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ON THE DIRECT THEOREM OF SATURATION*

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We denote by $C^*[-\pi,\pi]$, the space of 2π periodic continuous functions and $\|\cdot\|$, the supremum norm. Let (L_n) be a sequence of convolution operators on C^* given by

$$(1) \qquad L_n(f,x) = (f * d\mu_n)(x) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x+t)\,d\mu_n(t)$$

where $d\mu_n$ is an even Borel measure on $[-\pi,\pi]$ with $\frac{1}{\pi}\int_{-\pi}^{\pi} d\mu_n(t) = 1$. For $d\mu_n$, the real Fourier-Sttieltjes coefficients $\rho_{k,n}$ and the complex coefficients $\check{\mu}_n(k)$ are defined by

$$\rho_{k,n} = \frac{1}{\pi}\int_{-\pi}^{\pi} \cos kt\; d\mu_n(t) = \frac{1}{\pi}\int_{-\pi}^{\pi} e^{-ikt} d\mu_n(t) = 2\check{\mu}_n(k).$$

Thus, the measure $d\mu_n$ has as its Fourier-Stieltjes series

$$d\mu_n(t) \sim \sum_{k=-\infty}^{\infty} \check{\mu}_n(k) e^{ikt} \sim \frac{1}{2} + \sum_{k=1}^{\infty} \rho_{k,n} \cos kt.$$

Similarly, for a function $f \in C^*$, we let $\hat{f}(k) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-ikt} f(t)\; dt$ and $A_k(f,x) = a_k \cos kx + b_k \sin kx$ with

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos kt\; dt,\; b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin kt\; dt.$$

Then f has the Fourier series

$$f \sim \sum_{k=-\infty}^{\infty} \hat{f}(k) e^{ikt} \sim \frac{A_0(f,t)}{2} + \sum_{k=1}^{\infty} A_k(f,t).$$

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We shall consider the problem of determining the saturation properties of (L_n) from the Fourier coefficients of $d\mu_n$. For this purpose, assume that there is a sequence $(\phi(n))$ of positive numbers converging to 0 such that for each $k=1,2,\cdots$,

$$\lim_{n\to\infty}\frac{1-\rho_{k,n}}{\phi(n)}=\psi_k\neq 0. \tag{2}$$

Then, we have the following saturation theorem.

THEOREM 1. *If (L_n) is a sequence of convolution operators of the form* (1) *which satisfies* (2), *then*

i. *For $f\in C^*$, $\|f-L_n(f)\|=o(n)$, if and only if f is constant.*

ii. *If $f\in C^*$, and $\|f-L_n(f)\|=O(\phi(n))$ then*

$$\sum_{k=1}^{\infty}\psi_k A_k(f,x)\in L_\infty. \tag{3}$$

iii. *Suppose in addition that*

$$\text{the sequences}\left(\frac{1-\rho_{k,n}}{\psi_k\phi(n)}\right)_{k=1}^{\infty}\text{are multipliers} \tag{4}$$

from L_∞ to L_∞ with norms uniformly bounded in n. Then if $f\in C^*$ with $\sum_{k=1}^{\infty}\psi_k A_k(f,x)\in L_\infty$, we have

$$\|f-L_n(f)\|=O(\phi(n))$$

Part ii of Theorem 1 is the so-called converse theorem of saturation which was given by Sunouchi and Watari [2] and also independently by Harsiladge [1]. Part iii is called the direct theorem of saturation and was given by Sunouchi [3]. The object of this theorem is to characterize the saturation class $S(L_n)=\{f:\|f-L_n(f)\|=O(\phi(n))\}$. Of course, this is achieved under the additional hypothesis (4) since in this case $S(L_n)=\left\{f\in C^*:\sum_{k=1}^{\infty}\psi_k A_k(f,x)\in L_\infty\right\}$. Actually, if (ψ_k^{-1}) is a multiplier from C^* to C^* then (4) is necessary for the saturation class to be characterized by (3) (see the proof of theorem 2). For this reason, much interest has centered about determining the true nature of (4).

Tureckii [5] has claimed that if the measures $d\mu_n$ have uniformly bounded norms i.e., there is a constant $M>0$ such that $\int_{-\pi}^{\pi} |d\mu_n(t)| \leqq M$ for $n=1,2\cdots$, then (4) is satisfied. However, his proof was not correct and Sunouchi has given a counterexample to this claim. Sunouchi [4] has shown that if $\alpha>0$, there is a sequence of even functions (h_n), $h_n \in L_1[-\pi,\pi]$, with uniformly bounded L_1 norms such that

$$\lim_{n\to\infty} n^\alpha \hat{h}_n(k) = k^\alpha \qquad k=1,2\cdots$$

and yet there is a function $f_0 \in C^*$ such that

$$\sum_{k=1}^{\infty} k^\alpha A_k(f_0, x) \in L_\infty$$

and

$$\|f_0 - f_0 * h_n\| \neq O(n^{-\alpha}).$$

Now, if we assume that the measures $d\mu_n$ are positive then the situation changes. For one thing, if (L_n) is a sequence of positive convolution operators which satisfies (2) with $\psi_k = k^2$ then Tureckii [6] has shown that $f \in S(L_n)$ if and only if $\sum_{k=1}^{\infty} k^2 A_k(f,x) \in L_\infty$. Secondly, the examples given by Sunouchi are not positive operators. Thus the question as to whether (3) always characterizes the saturation class for positive operators that satisfy (2) has not been settled.

In this paper, we shall give a general procedure for constructing examples of positive operators which satisfy (2) but nevertheless (3) does not characterize the saturation class. Of course, we must avoid the case $\psi_k = k^2$. We do this by assuming that

$$\psi_k = o(k^2). \tag{5}$$

It is important to note, that for positive operators we always have $\psi_k = O(k^2)$ since

$$1-\rho_{k,n} = \frac{1}{\pi}\int_{-\pi}^{\pi} (1-\cos kt)\, d\mu_n(t) = \frac{2}{\pi}\int_{-\pi}^{\pi} \sin^2 \frac{kt}{2}\, d\mu_n(t)$$

$$\leqq \frac{2k^2}{\pi}\int_{-\pi}^{\pi} \sin^2 \frac{t}{2}\, d\mu_n(t) = k^2(1-\rho_{1,n}).$$

We will also assume that

(6) (ψ_k^{-1}) is a multipliers from C^* to C^*.

This means that for each $f \in C^*, \sum_{k=1}^{\infty} \psi_k^{-1} A_k(f,x) \in C^*$. We also wish to state our theorem for a more general class of operators than those that are positive. For this purpose, we will assume that for $d\mu_n$ we have

(7) $(1-\rho_{k,n}) \geqq 0$ for each k and n.

Of course (7) is always satisfied when $d\mu_n$ is positive.

THEOREM 2. *Let (L_n) be a sequence of convolution operators for which (2) and (7) hold. Also let (ψ_k) satisfy conditions (5) and (6). Then there exists a sequence (ξ_n) of positive numbers converging to 0 such that the operators $\widetilde{L}_n$ defined by*

$$\widetilde{L}_n(f,x) = f * d\widetilde{\mu}_n$$

with $d\widetilde{\mu}_n(t) = \frac{1}{2}[d\mu_n(t-\xi_n) + d\mu_n(t+\xi_n)]$ *satisfy*

(8) $$\lim_{n\to\infty} \frac{1-\widetilde{\rho}_{k,n}}{\phi(n)} = \psi_k$$

and

(9) *there is a function $f_0 \in C^*$ with*

$$\sum_{k=1}^{\infty} \psi_k A_k(f_0,x) \in L_\infty$$

and

$$\|f_0 - \widetilde{L}_n(f_0)\| \neq O(\phi(n)).$$

PROOF. We first note that (7) implies that $\psi_m > 0$ for all m.

$$\text{Let } 0 < a_n = \max\{m^{-2}\psi_m : m > (\phi(n))^{-1/2}\}$$

Then (5) shows that $a_n \to 0$. We shall show that the theorem holds with $\xi_n = a_n^{1/4}(\phi(n))^{1/2}$.

We first compute the Fourier-Stieltjes coefficients $\tilde{\rho}_{k,n}$ of $d\tilde{\mu}_n$. We have

$$\tilde{\rho}_{k,n} = \frac{1}{\pi}\int_{-\pi}^{\pi} \cos\, kt\ d\tilde{\mu}_n(t) = \frac{1}{\pi}\int_{-\pi}^{\pi} \cos\, kt\ d\mu_n(t-\xi_n)$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} [\cos\, k(t-\xi_n)\cos\, k\xi_n - \sin\, k(t-\xi_n)\ \sin\, k\xi_n]\, d\mu_n(t-\xi_n)$$

$$= \cos\, k\xi_n(\rho_{k,n})$$

where we have used the fact that $d\mu_n$ is even. Hence,

$$1-\tilde{\rho}_{k,n} = 1-\cos\, k\xi_n + \cos\, k\xi_n(1-\rho_{k,n}) = 2\sin^2\frac{k\xi_n}{2} + \cos\, k\xi_n(1-\rho_{k,n})$$

Thus, for $k=1,2,\cdots$

$$\lim_{n\to\infty}\frac{1-\tilde{\rho}_{k,n}}{\phi(n)} = \lim_{n\to\infty}\frac{2\sin^2\dfrac{k\xi_n}{2}}{\phi(n)} + \lim_{n\to\infty}\cos\, k\xi_n\frac{1-\rho_{k,n}}{\phi(n)} = \Psi_k\,.$$

where we have used the fact that $\sin^2\dfrac{k\xi_n}{2}=O(\xi_n^2)=o(\phi(n))$. This establishes (8).

Now, for each n, let $d\lambda_n$ be the even Borel measure whose Fourier coefficients are

$$\check{\lambda}_n(k) = \frac{1-\tilde{\rho}_{k,n}}{\Psi_k\phi(n)}$$

That such a measure exists follows from (6) and the fact that $(1-\tilde{\rho}_{k,n})$ is a Fourier-Stieltjes sequence. If $m_n=\left[\dfrac{1}{\xi_n}\right]+1$, then $\dfrac{1}{\xi_n}\leqq m_n\leqq\dfrac{1}{\xi_n}+1$. Thus, for n sufficiently large

$$\check{\lambda}_n(m_n) = \frac{2\sin^2\dfrac{m_n\xi_n}{2}}{\Psi_{m_n}\phi(n)} + \frac{(\cos\, m_n\xi_n)(1-\rho_{m_n,n})}{\Psi_{m_n}\phi(n)} \geqq \frac{2\sin^2\dfrac{1}{2}}{\Psi_{m_n}\phi(n)}$$

where we were able to omit the last term since $\cos\, m_n\xi_n\to\cos 1>0$.

Since $m_n \geqq \xi_n^{-1} \geqq (\phi(n))^{-1/2}$, we have

$$\Psi_{m_n}\phi(n) \leqq a_n m_n^2 \phi(n) \leqq a_n(a_n^{-1/4}(\phi(n))^{-1/2}+1)^2\phi(n) \to 0.$$

Therefore, $\check{\lambda}_n(m_n) \to \infty$. Thus, we must have

$$\int_{-\pi}^{\pi} |d\lambda_n(t)| \to \infty .$$

From the uniform boundedness principle we can conclude that there is a function $g \in C^*$ for which

$$\|g*d\lambda_n\| \to \infty .$$

Let f_0 be the function with Fourier coefficients $\hat{f}_0(k) = \frac{\hat{\sigma}(k)}{\Psi_k}$. Then $f_0 \in C^*$ by virtue of (6). It is easy to see that

$$2\frac{f_0 - f_0*d\tilde{\mu}_n}{\phi(n)} = g*d\lambda_n$$

by merely checking Fourier coefficients. Therefore,

$$2\left\|\frac{f_0 - f_0*d\tilde{\mu}_n}{\phi(n)}\right\| = \|g*d\lambda_n\| \neq O(1)$$

and the theorem is proved.

One interesting example occurs when we let $d\mu_n(t) = F_n(t)dt$ where F_n is the Fejér kernel

$$F_n(t) = \frac{1}{2\pi n}\left(\frac{\sin\frac{nt}{2}}{\sin\frac{t}{2}}\right)^2$$

Then the Fejér operators $\sigma_n(f) = f*F_n$ are saturated with order (n^{-1}). A simple check shows that we can take $\xi_n = \eta^{-5/8}$. Thus the operators

$$L_n(f) = f*\Lambda_n$$

where

$$\Lambda_n(t)=\frac{1}{2}[F_n(t-n^{-5/8})+F_n(t+n^{-5/8})]$$

are saturated with order (n^{-1}) and the Fourier coefficients of (Λ_n) satisfy

$$\lim_{n\to\infty} n(1-2\hat{\Lambda}_n(k))=k\,.$$

Yet, the saturation class $S(L_n)\neq\left\{f:\sum kA_k(f,x)\in L_\infty\right\}$.

More generally, we can consider the typical means of the Fourier series, for $0<\lambda<2$,

$$R_n^\lambda(f,x)=\sum_{k=0}^{n-1}\left(1-\left(\frac{k}{n}\right)^\lambda\right)A_k(f,x)=f*X_n^\lambda$$

which are saturated with order $(n^{-\lambda})$ and saturation class

$S(R_n^\lambda)=\left\{f:\sum_1^\infty k^\lambda A_k(f,x)\in L_\infty\right\}$. The sequence $(k^{-\lambda})$ is a multiplier from C^* to C^*. In this case, we can take $\xi_n^{(\lambda)}=n^{\frac{\lambda^2}{8}-\frac{3\lambda}{4}}$. Then the modified operators

$$L_n^\lambda(f)=f*\Lambda_n^\lambda$$

where

$$\Lambda_n^\lambda(t)=\frac{1}{2}[X_n^\lambda(t-\xi_n^{(\lambda)})+X_n^\lambda(t+\xi_n^{(\lambda)})]$$

satisfy

$$\lim_{n\to\infty} n^\lambda(1-2\widehat{\Lambda_n^\lambda}(k))=k^\lambda$$

and yet

$$S(L_n^\lambda)\neq\left\{f:\sum_1^\infty k^\lambda A_k(f,x)\in L_\infty\right\}.$$

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