JOURNAL OF APPROXIMATION THEORY 1, 314–318 (1968)

On Jackson's Theorem

RONALD DEVORE

The Ohio State University, Columbus, Ohio 43210*

If f is a continuous function on $[a,b]$ and $\omega(f,h)$ is its modulus of continuity, then the classical theorem of Jackson ([*1*], p. 15) states that for each positive n there is an algebraic polynomial P of degree $\leqslant n$ such that

$$\max_{a\leqslant x\leqslant b} |f(x)-P(x)| \leqslant C\omega(f,1/n) \tag{1}$$

where C is a constant independent of f and n. Jackson's proof of this theorem consisted of proving the analogous result for approximation of 2π-periodic functions by trigonometric polynomials and using a standard transformation to obtain (1). At the Oberwolfach Conference on Approximation Theory in 1963 ([*2*], p. 180), P. L. Butzer has pointed out the desirability for a direct proof of (1). Subsequently, G. Freud [*3*] and R. B. Saxena [*4*] have both constructed interpolation procedures which lead to (1). Freud's results were later generalized by M. Sallay [*7*].

It is natural to ask whether one can obtain Jackson's theorem by considering convolution with non-negative algebraic polynomials. For example, the Landau polynomials

$$C_n\int_{-1}^{1} f(t)(1-(t-x)^2)^n\,dt, \qquad 1/C_n=\int_{-1}^{1}(1-x^2)^n\,dx$$

converge uniformly to f on $[-\delta,\delta]$ if $\delta<1$, provided that f is continuous on $[-1,1]$. However, they do not provide the order of approximation (1).

In this paper, we shall consider the nonnegative polynomials

$$\Lambda_n(t)=C_n\frac{(P_{2n}(t))^2}{(t^2-x_{n+1}^2)^2}, \qquad 1/C_n=\int_{-1}^{1}\frac{(P_{2n}(t))^2\,dt}{(t^2-x_{n+1}^2)^2},$$

where $P_{2n}(t)$ is the Legendre polynomial of degree $2n$, and x_{n+1} is its smallest positive zero. Our main result is the following

THEOREM 1. *If f is a continuous function on $[-\frac{1}{2},\frac{1}{2}]$ and $f(-\frac{1}{2})=f(\frac{1}{2})=0$, then the polynomial*

$$L_n(f,x)=\int_{-1/2}^{1/2} f(t)\,\Lambda_n(t-x)\,dt$$

*Present address: Oakland University, Rochester, Michigan.

of degree $\leqslant 4n-4$, *satisfies*

$$\max_{-\frac{1}{2}\leqslant x\leqslant\frac{1}{2}} |f(x)-L_n(f,x)| \leqslant 40\omega(f,1/n), \qquad n=1,2,3,\ldots. \tag{2}$$

It is readily seen that the restriction that f vanishes at the end-points is necessary to guarantee uniform convergence of polynomials which arise from convoluting f with an even polynomial. Essentially, this is due to the fact that the integrals of the kernels over $[-1,0]$ and $[0,1]$ are only $\frac{1}{2}$. It is easy to obtain Jackson's theorem from Theorem 1. The polynomial

$$\bar{L}_n(f,x) = l(x) + L_n(f-l,x),$$

where

$$l(x) = f(-\tfrac{1}{2}) + [f(\tfrac{1}{2}) - f(-\tfrac{1}{2})](x+\tfrac{1}{2}),$$

provides the estimate (1) for an arbitrary continuous function when $a=-\frac{1}{2}$, $b=\frac{1}{2}$.

We shall need the following.

LEMMA 1. *Let* $\gamma(t)$ *have total variation* $\leqslant A$ *on* $[-\frac{1}{2},\frac{1}{2}]$, $A\geqslant 1$.

If f *is continuous on* $[-\frac{1}{2},\frac{1}{2}]$ *and* $f(-\frac{1}{2})=f(\frac{1}{2})=0$, *then*

$$\left|\int_{-1/2}^{1/2} f(t)\,d\gamma(t)\right| \leqslant 4A\omega(f,\delta),$$

where

$$\delta = \int_{-1/2}^{1/2} |\gamma(t)|\,dt.$$

Proof. Let us first consider an arbitrary function g in $\mathrm{Lip}_M 1$ which vanishes at $-\frac{1}{2}$ and $\frac{1}{2}$. We have

$$g(x) = \int_{-1/2}^{x} g'(t)\,dt, \qquad \text{with } |g'(t)| \leqslant M \text{ a.e. on } [-\tfrac{1}{2},\tfrac{1}{2}].$$

Thus,

$$\left|\int_{-1/2}^{1/2} g(t)\,d\gamma(t)\right| = \left|\int_{-1/2}^{1/2} g'(t)\gamma(t)\,dt\right| \leqslant M\int_{-1/2}^{1/2} |\gamma(t)|\,dt = M\delta.$$

Suppose now that f is an arbitrary continuous function on $[-\frac{1}{2},\frac{1}{2}]$. We can find a concave modulus of continuity $\omega_1(t)$ (see [5], p. 45) satisfying

$$\omega(f,h) \leqslant \omega_1(h) \leqslant 2\omega(f,h), \qquad 0\leqslant h\leqslant 1. \tag{4}$$

We shall use the following result on approximation by functions in $\mathrm{Lip}_M 1$ (see [5], pp. 122–123).

PROPOSITION 1. *Let* $0<\eta\leqslant 1$ *and let* ω_1 *be a concave modulus of continuity. Then, there exists an* $M>0$ *such that for each continuous function* f *whose modulus of continuity* ω *satisfies*

$$\omega(f,h) \leqslant \omega_1(h), \qquad h>0$$

we can find a function g in $\mathrm{Lip}_M 1$ *for which*

$$\max_{-\frac{1}{2} \leqslant x \leqslant \frac{1}{2}} |f(x) - g(x)| \leqslant \omega_1(\eta) - M\eta.$$

Now the rest of the proof of Lemma 1 is simple. Let $\eta = \delta$ in the above proposition. Suppose f satisfies the hypothesis of Lemma 1, and g is the function given by Proposition 1. The function

$$\bar{g}(x) = g(x) - [g(-\tfrac{1}{2}) + (g(\tfrac{1}{2}) - g(-\tfrac{1}{2}))(x + \tfrac{1}{2})]$$

is in $\mathrm{Lip}_{2M} 1$ and satisfies

$$\begin{aligned}\max_x |f(x) - \bar{g}(x)| &\leqslant \max_x |f(x) - g(x)| + \max_x |g(x) - \bar{g}(x)| \\ &\leqslant \omega_1(\delta) - M\delta + \omega_1(\delta) - M\delta = 2(\omega_1(\delta) - M\delta), \qquad (5)\end{aligned}$$

since

$$|g(-\tfrac{1}{2})| \leqslant \omega_1(\delta) - M\delta \quad \text{and} \quad |g(\tfrac{1}{2})| \leqslant \omega_1(\delta) - M\delta.$$

Thus,

$$\begin{aligned}\left|\int_{-1/2}^{1/2} f(t)\, d\gamma(t)\right| &\leqslant \int_{-1/2}^{1/2} |f(t) - \bar{g}(t)| \cdot |d\gamma(t)| + \left|\int_{-1/2}^{1/2} \bar{g}(t)\, d\gamma(t)\right| \\ &\leqslant (2\omega_1(\delta) - 2M\delta) A + 2M\delta \leqslant 2A\omega_1(\delta),\end{aligned}$$

where the first term was estimated by (5) and the second term by (3). The proof is complete by invoking (4).

We note two elementary properties of the polynomials P_{2n} which can be found in [*6*].

PROPOSITION 2. ([*6*], *p.* 121). *If* x_{n+1} *is the smallest positive zero of* P_{2n}, *then* $x_{n+1} \leqslant 2/n$.

PROPOSITION 3. (*The Gauss Quadrature Formula* [*6*], *p.* 97.) *Let* $x_1, \ldots, x_{2n}$ *be the zeros of* P_{2n} *written in increasing order. Then, there exist real positive constants* $A_k^{(n)}$, $k = 1, 2, \ldots, 2n$, *such that for each polynomial* Q *of degree* $\leqslant 4n - 1$, *we have*

$$\int_{-1}^{1} Q(t)\, dt = \sum_{k=1}^{2n} A_k^{(n)} Q(x_k).$$

Note: *Since* P_{2n} *is an even polynomial,* $x_n = -x_{n+1}$ *and* $A_n^{(n)} = A_{n+1}^{(n)}$.

Now to the proof of Theorem 1. Let u denote the Dirac measure having unit mass at 0. If f is continuous on $[-\frac{1}{2}, \frac{1}{2}]$, we have the representation

$$f(x) - L_n(f, x) = \int_{-1/2}^{1/2} f(t)[du(t - x) - \Lambda_n(t - x)\, dt] = \int_{-1/2}^{1/2} f(t)\, d\gamma_n(t - x),$$

where

$$\gamma_n(t) = u(t) - \int_{-1}^{t} \Lambda_n(x)\, dx.$$

Also, for $|x| \leqslant \frac{1}{2}$,

$$\int_{-1/2}^{1/2} |\gamma_n(t-x)|\, dt \leqslant \int_{-1}^{1} |\gamma_n(t)|\, dt. \tag{6}$$

If we integrate by parts, we find

$$\int_{-1}^{1} |\gamma_n(t)|\, dt = \int_{-1}^{1} |t|\, \Lambda_n(t)\, dt.$$

Now,

$$\int_{-1/n}^{1/n} |t|\, \Lambda_n(t)\, dt \leqslant 1/n \int_{-1/n}^{1/n} \Lambda_n(t)\, dt \leqslant 1/n.$$

Using Proposition 3 and observing that Λ_n is a polynomial of degree $4n-4$, we have

$$\begin{aligned}\int_{[-1,1]-[-n^{-1},n^{-1}]} |t|\, \Lambda_n(t)\, dt &\leqslant n \int_{-1}^{1} t^2 \Lambda_n(t)\, dt \\ &= n \sum_{k=1}^{2n} A_k^{(n)} x_k^{\,2} \Lambda_n(x_n) \\ &= n(A_n^{(n)} x_n^{\,2} \Lambda_n(x_n) + A_n^{(n+1)} x_{n+1}^2) \\ &= n x_{n+1}^2 (A_n^{(n+1)} \Lambda_n(x_{n+1}) + A_n^{(n)} \Lambda_n(x_n)) \\ &= n x_{n+1}^2 \int_{-1}^{1} \Lambda_n(t)\, dt = n x_{n+1}^2.\end{aligned}$$

Thus from Proposition 2, we find

$$\int_{[-1,1]-[-n^{-1},\, n^{-1}]} |t|\, \Lambda_n(t)\, dt \leqslant n x_{n+1}^2 \leqq 4/n.$$

By virtue of (6), we have for $|x| \leqslant \frac{1}{2}$,

$$\int_{-1/2}^{1/2} |\gamma_n(t-x)|\, dt \leqslant \int_{-1}^{1} |t|\, \Lambda_n(t)\, dt \leqslant 1/n + 4/n = 5/n. \tag{7}$$

Finally, since the total variation of $\gamma_n(t-x)$ is $\leqslant 2$ on $[-\frac{1}{2}, \frac{1}{2}]$, and $\omega(f, 5/n) \leqslant 5\omega(f, 1/n)$, Theorem 1 follows immediately from Lemma 1 and from (7).

Acknowledgment

The author would like to express his gratitude to Professor R. Bojanic and to Professor P. L. Butzer for their valuable suggestions.

References

1. Dunham Jackson, "The Theory of Approximation," Vol. XI. Amer. Math. Soc. Colloquium Publications, New York, 1930.
2. On approximation theory. Proceedings of the Conference held in the Mathematical Research Institute at Oberwolfach, 1963, Basel: Birkhauser, 1964.

3. G. FREUD, Über ein Jacksonisches Interpolationsverfahren, in ref. *2*, 227–232.
4. R. B. SAXENA, On a polynomial of interpolation. *Studia Sci. Math. Hungar.* **2** (1967), 167–183.
5. G. G. LORENTZ, "Approximation of Functions." Holt, New York, 1966.
6. GABOR SZEGÖ, "Orthogonal Polynomials," Vol. XXIII. Amer. Math. Soc. Colloquium Publications, New York, 1959.
7. M. SALLAY, "Über ein Interpolationsverfahren." Publ. Math. Inst., Hungar. Acad. Sci., Vol. IX, Series A, Fasc. 3 (1964–65), 607–615.