2).$$

So that,

$$f(x) - L(f, t_1, t_2, \ldots, t_{2n}, x) = h(x^2) - Q(x^2) = \frac{h^{(n)}(\xi_{x^2})}{n!}(x - t_1) \ldots (x - t_{2n})$$

for $x \in (-1, 1)$ $x \neq 0$. Thus $f(x) - L(f, t_1, t_2, \ldots, t_{2n}, x)$ changes sign at $t_1, t_2, \ldots, t_{2n}$ and only these points on $[-1, 1] \setminus \{0\}$. Since the function $f(x) - L(f, t_1, t_2, \ldots, t_{2n}, x)$ is even, it does not change sign at 0. Finally, since the degree of $Q(x^2)$ is $2n-2$, the theorem is proved.

If we consider the function $f(x)=|x|^s$ $(s>-1)$, Theorem 2 gives the following corollary which was proved by NIKOLSKI using a different method based on Descartes' rule of signs.

COROLLARY 1. *The polynomial of best L_1 approximation to $|x|^s$ $(s>-1)$ on $[-1,1]$ of degree $\leqq 2n-2, 2n-1$ is $L(|x^s|, t_1, t_2, \ldots, t_{2n}, x)$ where*

$$t_k = -\cos\left(\frac{k\pi}{2n+1}\right) \qquad k=1,2,\ldots,2n.$$

Also,

$$E_1(|x|^s, \mathbf{P}_{2n-2}) = E_1(|x^s|, \mathbf{P}_{2n-1}) = \frac{2}{s+1}\left|2\sum_{k=1}^{n}(-1)^k\left(\cos\left(\frac{k\pi}{2n+1}\right)\right)^{s+1}+1\right|.$$

Let us now consider the function $f(x)=x^{m-1}|x|$ when m is an integer $\geqq -1$. Since $x_+^m=\frac{1}{2}(|x|x^{m-1}+x^m)$, we have $E_1(x_+^m, \mathbf{P}_n)=\frac{1}{2}E_1(f,\mathbf{P}_n)$ for $n\geqq m$. Therefore, we have the following two corollaries to Theorems 2 and 3.

COROLLARY 2. *For m an odd positive integer and $n\geqq m$*

$$E_1(x_+^m, \mathbf{P}_n) = \frac{1}{m+1}\left|1+2\sum_{k=1}^{[\frac{1}{2}n]+1}(-1)^k\left(\cos\left(\frac{k\pi}{2[\frac{1}{2}n]+1}\right)\right)^{m+1}\right|.$$

COROLLARY 3. *For m an even non negative integer and $n\geqq m$*

$$E_1(x_+^m, \mathbf{P}_n) = \frac{1}{m+1}\left|1+2\sum_{k=1}^{[\frac{1}{2}(n+1)]+1}(-1)^k\left(\cos\left(\frac{k\pi}{2[\frac{1}{2}(n+1)]+1}\right)\right)^{m+1}\right|.$$

NIKOLSKI [3] has shown that $M_1(m,\mathbf{P}_n)=1+O\left(\frac{\log n}{n}\right)$. Thus in the case $p=1$, Theorem 1 becomes:

COROLLARY 4. *For $n\geqq m, m=1,2,\ldots$*

$$E_1(\mathbf{A}_m, \mathbf{P}_n) = \left(1+O\left(\frac{\log n}{n}\right)\right)\frac{E_1(x_+^m, \mathbf{P}_n)}{m!}$$

where $E_1(x_+^m, \mathbf{P}_n)$ is given in corollaries 2 and 3.

3. Estimates on the coefficients of polynomial approximations to functions in $\mathbf{A}_m$

The principal result of this section is the following theorem.

THEOREM 4. *If $f\in\mathbf{A}_m$, there is a polynomial*

$$P_n(x) = A_0(n)+A_1(n)x+\ldots+A_n(n)x^n$$

satisfying

$$1^\circ \quad \int_{-1}^{1}|f(x)-P_n(x)|\,dx \leqq \frac{C_1}{n^{m+1}}$$

$$2^\circ \quad |A_k(n)|\leqq C_2 3^n \qquad k=0,1,\ldots,n$$

where C_1 and C_2 depend only on m.

Proof. We consider only the case when both m and n are odd. Other cases are handled in a similar manner. We can also assume that $n \geqq m$. Let Q_n denote the polynomial of best L_1 approximation to x_+^m on $[-1, 1]$ of degree $\leqq n$ which is given by Theorem 2. Then for $|a| \leqq 1$

$$\int_{-1}^{1} \left| (x-a)_+^m - 2^m Q_n\left(\frac{x-a}{2}\right) \right| dx = \int_{-1-a}^{1-a} \left| x_+^m - 2^m Q_n\left(\frac{x}{2}\right) \right| dx \leqq \tag{3.1}$$

$$\leqq \int_{-2}^{2} \left| x_+^m - 2^m Q_n\left(\frac{x}{2}\right) \right| dx = 2^{m+1} E_1(x_+^m, \mathbf{P}_n).$$

Let $f \in \mathbf{A}_m$. Using the notation introduced in the proof of Theorem 1, for a suitable partition $-1 = x_0 < x_1 < \ldots < x_r = 1$, we have

$$\int_{-1}^{1} |f(x) - l_m(x)|\, dx \leqq \varepsilon \leqq E_1(x_+^m, \mathbf{P}_n). \tag{3.2}$$

We define

$$P_n(x) = P(x) + \frac{1}{m!} \sum_{k=1}^{r} (g(x_k) - g(x_{k-1})) 2^m Q_n\left(\frac{x - x_k}{2}\right) = \tag{3.3}$$

$$= A_0(n) + A_1(n)x + \ldots + A_n(n)x^n.$$

Using (2.1), (3.1), and (3.3), we have

$$\int_{-1}^{1} |l_m(x) - P_n(x)|\, dx \leqq \tag{3.4}$$

$$\leqq \frac{1}{m!} \sum_{k=1}^{r} \left(|g(x_k) - g(x_{k-1})| \int_{-1}^{1} \left| (x - x_k)_+^m - 2^m Q_n\left(\frac{x - x_k}{2}\right) \right| dx \right) \leqq \frac{2^{m+1}}{m!} E_1(x_+^m, \mathbf{P}_n).$$

Thus, by (3.2), (3.4), and (1.4)

$$\int_{-1}^{1} |f(x) - P_n(x)|\, dx \leqq \left(\frac{2^m}{m!} + 1\right) E_1(x_+^m, \mathbf{P}_n) \leqq \frac{C_1}{n^{m+1}}.$$

We now estimate the coefficients of P_n. From Theorem 2 and the Lagrange interpolation formula, it follows that

$$Q_n\left(\frac{x-a}{2}\right) = L\left(|x|_+^m, t_1, t_2, \ldots, t_{n+1}, \frac{x-a}{2}\right) = \sum_{t_k > 0} \frac{t_k^m U_{n+1}\left(\frac{x-a}{2}\right)}{U'_{n+1}(t_k)\left(\frac{x-a}{2} - t_k\right)},$$

i. e.

(3.5) $$Q_n\left(\frac{x-a}{2}\right)=\sum_{k=\frac{n+1}{2}+1}^{n+1}\frac{(-1)^{k+1}(n+2)\left(-\cos\left(\frac{k\pi}{n+2}\right)\right)^m U_{n+1}\left(\frac{x-a}{2}\right)}{\sin^2\left(\frac{k\pi}{n+2}\right)\left(\frac{x-a}{2}-t_k\right)}=$$

$$=B_0(n,a)+B_1(n,a)x+\ldots+B_n(n,a)x^n.$$

Thus, by Cauchy's inequality and the maximum modulus principle we have for $|a|\leqq 1$

$$|B_k(n,a)|\leqq\sup_{|z|=1}\left|Q_n\left(\frac{z-a}{2}\right)\right|\leqq\sup_{|z|=1}|Q_n(z)|\leqq$$

$$\leqq\left(\frac{(n+2)^2}{2}\frac{1}{\sin^2\left(\frac{\pi}{n+2}\right)}\max_k\left\{\sup_{|z|=1}\left|\frac{U_{n+1}(z)}{z-t_k}\right|\right\}\right).$$

Since

$$\sin^{-2}\left(\frac{\pi}{n+2}\right)\leqq\frac{1}{4}(n+2)^2\quad\text{and for}\quad |z|=1$$

$$|z-t_k|\geqq 1-|t_k|\geqq 1-\cos\left(\frac{\pi}{n+2}\right)\geqq\frac{2}{(n+2)^2},$$

we have for $|a|\leqq 1$ and $k=0,1,\ldots,n$

(3.6) $$|B_k(n,a)|\leqq\frac{(n+2)^6}{16}\sup_{|z|=1}|U_{n+1}(z)|.$$

The leading coefficient of U_{n+1} is 2^{n+1} and thus

$$\sup_{|z|=1}|U_{n+1}(z)|=2^{n+1}\sup_{|z|=1}|(z-t_1)(z-t_2)\ldots(z-t_{n+1})|=$$

$$=2^{n+1}\sup\left|(z^2-t_1^2)(z^2-t_2^2)\ldots\left(z^2-t_{\frac{n+1}{2}}^2\right)\right|\leqq 2^{n+1}(1+t_1^2)(1+t_2^2)\ldots\left(1+t_{\frac{n+1}{2}}^2\right).$$

Since

$$(1+t_1^2)(1+t_2^2)\ldots\left(1+t_{\frac{n+1}{2}}^2\right)\leqq\exp\left(\sum_{k=1}^{\frac{n+1}{2}}\cos^2\left(\frac{k\pi}{n+2}\right)\right)=\exp\left(\frac{n+2}{4}\right)$$

it follows that

$$\sup_{|z|=1}|U_{n+1}(z)|\leqq 2^{n+1}\exp\left(\frac{1}{4}(n+2)\right).$$

Using (3.6), we find finally, that

(3.7) $$|B_k(n,a)|\leqq(n+2)^6 2^n\exp\left(\frac{n+2}{4}\right)\leqq C_0 3^n$$

where C_0 is a constant independent of n, and $k=0,1,\ldots,n$.

Next, from (3. 3) and (3. 5), it follows that

$$A_j(n) = a_j + \frac{1}{m!} \sum_{k=1}^{r} (g(x_k) - g(x_{k-1})) 2^m B_j(n, a)$$

where a_j is the j^{th} coefficient of P. Hence, from (3. 7) it follows that

$$|A_j(n)| \leqq |a_j| + \frac{2^m}{m!} \sum_{k=1}^{r} |g(x_k) - g(x_{k-1})| C_0 3^n \leqq |a_j| + C_0 3^n \leqq C_2 3^n$$

for $j = 0, 1, \ldots, n$ and the theorem is proved.

REFERENCES

[1] FAVARD, J.: Sur les meilleures procedes d'approximation de certain classes des fonctions par des polynomes trigonometriques, *Bull. Sci. Math.* 61 (1937) 209—224.

[2] NIKOLSKI, S.: On the best approximation of differentiable non-periodic functions by polynomials, *Acta Sci. Math. (Szeged)* 12 (1950) 185—197.

[3] NIKOLSKI, S.: Sur la meilleure approximation en moyenne par polynomes des fonctions ayant des singularites des la forme $|a-x|^r$, *Dokl. Akad. Nauk SSSR* 55 (1947) 191—194.

[4] KOREVAAR, J.: Best L_1 approximation and the remainder in Littlewood's Theorem, *Nederl. Akad. Wetensch. Indag. Math.* 15 (1953) 281—293.

Oakland University Rochester, Michigan, USA

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