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OPTIMAL CONVERGENCE OF POSITIVE LINEAR OPERATORS

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1. Introduction

Let $C[a, b]$ denote the space of continuous functions on the interval $[a, b]$ and for $f \in C[a, b]$ let $\|f\| [a, b] = \sup_{a \le x \le b} |f(x)|$ (we will sometimes write $\|f\|_\delta$ for $\|f\| [-\delta, \delta]$). If (L_n) is a sequence of positive linear operators mapping $C[-1, 1]$ into $C[a, b]$ with $[a, b] \subseteq [-1, 1]$ then a famous theorem of Bohman–Korovkin [1] states that a necessary and sufficient condition that

$$\|f - L_n(f)\| [a, b] = o(1) \quad (n \to \infty)$$

should hold for each f in $C[-1, 1]$ is that for $i = 1, 2, 3$ we have

$$(1.1) \qquad \|e_i - L_n(e_i)\| [a, b] = o(1) \quad (n \to \infty)$$

where $e_i(x) = x^{i-1}$.

This result was recently put into a quantitative form by Shisha and Mond [2] in the following theorem.

Theorem A. *If (L_n) is a sequence of positive operators mapping $C[-1, 1]$ into $C[a, b]$ with $[a, b] \subseteq [-1, 1]$ and f is a continuous function on $[-1, 1]$ with modulus of continuity $\omega(f, h)$ then*

$$(1.2) \qquad \|f - L_n(f)\| [a, b] \le (\|f\| [a, b]) (\|e_1 - L_n(e_1)\| a, b]) + \\ + (\|e_1 + L_n(e_1)\| [a, b]) \, \omega(f, \mu_n)$$

where

$$(1.3) \qquad \mu_n^2 = \|L_n(t - x)^2, x)\| [a, b] \qquad n = 1, 2, \ldots .$$

In particular if

$$(1.4) \qquad \|e_i - L_n(e_i)\| [a, b] = O(\lambda_n^2) \quad (n \to \infty)$$

for $i = 1, 2, 3$ then there is a constant $C > 0$ such that

$$(1.5) \quad \|f - L_n(f)\| [a, b] \le C(1 + \|f\| [a, b]) \left(\omega(f, \lambda_n) + \lambda_n^2\right), \quad n = 1, 2, \ldots .$$

Note. In the expression (1.3) the operator L_n is applied to the variable t and the norm is taken with respect to the variable x. This convention will be used throughout.

This theorem gives an explicit method for determining an upper bound for the order of approximation by means of positive linear operators which in many cases is the best possible [2], [3].

It is easy to extend Theorem A to include estimates for the approximation of differentiable functions. In Section 2, we will show that if f is a continuously differentiable function on $[a, b]$ then

$$(1.6)\qquad \|f - L_n(f)\|\,[a,b] \leq (\|f\|\,[a,b])\,(\|e_1 - L_n(e_1)\|\,[a,b]) + \\ + (\|f'\|\,[a,b])\,(\|L_n(t-x,x))\,[a,b]) + (1 + (|L_n(e_1)\|\,[a,b])^{1/2})\,\mu_n\,\omega(f',\mu_n)\,.$$

In particular if

$$\|e_i - L_n(e_i)\|\,[a,b] = O(\lambda_n^2) \quad (n \to \infty) \quad i = 1, 2, 3$$

then

$$(1.7)\qquad \|f - L_n(f)\|\,[a,b] \leq \\ \leq C(1 + \|f\|\,[a,b] + \|f'\|\,[a,b])\,(\lambda_n\,\omega(f',\lambda_n) + \lambda_n^2), \quad n = 1, 2, \ldots .$$

Let us now consider the particular case when (L_n) is also a sequence of polynomial operators (i.e. for each n the range of L_n is contained in P_n the space of algebraic polynomials of degree $\leq n$). Then a lower bound for the order of approximation by the sequence (L_n) was also given by KOROVKIN [1] in the following theorem.

THEOREM B. *If (L_n) is a sequence of positive polynomial operators and $-1 \leq a < b \leq 1$ then for one of the functions e_i, $i = 1, 2, 3$ we have*

$$\|e_i - L_n(e_i)\|\,[a,b] \neq o(1/n^2) \qquad (n \to \infty)\,.$$

Actually, Theorem B can be derived from Theorem A and the fact that the function $f(x) = |\,x - (a+b)/2\,|$ is in Lip 1 and cannot be approximated by algebraic polynomials with order $o(1/n)$ $(n \to \infty)$ (see [4], p. 44). Theorem B shows that when the sequence (λ_n) given by (1.4) is $O(1/n^2)$ $(n \to \infty)$ then the estimates (1.5) and (1.7) are of the best possible order.

A slightly stronger form of Theorem B is given by Corollary 3.2 which shows that if for $\delta > 0$, $i = 1, 2$

$$\|e_i - L_n(e_i)\|_\delta = O(1/n^2) \quad (n \to \infty)$$

then

$$\|e_3 - L_n(e_3)\|_\delta \neq o(1/n^2) \quad (n \to \infty)\,.$$

Thus if (L_n) is a sequence of positive polynomial operators defined on $C[-1, 1]$ then an optimal behavior in terms of approximation on $[-\delta, \delta]$ occurs when

$$(1.8)\qquad \|e_i - L_n(e_i)\|_\delta = o(1/n^2) \quad (n \to \infty)\ i = 1, 2$$

$$(1.9)\qquad \|e_3 - L_n(e_3)\|_\delta = O(1/n^2) \quad (n \to \infty)\,.$$

If (1.8) and (1.9) hold we will say the sequence (L_n) is optimal on $[-\delta, \delta]$. In Section 5, we will give some examples of optimal operators.

Theorem B indicates that the order of approximation by means of the sequence (L_n) is limited by $1/n^2$ at least in the sense that no differentiability condition can guarantee better approximation than $O(1/n^2)$. Of course, Theorem B does not preclude the possibility of having $(L_n(f))$ approximate f with order $o(1/n^2)$ for certain functions f. Our main result is to strengthen Korovkin's theorem for the case of optimal sequences of operators by showing that if (L_n) is optimal on $[-\delta, \delta]$ for each $\delta < 1$ and if $\|f - L_n(f)\|_\delta = o(1/n^2)$ $(n \to \infty)$ for each $\delta < 1$ then f is linear on $[-1, 1]$. In particular, we have that if (L_n) is optimal on $[-1, 1]$ then (L_n) is saturated on $[-1, 1]$ with order $1/n^2$. A similar result for 2π-periodic functions was given by CURTIS [5] who showed that if $L_n(f, x) = \int_{-\pi}^{\pi} f(t)\, T_n(t - x)\, dt$ with T_n a non-negative trigonometric polynomial of degree $\leq n$, $n = 1, 2, \ldots$ and if

$$\|e_1 - L_n(e_1)\|_\pi = o(1/n^2) \quad (n \to \infty)$$

then

$$\|f - L_n(f)\|_\pi = o(1/n^2) \quad (n \to \infty)$$

holds for a 2π-periodic function f if and only if f is constant. Of course, this result covers only the special case when the operators are given by convolution with a trigonometric polynomial. No result as general as those given in this paper for the algebraic case are known for the trigonometric case. There appears to be a difficulty in carrying our result directly over to the trigonometric case.

There are numerous examples of saturation theorems in the literature. We mention in particular the paper of BAJSANSKI and BOJANIC [6] on the saturation of Bernstein polynomials and an extension of this paper as given by KARAMATA and VUILLEUMEIR [7] since these papers contain some techniques which we will use.

2. Approximation of differentiable functions by means of positive linear operators

In this section we will prove the following extension of Theorem A.

THEOREM 2.1. *If (L_n) is a sequence of positive linear operators mapping $C[-1, 1]$ into $C[a, b]$ with $[a, b] \subseteq [-1, 1]$ and f is a continuously differentiable function then*

$$\begin{aligned}(2.1)\qquad \|f - L_n(f)\|\,[a, b] &\leq (\|f\|\,[a, b])(\|e_1 - L_n(e_1)\|\,[a, b]) + \\ &+ (\|f'\|\,[a, b])(\|L_n((t - x), x)\|\,[a, b]) + \\ &+ \left(1 + (\|L_n(e_1)\|\,[a, b])^{1/2}\right)\mu_n\, \omega(f', \mu_n) \qquad n = 1, 2, \ldots\end{aligned}$$

where $\mu_n^2 = \|L_n((t - x)^2, x)\|\,[a, b]$.

In particular, if for $i = 1, 2, 3$

$$(2.2)\qquad \|e_i - L_n(e_i)\|\,[a, b] = O(\lambda_n^2) \quad (x \to \infty).$$

There is a constant $C > 0$ such that for each continuously differentiable unc tion f

(2.3)
$$\|f - L_n(f)\|[a,b] \leq$$
$$\leq C(1 + \|f\|[a,b] + \|f'\|[a,b])\left(\lambda_n\,\omega(f',\lambda_n) + \lambda_n^2\right) \qquad n = 1, 2, \ldots .$$

PROOF. Let $x \in [a, b]$ and $t \in [-1, 1]$. We have from the mean value theorem that

$$|f(t) - f(x) - f'(x)(t - x)| \leqslant |t - x|\,\omega(f', |t - x|) .$$

Thus from elementary properties of ω have

(2.4)
$$|f(t) - f(x) - f'(x)(t - x)| \leq |t - x|\,\omega\left(f', \frac{\mu_n}{\mu_n}|t - x|\right) \leq$$
$$\leq |t - x|\left(1 + \frac{|t - x|}{\mu_n}\right)\omega(f', \mu_n) .$$

Using the Cauchy–Schwartz inequality for positive operators we have

$$L_n(|t - x|, x) \leq \left(L_n((t - x)^2, x)\right)^{1/2}\left(L_n(e_1, x)\right)^{1/2} \leq \mu_n\left(\|L_n(e_1)\|[a, b]\right)^{1/2} .$$

So that (2.4) becomes

$$L_n\left(|f(t) - f(x) - f'(x)(t - x)|, x\right) \leq \left(1 + (\|L_n(e_1)\|[a, b])^{1/2}\right)\mu_n\,\omega(f', \mu_n) .$$

Finally,

$$|f(x) - L_n(f, x)| \leq |f(x) - L_n(f(x), x)| + |f'(x)\,L_n(t - x, x)| +$$
$$+ |L_n(f(t) - f(x) - f'(x)(t - x), x)| \leq (\|f\|[a, b])(\|e_1 - L_n(e_1)\|[a, b]) +$$
$$+ (\|f'\|[a, b])(\|L_n(t - x, x)\|[a, b]) + \left(1 + (\|L_n(e_1)\|[a, b])^{1/2}\right)\mu_n\,\omega(f', \mu_n).$$

The inequality (2.3) follows easily from (2.1) and the fact that $\left(\|L_n(e_1)\|[a,b]\right)$ is a bounded sequence.

3. Local approximation by positive linear operators

From Theorem B we see that if (L_n) is a sequence of positive polynomial operators then for any interval $[a, b] \subsetneqq [-1, 1]$ there is a constant $C > 0$ such that

$$\|L_n((t - x)^2, x)\|[a, b] \geq \frac{C}{n^2} \qquad n = 1, 2, \ldots .$$

The constant C depends on the interval $[a, b]$. In this section we shall give an analysis of this dependence which we shall use in Section 4.

It is well known ([4], p. 94) that there is a constant $C > 0$ such that for any function f, in $\mathrm{Lip}_1\, 1$ and any integer $n \geq 0$ there exists an algebraic polynomial P of degree $\leq n$ satisfying

$$|f(x) - P(x)| \leq C \left(\frac{\sqrt{1-x^2}}{n} + \frac{1}{n^2} \right) \quad |x| \leq 1.$$

Since the best order of approximation for the class $\mathrm{Lip}_1\, 1$ is $O\left(\frac{1}{n}\right)$, this result shows that it is possible to approximate functions in $\mathrm{Lip}_1\, 1$ by algebraic polynomials with better local approximation at the end points of the interval while retaining the best order on the whole interval. We shall need the following result which shows that this phenomenon is unique for the end points of the interval.

THEOREM 3.1. *If $M > 0$ is any constant then there are positive constants C_1 and C_2 such that for any $x_0 \in [-1, 1]$ and n a positive integer there exists a function $f_n \in \mathrm{Lip}_1\, 1$ for which there is no algebraic polynomial P of degree $\leq n$ satisfying*

$$(3.1) \qquad \|f_n - P\|_1 \leq \frac{M}{n}$$

and

$$(3.2) \qquad \|f_n - P\| \left[x_0 - \frac{C_1}{n},\, x_0 + \frac{C_1}{n} \right] \leq \frac{C_2\sqrt{1-x_0^2}}{n}.$$

We shall first establish the analogous result for trigonometric approximation.

THEOREM 3.2. *If $M^* > 0$ is any constant then there are positive constants C_1^* and C_2^* such that for any $x_0 \in [-\pi, \pi]$ and n a positive integer there exists a 2π periodic even function g_n in $\mathrm{Lip}_1\, 1$ for which there is no trigonometric polynomial T of degree $\leq n$ satisfying*

$$(3.3) \qquad \|g_n - T\|_\pi \leq \frac{M^*}{n}$$

and

$$(3.4) \qquad \|g_n - T\| \left[x_0 - \frac{C_1^*}{n},\, x_0 + \frac{C_1^*}{n} \right] \leq \frac{C_2^*}{n}.$$

PROOF OF THEOREM 3.2. We shall need some preliminary results concerning representation of trigonometric polynomials. Let $x_k = \frac{2k\pi}{3n}$, $k = 0, 1, \ldots, 3n-1$. Then if T is a trigonometric polynomial of degree $\leq n$, we have ([9], p. 33)

$$(3.5) \qquad T(x) = \frac{2}{3n} \sum_{k=0}^{3n-1} T(x_k) K_n(x - x_k)$$

where

$$(3.6)\qquad K_n(t) = \frac{1}{2n}\,\frac{\sin\left(\dfrac{3nt}{2}\right)\sin\left(\dfrac{nt}{2}\right)}{\sin^2\dfrac{t}{2}}.$$

Now to the proof of Theorem 3.2. We shall assume that $x_0 \in [0, \pi]$, the other case is handled similarly. Let k_0 be that integer for which $x_0 \in \left[\dfrac{2\pi k_0}{3n}, \dfrac{2\pi(k_0+1)}{3n}\right] = I_0$ and let y be the midpoint of I_0. If $0 \leq k \leq$ $\leq 3n-1$ we have from (3.6) that

$$(3.7)\qquad |K_n(y-x_k)| \leq \frac{1}{2n}\,\frac{1}{\left(\dfrac{1}{\pi}\,\dfrac{y-x_k}{2}\right)^2} = \frac{9n}{2\left(k-k_0-\dfrac{1}{2}\right)^2}.$$

Consider the 2π periodic even function $g_n(x_0, x)$ which is zero on $[0, \pi]\setminus I_0$ and the roof function $\dfrac{\pi}{3n} - |x-y|$ for $x \in I_0$. We shall show that $g_n(x) =$ $= g_n(x_0, x)$ satisfies the conclusions of Theorem 3.2 with C_1^* and C_2^* to be determined below.

Suppose T is a trigonometric polynomial of degree $\leq n$ satisfying (3.3) and (3.4). From (3.5) and (3.7) we have

$$|T(y)| \leq \frac{2}{3n} \sum_{|x_k - x_0| \leq \frac{C_1^*}{n}} |T(x_k)| \frac{9n}{2\left(k-k_0-\dfrac{1}{2}\right)^2} +$$

$$+ \frac{2}{3n} \sum_{|x_k - x_0| > \frac{C_1^*}{n}} |T(x_k)| \frac{9n}{2\left(k-k_0-\dfrac{1}{2}\right)^2}.$$

Since $g_n(x_k) = 0$, $0 \leq k \leq 3n-1$, we have

$$|T(y)| \leq \frac{3C_2^*}{n} \sum_{|x_k - x_0| \leq \frac{C_1^*}{n}} \frac{1}{\left(k-k_0-\dfrac{1}{2}\right)^2} + \frac{3M}{n} \sum_{|x_k - x_0| > \frac{C_1^*}{n}} \frac{1}{\left(k-k_0-\dfrac{1}{2}\right)^2} \leq$$

(3.8)

$$\leq \frac{6C_2^*}{n} \sum_{k=0}^{\infty} \frac{1}{\left(k-\dfrac{1}{2}\right)^2} + \frac{6M}{n} \sum_{k > \frac{3C_1^*}{2\pi}} \frac{1}{\left(k-\dfrac{1}{2}\right)^2}.$$

We now choose C_1^* so small that $C_2^* < \dfrac{4}{5}$ and

$$6C_2^* \sum_{k=0}^{\infty} \frac{1}{\left(k-\dfrac{1}{2}\right)^2} < \frac{1}{10}$$

and we choose C_1^* large that

$$6M \sum_{k > \frac{3C_1^*}{2\pi}} \frac{1}{\left(k - \frac{1}{2}\right)^2} < \frac{1}{10}.$$

We then have by virtue of (3.8) that

$$|T(y)| < \frac{1}{10n} + \frac{1}{10n} = \frac{1}{5n}.$$

Since $g_n(y) = \dfrac{\pi}{3n}$ we see that

$$|g_n(y) - T(y)| \geq \frac{\pi}{3n} - \frac{1}{5n} \geq \frac{4}{5n} > \frac{C_2^*}{n}$$

the desired contradiction.

Proof of Theorem 3.1. Let C_1^* and C_2^* be the constants of Theorem 3.2 corresponding to $M^* = M$. We shall show Theorem 3.1 holds with $C_1 = 2C_1^*$ and $C_2 = C_2^*$. We can assume $C_1^* \geq \pi$ then letting

$$f_n(x_0, x) = g_n\left(\cos^{-1}\left(x_0 - \frac{C_1^*}{n}\right), \cos^{-1} x\right)$$

where g_n is the function given in the proof of Theorem 3.2, we have that

$$f_n(x_0, x) = 0 \quad \text{if} \quad |x| \geq x_0.$$

Thus

$$|f_n'(x_0, x)| = \left| g_n'\left(\cos^{-1}\left(x_0 - \frac{C_1^*}{n}\right), \cos^{-1} x\right) \frac{1}{(1 - x_0^2)^{1/2}} \right| \leq$$

$$\leq \frac{1}{(1 - x_0^2)^{1/2}} \quad \text{if} \quad |x| \leq 1.$$

Therefore the function $(1 - x_0^2)^{1/2} f_n(x_0, x)$ is in $\mathrm{Lip}_1\, 1$. We shall now show that $(1 - x_0^2)^{1/2} f_n(x_0, x)$ satisfies the conclusion of the theorem with C_1 and C_2 as described above. Suppose that P is a polynomial of degree $\leq n$ and

$$\|f_n - P\|_1 \leq \frac{M}{n}.$$

Then

$$\|f_n(\cos x) - P(\cos x)\|_1 \leq \frac{M^*}{n}.$$

Since $P(\cos x)$ is a trigonometric polynomial of degree $\leq n$ we have as in Theorem 3.2 that

$$\left\| g_n\left(\cos^{-1} x_0 - \frac{C_1^*}{n}, x\right) - P(\cos x)\right\| \left[\cos^{-1} x_0 - \frac{2C_1^*}{n}, \cos^{-1} x_0\right] \geq \frac{C_2^*}{n}$$

or

$$\|f_n(x_0,x) - P(x)\| \left[\cos\left(\cos^{-1} x_0 - \frac{2C_1^*}{n}\right), x_0\right] \geq \frac{C_2^*}{n}.$$

If we write $\cos\left(\cos^{-1} x_0 - \frac{2C_1^*}{n}\right) = x_0 - \frac{\lambda(x_0)}{n}$ we see that $|\lambda(x_0)| \leq |2C_1^*|$ thus

$$\|f_n(x_0,x) - P(x)\| \left[x_0 - \frac{2C_1^*}{n}, x_0 + \frac{2C_1^*}{n}\right] \geq \frac{C_2^*}{n}$$

the desired conclusion and Theorem 3.1 is proved.

COROLLARY 3.1. *Let (L_n) be a sequence of positive polynomial operators and let $0 < \delta \leq 1$. If for $i = 1, 2, 3$, we have*

$$\|e_i - L_n(e_i)\|_\delta = O\left(\frac{1}{n^2}\right) \quad (n \to \infty)$$

then there are constants C_1, C_2 and C_3 depending only on δ such that if $x_0 \in (-\delta, \delta)$

$$\delta_n = \| L_n((t-x)^2, x)\| \left[x_0 - \frac{C_1}{n},\ x_0 + \frac{C_1}{n}\right] \geq \frac{C_2(\delta^2 - x_0^2)}{n^2} - \frac{C_3}{n^3}.$$

PROOF. We shall prove the Corollary only in the case $\delta = 1$, the case $\delta < 1$ is obtained by using the usual transformation from $[-\delta, \delta]$ onto $[-1, 1]$. From Theorem A, we have for $f \in \mathrm{Lip}_1 1$

$$\|f - L_n(f)\|_1 \leq C\left(\frac{\|f\|}{n^2} + \frac{1}{n}\right).$$

Choose C_1' and C_2' as in Theorem 3.1 for $M' = C + 1$. Then from Theorem A

$$\|f - L_n(f)\| \left[x_0 - \frac{C_1'}{n},\ x_0 + \frac{C_1'}{n}\right] \leq C\left(\frac{\|f\|}{n^2} + \delta_n^{1/2}\right).$$

Since the function f_n of Theorem 3.1 is in $\mathrm{Lip}_1 1$ and satisfies $\|f_n\| \leq 1$ we must have

$$\frac{C}{n^2} + C\delta_n^{1/2} \geq \frac{C_2'\sqrt{1-x_0^2}}{n} \quad \text{for} \quad n \geq N. \tag{3.9}$$

Squaring both sides of (3.9), we have

$$\delta_n \geq \left(\frac{C_2'}{C}\right)^2 \left(\frac{1-x_0^2}{n^2}\right) - \frac{1}{n^4} - \frac{2\delta_n^{1/2}}{n^2} \geq \frac{C_2(1-x_0^2)}{n^2} - \frac{C_3}{n^3}$$

where

$$C_2 = \left(\frac{C_2'}{C}\right)^2 \quad \text{and} \quad C_3 = 1 + 2M$$

with

$$M^2 = \sup_n \left(n^2 \| L_n((t-x)^2, x)\|\right) < +\infty .$$

COROLLARY 3.2. *Let (L_n) be a sequence of positive polynomial operators and $0 < \delta \leq 1$. If for $i = 1, 2, 3$ we have*

$$\| e_i - L_n(e_i) \|_\delta = O\left(\frac{1}{n^2}\right) \quad (n \to \infty)$$

then

$$\| e_3 - L_n(e_3) \|_\delta \neq o\left(\frac{1}{n^2}\right) \quad (n \to \infty).$$

PROOF. We need only consider the case when $\| e_3 - L_n(e_3) \|_\delta = O\left(\frac{1}{n^2}\right)$ $(n \to \infty)$. Since $(t-x)^2 = e_3(t) - 2xe_2(t) + x^2 e_1(t)$ we have

$$(3.10) \qquad \| L_n((t-x)^2, x) \|_\delta \leq \| e_3 - L_n(e_3) \|_\delta + 2\delta \| e_2 - L_n(e_2) \|_\delta + $$
$$+ \delta^2 \| e_1 - L_n(e_1) \|_\delta \leq \frac{M}{n^2}.$$

Choosing C_1 and C_2 from Corollary 3.1, we have that for n sufficiently large

$$\| L_n((t-x)^2, x) \|_{\frac{C_1}{n}} \geq \frac{C_2}{2n^2}.$$

However (3.10)

$$\| L_n((t-x)^2, x) \|_{\frac{C_1}{n}} \leq \| e_3 - L_n(e_3) \|_\delta + \frac{2C_1}{n} \| e_2 - L_n(e_2) \|_\delta +$$
$$+ \frac{C_1^2}{n^2} \| e_1 - L_n(e_1) \|_\delta .$$

Since the last two terms on the right hand side of the inequality (3.1) are both $o\left(\frac{1}{n^2}\right)$ $(n \to \infty)$, we must have

$$\| e_3 - L_n(e_3) \|_\delta \neq o\left(\frac{1}{n^2}\right) \quad (n \to \infty).$$

4. Saturation order of optimal sequences of positive operators

In this section we will show that any optimal sequence of positive operators on $[-1,1]$ is saturated with order $\frac{1}{n^2}$. First, we will note some results on smooth functions which we will need. A function f is said to be smooth on $[a, b]$ if $\| f(x+h) + f(x-h) - 2f(x) \| [a, b] = o(h)$ $(h \to 0)$. We will need the following inverse theorem on approximation by algebraic polynomials.

PROPOSITION 1. *Suppose f is a continuous function on $[-1, 1]$ such that there is a sequence of algebraic polynomials (P_n) with P_n of degree $\leq n$ satisfying*

$$\|f - P_n\|_1 = o\left(\frac{1}{n^2}\right) \quad (n \to \infty)$$

then for each $\delta < 1$, f' is smooth on $[-\delta, \delta]$.

The proof of this proposition can be obtained via the corresponding result of ZYGMUND [10] for trigonometric approximation of 2π periodic functions. One finds from Zygmund's result that the function $(\sin t)f'(\cos t)$ is smooth for $t \in [-\pi, \pi]$ from which one can derive that $f'(x)$ is smooth for $x \in \in [-\delta, \delta]$, if $\delta < 1$.

Suppose f is a continuous function on $[a, b]$ extend f so as to be continuous on $(-\infty, \infty)$ and constant on (b, ∞) and $(-\infty, a)$ and define

$$f_n(x) = \frac{n^2}{4} \int_{-\frac{1}{n}}^{\frac{1}{n}} \int_{-\frac{1}{n}}^{\frac{1}{n}} f(x + u + v)\, du\, dv .$$

Also let $g_n = f - f_n$.

The proof of the following proposition is contained in ([8], p. 117).

PROPOSITION 2. *Let f be a continuously differentiable function on $[a, b]$ such that f' is smooth on $[a, b]$ then if $a < a' < b' < b$ there is a constant N such that for $n \geq N$*

(i) $$\|f_n^{(3)}\|\,[a', b'] \leq n\,\frac{\varepsilon(n)}{2}$$

(ii) $$\|g_n'\|\,[a', b'] \leq \frac{\varepsilon(n)}{2n}$$

(iii) $$\|g_n\|\,[a', b'] \to 0 \quad (n \to \infty)$$

where

(4.1) $$\varepsilon(n) = \sup_{0 \leq h \leq \frac{2}{n}} \|f'(x + h) + f'(x - h) - 2f'(x)\|\,[a, b].$$

The following lemma embodies most of the technical difficulties in the proof of the saturation order. This lemma is a refinement of a result of BAJSANSKI and BOJANIC [6].

LEMMA 4.1. *Let f be a continuous function on $[-1, 1]$ which has a smooth derivative on $[-\delta, \delta]$ for each $\delta < 1$ and suppose that $f(-1) = f(1) = 0$ and $f(x_0)$ is positive for some $x_0 \in (-1, 1)$. If C is any positive real number then there exists a positive integer N, a real number $a < 0$, and $0 < \delta_0 < 1$ such that for each $n \geq N$ there is a point $x_n \in [-\delta_0, \delta_0]$ for which the family* *of parabolas*

$$Q_n(x, y) = a(x-y)^2 + f_n'(y)\ (x-y) + f_n(y)$$

satisfy

$$Q_n(x, y) \geq f_n(x) \quad \text{for} \quad -1 \leq x \leq 1 \quad \text{and} \quad |y - x_n| < \frac{C}{n}.$$

PROOF. Let $x_0 \notin (-1, 1)$ and $0 < \lambda < f(x_0)$ and let $\delta_0 < 1$ be such that

$$\sup_{x \notin [-\delta_0, \delta_0]} |f(x)| < \frac{\lambda}{3}.$$

We have by virtue of Proposition 2 (iii) that for n sufficiently large say $n \geq N_1$

$$\sup_{x \notin [-\delta_0, \delta_0]} |f_n(x)| \leq \frac{\lambda}{3}$$

and

$$f_n(x_0) \geq \lambda.$$

Thus we can choose $\alpha < 0$ sufficiently small and $\beta > 0$ sufficiently large so that the parabola

$$Q(x) = \alpha(x - x_0)^2 + \beta$$

satisfies

$$Q(x) \geq f_n(x) \qquad x \in [-1, 1] \quad \text{and} \quad n \geq N_1$$

and the $\inf\limits_{-1 \leq x \leq 1} |f_n(x) - Q(x)|$ will be attained on $[-\delta_0, \delta_0]$ provided $n \geq N_1$. For each $n \geq N_1$ let x_n be a point in $[-\delta_0, \delta_0]$ where

$$\inf_{-1 \leq x \leq 1} |f_n(x) - Q(x)| = |f_n(x_n) - Q(x_n)|.$$

then

$$\alpha(x - x_n)^2 + f_n'(x_n)\,(x - x_n) + f_n(x_n) \geq f_n(x) \qquad x \in [-1, 1]. \tag{4.2}$$

We will now show that the conclusions of the lemma are satisfied with the above choice for (x_n) and for $a = \dfrac{\alpha}{2}$. We must show that

$$\frac{\alpha}{2}(x - y)^2 + f_n'(y)\,(x - y) + f_n(y) \geq f_n(x) \tag{4.3}$$

if $$x \in [-1, 1], \quad |y - x_n| < \frac{C}{n}$$

provided n is sufficiently large. To this end let $\eta = \frac{1-\delta_0}{2}$. We have

$$\frac{\alpha}{2}(x-y)^2 + f_n'(y)(x-y) + f_n(y) =$$

$$= \frac{\alpha}{2}\left((x-y)^2 - (x-x_n)^2\right) + f_n'(y)(x-y) - f_n'(x_n)(x-x_n) +$$

$$+ \left(f_n(y) - f_n(x_n)\right) + \frac{\alpha}{2}(x-x_n)^2 + f_n'(x_n)(x-x_n) + f_n(x_n).$$

Now for n sufficiently large say $n \geq N_2 \geq N_1$ we have

$$(4.4) \qquad \frac{\alpha}{2}[(x-y)^2 - (x-x_n)^2] + f_n'(y)(x-y) - f_n'(x_n)(x-x_n) +$$

$$+ f_n(y) - f_n(x_n) \geq \frac{\alpha}{2}\eta^2 \quad \text{for } -1 \leq x \leq 1 \quad \text{and} \quad |y - x_n| < \frac{C}{n}.$$

This follows simply from the equicontinuity of the functions f_n and f_n', $n = 1, 2, \ldots$ and the fact that $\alpha < 0$.

If $x \notin [-1+\eta, 1-\eta]$ we have $|x - x_n| \geq \eta$ thus for $n \geq N_2$ we have using (4.2) and (4.4)

$$\frac{\alpha}{2}(x-y)^2 + f_n'(y)(x-y) + f_n(y) \geq$$

$$\geq \frac{\alpha}{2}\eta^2 + \frac{\alpha}{2}(x-x_n)^2 + f_n'(x_n)(x-x_n) + f_n(x_n) \geq$$

$$\geq \alpha(x-x_n)^2 + f_n'(x_n)(x-x_n) + f_n(x_n) \geq f(x).$$

This proves (4.3) provided $x \notin [-1+\eta, 1-\eta]$.

To see that (4.3) is valid for $x \in [-1+\eta, 1-\eta]$ consider the function

$$h_n(x, y) = \frac{f_n(x) - f_n(y) - f_n'(y)(x-y)}{(x-y)^2}.$$

It is sufficient to show that for n sufficiently large

$$h_n(x, y) \leq \frac{\alpha}{2}$$

provided $x \in [-1+\eta, 1-\eta]$ and $|y - x_n| < \frac{C}{n}$. We have for $x \neq y$

$$|h_n(x+\delta, y+\delta) - h_n(x, y)| =$$

$$= \left| \frac{1}{(x-y)^2} \int_y^x \int_t^{t+\delta} [f_n''(s) - f_n''(s+y-t)]\, ds\, dt \right| \leq \frac{\delta}{|x-y|}\, \omega(f_n'', |x-y|)$$

which by virtue of Proposition 2 (i) is

$$\leq \frac{n\,\varepsilon(n)\delta}{2}$$

provided n is sufficiently large say $n \geq N_3 \geq N_2$, where $\varepsilon(n)$ is obtained as in Proposition 2 by considering f on $\left[-1+\frac{\eta}{2}, 1-\frac{\eta}{2}\right]$. Thus

$$|h_n(x+\delta, y+\delta) - h_n(x, y)| \leq C\,\varepsilon(n) \quad \text{if} \quad \delta < \frac{C}{n}.$$

Finally, choosing n sufficiently large say $n \geq N \geq N_3$ such that $C\,\varepsilon(n) < \frac{|\alpha|}{2}$ and $\frac{C}{N} < \frac{\eta}{2}$ we have for $|y - x_n| \leq \frac{C}{n}$ and $x \in \left[-1+\frac{\eta}{2}, 1-\frac{\eta}{2}\right]$ that

$$h_n\big(x + (y - x_n), y\big) \leq h_n(x, x_n) + \frac{|\alpha|}{2} \leq \alpha + \frac{|\alpha|}{2} = \frac{\alpha}{2}$$

or

$$h_n(z, y) \leq \frac{\alpha}{2} \quad \text{if} \quad |y - x_n| \leq \frac{C}{n} \quad \text{and} \quad z \in [-1+\eta, 1-\eta]$$

and the lemma is proved.

THEOREM 4.1. *If (L_n) is a sequence of positive polynomial operators which is optimal on $[-\delta, \delta]$ for each $\delta < 1$ and if for each $\delta < 1$*

$$\|f - L_n(f)\|_\delta = o\left(\frac{1}{n^2}\right) \qquad (n \to \infty).$$

then f is linear.

PROOF. Let $l(x) = f(-1) + \left(\frac{f(1) - f(-1)}{2}\right)(x+1)$. We wish to show that $\hat{f} = f - l$ is identically zero on $[-1, 1]$. Now $\hat{f}$ has a smooth derivative by virtue of Proposition 1. Suppose $\hat{f}(x_0) > 0$ for some $x_0 \in (-1, 1)$ then choose δ_0 as in Lemma 4.1 and let $\hat{f}_n$ and $\hat{g}_n$ be defined as in Proposition 2. The function $\hat{g}_n$ is in $\mathrm{Lip}_{\frac{\varepsilon(n)}{2n}}(1)$ for each n and $\|\hat{g}_n\|_\delta$ is a bounded sequence and thus from Theorem A we have

$$\|\hat{g}_n - L_n(\hat{g}_n)\|_{\delta_0} = o\left(\frac{1}{n^2}\right) \qquad (n \to \infty).$$

We shall show that

$$\|\hat{f}_n - L_n(\hat{f}_n)\|_{\delta_0} \neq o\left(\frac{1}{n^2}\right)$$

which will contradict our hypothesis and thus show $\hat{f}(x) \leq 0$ for all $x \in [-1, 1]$. One need only apply the same argument to $-\hat{f}$ to show $\hat{f}(x) \geq 0$ for all $x \in [-1, 1]$ and thus prove the theorem. Now, we have from Lemma 4.1 that for $n \geq N$

$$\hat{f}_n(x) \leq Q_n(x, y) \quad \text{for} \quad x \in [-1, 1] \quad |y - x_n| < \frac{C}{n}.$$

Using the fact that $(\|\hat{f}_n'\|_{\delta_0})$ is a bounded sequence we have

$$L_n(\hat{f}_n(x) - \hat{f}_n(y), y) \leq a\, L_n((x-y)^2, y) + \|\hat{f}_n'\|_{\delta_0}(\|e_2 - L_n(e_2)\|_{\delta_0}) =$$
$$= a\, L_n((x-y)^2, y) + o\left(\frac{1}{n^2}\right) \qquad (n \to \infty).$$

Thus,

$$\|L_n(\hat{f}_n(x) - \hat{f}_n(y), y)\|_{\delta_0} \geq$$
$$\geq |a|\, \|L_n((x-y)^2, y)\| \left[x_n - \frac{C}{n},\ x_n + \frac{C}{n}\right] + o\left(\frac{1}{n^2}\right) \qquad (n \to \infty).$$

Using Corollary 3.1, we find

$$\|L_n(\hat{f}_n(x) - \hat{f}_n(y), y\|_{\delta_0} \geq |a|\ \frac{C_2(1 - x_n^2)}{n^2} + o\left(\frac{1}{n^2}\right) \geq$$
$$\geq |a|\ \frac{C_2(1 - \delta_0^2)}{n^2} + o\left(\frac{1}{n^2}\right) \qquad (n \to \infty).$$

Finally, since $\|\hat{f}_n\|_{\delta_0}$ is a bounded sequence and $\|e_1 - L_n(e_1)\|_{\delta_0} = o\left(\frac{1}{n^2}\right)$ $(n \to \infty)$ we have

$$\|\hat{f}_n(y) - L_n(\hat{f}_n(y), y)\|_{\delta_0} = o\left(\frac{1}{n^2}\right) \qquad (n \to \infty)$$

thus

$$\|L_n(\hat{f}_n) - \hat{f}_n\|_{\delta_0} \geq \frac{|a|}{2}\ \frac{C_2(1 - \delta_0^2)}{n^2} + o\left(\frac{1}{n^2}\right) \qquad (n \to \infty)$$

as was desired.

The proof of Theorem 4.1 is considerably simpler if the sequence of operators (L_n) has the property that for each $x \in (-1, 1)$ $L_n((t-x)^2, x) \neq o\left(\frac{1}{n^2}\right)$ $(n \to \infty)$. In this case, the result is contained in the paper of Karamata and Vuilleumeir [7].

If a sequence of positive polynomial operators is optimal on $[-1, 1]$ then it is optimal on $[-\delta, \delta]$ for each $\delta < 1$. Since, we also have by virtue of Theorem 2.1 that for each f such that $f' \in \mathrm{Lip}_1\, 1$

$$\|f - L_n(f)\|_1 = O\left(\frac{1}{n^2}\right) \qquad (n \to \infty)\,.$$

Thus, we have the following Corollary to Theorem 4.1.

COROLLARY 4.1. *If (L_n) is a sequence of positive polynomial operators which is optimal on $[-1, 1]$ then (L_n) is saturated with order $\frac{1}{n^2}$.*

In the case of Corollary 4.1, it is of interest to determine the saturation class of the operators (L_n) (i.e. to determine the class of functions f for which $\|f - L_n(f)\|_1 = O\left(\frac{1}{n^2}\right)$ $(n \to \infty)$. As we observed, this class contains $\{f : f' \in \mathrm{Lip}\, 1\}$. It is contained in the class of those functions whose first derivative is quasi-smooth on every proper subinterval of $[-1, 1]$ (see [4], p. 79). However, an exact characterization of this class is not known and this characterization might depend on the sequence (L_n).

5. Examples of sequences of optimal operators

Let P_{2n} denote the Legendre polynomial of degree $2n$ and $-1 < x_1 < x_2 < \ldots < x_{2n} < 1$ its zeros written in increasing order. These zeros are symmetric with respect to the origin and x_{n+1} is the smallest positive zero. Let

$$\Lambda_n(t) = C_n \frac{(P_{2n}(t))^2}{(t^2 - x_{n+1}^2)^2\,(t^2 - x_{n+2}^2)^2}$$

and $\bar{\Lambda}_n(t) = \Lambda_n\left(\frac{t}{2}\right)$ where C_n is chosen so that

$$2\int_{-1}^{1} \Lambda_n(t)\,dt = \int_{-2}^{2} \bar{\Lambda}_n(t)\,dt = 1\,.$$

We will show that the sequence of polynomial operator (L_n) defined by

$$L_n(f, x) = \int_{-1}^{1} f(t)\,\bar{\Lambda}_m(t - x)\,dt$$

with $m = \left[\frac{n}{4}\right] + 1$ is optimal on $[-\delta, \delta]$ for each $\delta < 1$. More precisely we have

THEOREM 5.1. *For* $0 < \delta < 1$

(5.1) $$\|e_i - L_n(e_i)\|_\delta = o\left(\frac{1}{n^2}\right) \qquad (n \to \infty) \quad i = 1,2$$

(5.2) $$\|e_3 - L_n(e_3)\|_\delta = O\left(\frac{1}{n^2}\right) \qquad (n \to \infty).$$

PROOF. We will use two elementary properties of Legendre polynomials whose proofs can be found in [11].

PROPERTY 1. (Gauss Quadrature Formula.) *There exist positive constants* $A_k^{(n)}$, $k = 1, 2, \ldots, 2n$ *such that for any polynomial* P *of degree* $\leq 4n - 1$ *we have*

$$\int_{-1}^{1} P(x)\,dx = \sum_{k=1}^{2n} A_k^{(n)} P(x_k).$$

NOTE. Since P_{2n} is any even polynomial $A_n^{(n)} = A_{n+1}^{(n)}$, and $A_{n-1}^{(n)} = A_{n+2}^{(n)}$.

PROPERTY 2. *The zeros* x_{n+1} *and* x_{n+2} *satisfy the inequalities*

$$0 < x_{n+1} < x_{n+2} < \frac{3}{n}.$$

Now to the proof of the theorem. We first wish to establish (5.1). To estimate $\| e_1 - L_{4n-4}(e_1) \|_\delta$ we have

$$1 - \int_{-1}^{1} \bar{A}_n(t-x)\,dt = \int_{-2}^{2} \bar{A}_n(t)\,dt - \int_{-1-x}^{1-x} \bar{A}_n(t)\,dt.$$

Thus for $x \in [-\delta, \delta]$

$$|1 - L_{4n-4}(e_1, x)| \leq \int_{-2}^{-1+\delta} \bar{A}_n(t)\,dt + \int_{1-\delta}^{2} \bar{A}_n(t)\,dt = 2\int_{1-\delta}^{2} \bar{A}_n(t)\,dt$$

or

(5.3) $$\|e_1 - L_{4n-4}(e_1)\|_\delta \leq 2\int_{1-\delta}^{2} \bar{A}_n(t)\,dt.$$

Similarly for $\| e_2 - L_{4n-4}(e_2) \|_\delta$ we have

$$x - \int_{-1}^{1} t\,\bar{A}_n(t-x)\,dt = \int_{-2}^{2} x\,\bar{A}_n(t)\,dt - \int_{-1-x}^{1-x} (t+x)\,\bar{A}_n(t)\,dt =$$

$$= \int_{-2}^{-1-x} x\,\bar{A}_n(t)\,dt + \int_{1-x}^{2} x\,\bar{A}_n(t)\,dt + \int_{-1-x}^{1-x} t\,\bar{A}_n(t)\,dt.$$

Since $\int_{-1+\delta}^{1-\delta} t\,\bar{A}_n(t) = 0$ we have that for $x \in [-\delta, \delta]$

$$|x - \int_{-1}^{1} t\,\bar{A}_n(t-x)\,dt| \leq 4\int_{-2}^{-1+\delta} \bar{A}_n(t)\,dt - 4\int_{1-\delta}^{2} A_n(t)\,dt = 8\int_{1-\delta}^{2} \bar{A}_n(t)\,dt$$

or

$$\| e_2 - L_{4n-4}(e_2) \|_\delta \le 8 \int_{1-\delta}^{2} \bar{A}_n(t)\, dt. \tag{5.4}$$

In view of (5.3) and (5.4) and the fact that $L_{4n+k} = L_{4n}$, $k = 1, 2, 3$, (5.1) will be verified if we show that

$$\int_{1-\delta}^{2} \bar{A}_n(t)\, dt = o\left(\frac{1}{n^2}\right) \qquad (n \to \infty).$$

To this end, using Property 1, we have

$$\int_{1-\delta}^{2} \bar{A}_n(t)\, dt \le \int_{-2}^{2} \frac{t^4}{(1-\delta)^4} \bar{A}_n(t)\, dt = \frac{32}{(1-\delta)^4} \int_{-1}^{1} t^4 \bar{A}_n(2t)\, dt =$$

$$= \frac{32}{(1-\delta)^4} \sum_{k=1}^{2n} A_k^{(n)} x_k^4 \bar{A}_n(2x_k) = \frac{32}{(1-\delta)^4} \sum_{k=1}^{2n} A_k^{(n)} x_k^4 \Lambda_n(x_k) =$$

$$= \frac{32}{(1-\delta)^4} \left(2A_{n+1}^{(n)} x_{n+1}^4 \Lambda_n(x_{n+1}) + 2A_{n+2}^{(n)} x_{n+2}^4 \Lambda_n(x_{n+2})\right)$$

which by Property 2 is

$$\le \frac{32}{(1-\delta)^4} \frac{81}{n^4} \left(2A_{n+1}^{(n)} \Lambda_n(x_{n+1}) + 2A_{n+2}^{(n)} \Lambda_n(x_{n+2})\right) =$$

$$= \frac{32}{(1-\delta)^4} \frac{81}{n^4} \int_{-1}^{1} \Lambda_n(t)\, dt = \frac{64}{(1-\delta)^4} \frac{81}{n^4} = O\left(\frac{1}{n^2}\right). \qquad (n \to \infty).$$

We now estimate $\| e_3 - L_{4n-4}(e_3) \|_\delta$. For $x \in [-\delta, \delta]$, we have

$$|x^2 - \int_{-1}^{1} t^2 \bar{A}_n(t - x)\, dt| = |x^2 \int_{-2}^{2} \bar{A}_n(t)\, dt - \int_{-1-x}^{1-x} (t + x)^2 \bar{A}_n(t)\, dt| \le$$

$$\le 2 \int_{1-\delta}^{2} x^2 \bar{A}_n(t)\, dt + | \int_{-1-x}^{1-x} (t^2 + 2xt) \bar{A}_n(t)\, dt | \le$$

$$\le 8 \int_{1-\delta}^{2} \bar{A}_n(t)\, dt + 16 \int_{1-\delta}^{2} \bar{A}_n(t)\, dt + \int_{-1-x}^{1-x} t^2 \bar{A}_n(t)\, dt.$$

Thus

$$\| e_3 - L_{4n-4}(e_3) \|_\delta \le 24 \int_{1-\delta}^{2} \bar{A}_n(t)\, dt + \int_{-2}^{2} t^2 \bar{A}_n(t)\, dt.$$

We have already seen that

$$\int_{1-\delta}^{2} \bar{A}_n(t)\,dt = o\left(\frac{1}{n^2}\right) \qquad (n \to \infty).$$

Thus we need only show

$$\int_{-2}^{2} t^2 \bar{A}_n(t)\,dt = O\left(\frac{1}{n^2}\right) \qquad (n \to \infty).$$

However,

$$\int_{-2}^{2} t^2 \bar{A}_n(t)\,dt = 8 \sum_{k=1}^{n} A_k^{(n)} x_k^2 \Lambda_n(x_k) =$$

$$= 8\left(2A_{n+1}^{(n)} x_{n+1}^2 \Lambda_n(x_{n+1}) + 2A_{n+2}^{(n)} x_{n+2}^2 \Lambda_n(x_{n+2})\right) \leq$$

$$\leq \frac{72}{n^2}\left(2A_{n+1}^{(n)} \Lambda_n(x_{n+1}) + 2A_{n+2}^{(n)} \Lambda_n(x_{n+2})\right) =$$

$$= \frac{72}{n^2} \int_{-1}^{1} \Lambda_n(t)\,dt = \frac{144}{n^2} = O\left(\frac{1}{n^2}\right) \qquad (n \to \infty).$$

This completes the proof of Theorem 5.1.

The sequence operators (L_n) in Theorem 5.1 are modifications of the operators given in [12]. A general method for constructing optimal sequences of operators is suggested by the paper of Bojanic [14]. Indeed, let (Q_n) be a sequence of orthogonal polynomials with respect to the weight function $\omega(t)$ on $[-1, 1]$ and y_{n+1} and y_{n+2} be the two smallest zeros of Q_n. If $0 < m \leq \omega(t) \leq M < +\infty$ for $t \in [-1, 1]$ then using arguments similar to those given in Theorem 5.1 together with the estimates given in [14] it can be shown that (L_n) is an optimal sequence on $[-\delta, \delta]$ for each $\delta < 1$ when (L_n) is defined by

$$L_n(f, x) = \int_{-1}^{1} f(t)\, K_m(t-x)\,dt, \quad m = \left[\frac{n}{4}\right] + 1$$

where

$$K_m(t) = C_m \frac{\left(Q_{2m}\left(\frac{t}{2}\right)\right)^2}{\left(\frac{t^2}{4} - y_{n+1}^2\right)^2 \left(\frac{t^2}{4} - y_{n+2}^2\right)^2}$$

with C_m chosen so that

$$\int_{-2}^{2} K_m(t)\,dt = 1.$$

Finally, we note that by virtue of Theorems A and 2.1 the estimates (5.1) and (5.2) provide a direct proof of the classical theorems of D. Jackson on the approximation of continuous functions by algebraic polynomials (see [12]).

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