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Tree Approximation and Optimal Encoding[1]

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Tree approximation is a new form of nonlinear approximation which appears naturally in some applications such as image processing and adaptive numerical methods. It is somewhat more restrictive than the usual n-term approximation. We show that the restrictions of tree approximation cost little in terms of rates of approximation. We then use that result to design encoders for compression. These encoders are universal (they apply to general functions) and progressive (increasing accuracy is obtained by sending bit stream increments). We show optimality of the encoders in the sense that they provide upper estimates for the Kolmogorov entropy of Besov balls. © 2001 Academic Press

Key Words: compression; n-term approximation; encoding; Kolmogorov entropy.

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1. INTRODUCTION

Wavelets are utilized in many applications including image/signal processing and numerical methods for PDEs. Their usefulness stems in part from the fact that they provide efficient decompositions of functions into simple building blocks. For example, they provide unconditional bases, consisting of the shifted dilates of a finite number of functions, for many function spaces such as the L_p, H_p, Besov, and Triebel–Lizorkin spaces. The present article is concerned with the following question: what is the most effective way to organize the terms in the wavelet decomposition of a function f. Of course the answer to this question depends on the potential application. We shall introduce a way of organizing the wavelet decomposition, by using tree-structures and certain ideas from nonlinear approximation, that is particularly well fitted to the application of data compression. In fact, the tree-structured algorithms underlying coders such as EZW [22] or SPIHT [21] were the motivation for the present mathematical analysis. Our approach will result in an optimal encoding technique, which is already being implemented for applications in terrain mapping [14]. The description of this encoder in Section 6 is self-contained (it requires some definitions from Section 1.1 below), so that we hope this article will be useful to readers who prefer implementation issues to mathematical proofs, as well as to more mathematically minded readers. We will also use our results to give simple proofs of upper estimates for the *Kolmogorov entropy* of Besov balls in L_p. The results from this article have already been utilized in [6] for the design of encoders with optimal rate distortion performance with respect to deterministic and stochastic models for the signals.

1.1. Background and Motivation

To describe the wavelet decompositions we have in mind, we introduce some notation which will be used throughout the paper. We let $\mathcal{D} := \mathcal{D}(\mathbb{R}^d)$ denote the set of all dyadic cubes in $\mathbb{R}^d$, i.e., cubes of the type $2^{-j}(k + [0, 1]^d)$, with $j \in \mathbb{Z}$ and $k \in \mathbb{Z}^d$, we let $\mathcal{D}_j := \mathcal{D}_j(\mathbb{R}^d)$ denote the set of dyadic cubes with sidelength 2^{-j}, and we let $\mathcal{D}_+ := \mathcal{D}_+(\mathbb{R}^d)$ denote the set of dyadic cubes with sidelength ≤ 1. We shall indicate the dependence of these sets on d only if there is a chance of confusion. If g is a function in $L_2(\mathbb{R}^d)$, and $I = 2^{-j}(k + [0, 1]^d)$ is in $\mathcal{D}$, we define

$$g_I := g_{I,2} := 2^{jd/2} g(2^j \cdot -k).$$

Then, g_I is normalized in $L_2(\mathbb{R}^d)$: $\|g_I\|_{L_2(\mathbb{R}^d)} = \|g\|_{L_2(\mathbb{R}^d)}$, for each $I \in \mathcal{D}$. We shall also need normalizations in $L_p(\mathbb{R}^d)$, $0 < p \leq \infty$, given by

$$g_{I,p} := 2^{jd/p} g(2^j \cdot -k).$$

In order to explain our results in their simplest setting, we shall limit ourselves in this Introduction to univariate decompositions using compactly supported orthogonal wavelets. Our results hold in much more generality and in fact will be developed in this article in the multivariate setting for a general class of biorthogonal wavelets.

Let ψ be a univariate, compactly supported, orthogonal wavelet obtained from a compactly supported scaling function ϕ. Denoting by $\langle f, g\rangle := \int_{\mathbb{R}} f(x)\bar{g}(x)\,dx$ the standard scalar product for $L_2(\mathbb{R})$, each function $f \in L_2(\mathbb{R})$ has the wavelet decomposition

$$f = \sum_{I \in \mathcal{D}_0} \langle f, \phi_I \rangle \phi_I + \sum_{I \in \mathcal{D}_+} a_I(f) \psi_I \tag{1.1}$$

with

$$a_I(f) := a_{I,2}(f) := \langle f, \psi_I \rangle.$$

The collection of functions Ψ appearing in (1.1) is an orthonormal basis for $L_2(\mathbb{R})$; it is also an unconditional basis for all of the $L_p(\mathbb{R})$ spaces, $1 < p < \infty$.

The most common way of organizing the decomposition (1.1) is

$$f = \sum_{I \in \mathcal{D}_0} \langle f, \phi_I \rangle \phi_I + \sum_{j=0}^{\infty} \sum_{I \in \mathcal{D}_j} a_I(f) \psi_I. \tag{1.2}$$

In other words, the terms are organized according to the *dyadic level* j (the frequency 2^j); low frequency terms appear first. This is analogous to the usual way of presenting Fourier decompositions. This organization of the wavelet series can be justified to a certain extent in that the membership of a function f in smoothness spaces (such as the Sobolev and Besov spaces) can be characterized by the decay of the wavelet coefficients of f with respect to frequency. Another justification comes from the viewpoint of approximation theory. Let V_n denote the linear space spanned by the functions $\phi(2^n \cdot -k)$, $k \in \mathbb{Z}$, or equivalently by the functions ϕ_I, $I \in \mathcal{D}_0$, and ψ_I, $I \in \mathcal{D}_j$, $j = 0, \ldots, n-1$. Then the partial sum

$$P_n(f) := \sum_{I \in \mathcal{D}_0} \langle f, \phi_I \rangle \phi_I + \sum_{j=0}^{n-1} \sum_{I \in \mathcal{D}_j} a_I(f) \psi_I$$

is the best $L_2(\mathbb{R})$-approximation to f from V_n. Also, it is a *near best approximation* to f in $L_p(\mathbb{R})$ whenever $1 < p < \infty$; i.e.,

$$\| f - P_n(f) \|_{L_p(\mathbb{R})} \le \mathcal{C}_p \operatorname{dist}(f, V_n)_{L_p(\mathbb{R})},$$

where $\mathcal{C}_p$ depends only on p.

In most applications, we are dealing with functions defined on a domain Ω. In the univariate case that we are now discussing, we shall take Ω to be a (possibly infinite) interval. In this case the wavelet decomposition (1.1) holds with $\mathcal{D}_0$ replaced by $\mathcal{D}_0(\Omega)$ and $\mathcal{D}_+$ replaced by $\mathcal{D}_+(\Omega)$ where the Ω indicates that we only take those I such that ϕ_I (respectively ψ_I) is not identically zero on Ω. It is also necessary to alter the definition of the coefficients $a_I(f)$ when the support of ψ_I intersects the boundary of Ω (see Section 2). If Ω is bounded, then the sets $\mathcal{D}_j(\Omega)$, $j \ge 0$, are finite.

A second way to organize the wavelet series comes forward when one considers the following problem of *nonlinear approximation*. For each positive integer $n \ge 1$, we let Σ_n denote the set of all functions which can be written as a linear combination of the scaling functions at level 0 together with at most n wavelets; i.e.,

$$S = \sum_{I \in \mathcal{D}_0(\Omega)} \langle f, \phi_I \rangle \phi_I + \sum_{I \in \Lambda} c_I \psi_I, \qquad \#\Lambda \le n,$$

where Λ is any subset of $\mathcal{D}_+(\Omega)$. (In order to simplify the presentation here in the Introduction, we will not count the scaling functions appearing in the representation of S.)

Note that Σ_n is *not* a linear space since, for example, the sum of two elements from Σ_n could require $2n$ wavelet terms in its representation. Approximation by the elements of Σ_n is called n-term approximation and is one of the simplest cases of nonlinear approximation. We define the error of n-term approximation in $L_p(\Omega)$ by

$$\sigma_n(f)_p := \inf_{S\in\Sigma_n} \|f - S\|_{L_p(\Omega)}. \tag{1.3}$$

In the case of approximation in $L_2(\mathbb{R})$, it is trivial to find best approximations to a function $f \in L_2(\mathbb{R})$ from Σ_n. Let $I_1 := I_1(f)$, $I_2 := I_2(f), \ldots$, be a rearrangement of the intervals in $\mathcal{D}_+(\mathbb{R})$ so that

$$|a_{I_1}(f)| \geq |a_{I_2}(f)| \geq \cdots.$$

Then,

$$S_n := S_n(f) := \sum_{I\in\mathcal{D}_0(\mathbb{R})} \langle f, \phi_I\rangle\phi_I + \sum_{\ell=1}^{n} a_{I_\ell}(f)\psi_{I_\ell} \tag{1.4}$$

is a best n-term approximation to f and

$$\sigma_n(f)_2^2 = \sum_{\ell=n+1}^{\infty} |a_{I_\ell}(f)|^2. \tag{1.5}$$

It is remarkable that this same result is almost true when approximating in L_p. Now, we choose the intervals $I_\ell := I_\ell(f, p)$ such that

$$|a_{I_1,p}(f)| \geq |a_{I_2,p}(f)| \geq \cdots,$$

with $a_{I,p}(f) := |I|^{1/2-1/p}a_I(f)$ the L_p-normalized coefficients. With this choice, Temlyakov [23] has shown that the corresponding approximant

$$S_{n,p} := \sum_{I\in\mathcal{D}_0(\mathbb{R})} \langle f, \phi_I\rangle\phi_I + \sum_{\ell=1}^{n} a_{I_\ell,p}(f)\psi_{I_\ell,p}$$

is a *near best approximation* to f in L_p. Here, "near best" means again that

$$\|f - S_{n,p}(f)\|_{L_p(\mathbb{R})} \leq \mathcal{C}_p\sigma_n(f)_p,$$

with the constant $\mathcal{C}_p$ depending only on p.

We return now to the question of efficient decompositions of a function. If $f \in L_2(\mathbb{R})$, we consider the following arrangement of the wavelet series:

$$f = \sum_{I\in\mathcal{D}_0(\mathbb{R})} \langle f, \phi_I\rangle\phi_I + \sum_{\ell=1}^{\infty} a_{I_\ell}(f)\psi_{I_\ell}. \tag{1.6}$$

This arrangement of the terms is optimal in the sense that each partial sum is a best *n-term approximation*. Therefore, no other choice of n-terms could reduce the error more than this partial sum. Similarly, for functions in $L_p(\mathbb{R})$, the arrangement

$$f = \sum_{I \in \mathcal{D}_0(\mathbb{R})} \langle f, \phi_I \rangle \phi_I + \sum_{\ell=1}^{\infty} a_{I_\ell,p}(f)\psi_{I_\ell,p} \tag{1.7}$$

is near optimal. (Note that, because the ordering of the $a_{I,p}$ depends on p, the i_ℓ in (1.7) are in general not the same as in (1.6).)

The decompositions (1.6), (1.7) have many impressive applications (see [12]). However, these decompositions do have the following deficiency. Consider the problem of encoding, where we want to transmit a finite number of bits which would allow the user to recover a good approximation to f with efficiency measured by the $L_p(\mathbb{R})$ error between f and the approximant. A natural way to proceed would be to send a certain number of bits for each coefficient $a_{I_\ell,p}$ (how one might assign bits will be spelled out in Section 4). However, for the receiver to reconstruct the approximant, he or she will also need to know the intervals I_ℓ and their correspondence with the bitstream. To send this additional information may be very costly and in fact may dominate the total number of bits sent. The situation can be ameliorated by imposing more organization on the decomposition (1.7). We shall accomplish this by requiring that the intervals I appearing in the n-term approximation (1.4) be identified with nodes on a *tree*. This leads us to the concept of *tree approximation* which we now describe.

We consider the case when Ω is a finite interval. Any dyadic interval I has a parent and two children. We say that a collection of dyadic intervals $\mathcal{T} \subset \mathcal{D}_+(\Omega)$ is a tree if whenever $I \in \mathcal{T}$, with $|I| < 1$, then its parent is also in $\mathcal{T}$. The cubes $I \in \mathcal{T}$, with $|I| = 1$, are called the roots of the tree $\mathcal{T}$.

Tree-based approximations are then defined as follows. Given a positive integer n, we define Σ_n^t as the collection of all $S \in \Sigma_n$ such that

$$S = \sum_{I \in \mathcal{D}_0(\Omega)} c_I \phi_I + \sum_{I \in \mathcal{T}} c_I \psi_I, \qquad \#\mathcal{T} \le n, \tag{1.8}$$

with $\mathcal{T}$ a tree with cardinality $\le n$. Given $f \in L_p(\Omega)$, the error in tree approximation is defined by

$$t_n(f)_p := \inf_{S \in \Sigma_n^t} \|f - S\|_{L_p(\Omega)}. \tag{1.9}$$

Then clearly

$$\Sigma_n^t \subset \Sigma_n \qquad \text{and} \qquad \sigma_n(f)_p \le t_n(f)_p. \tag{1.10}$$

Because we impose a tree structure on the dyadic intervals indexing the coefficients used in our approximant, the information about the "addressing" of the coefficients, i.e., about which intervals I_ℓ belong to the tree, can be sent in a number of bits proportional to the cardinality of the tree. (We shall come back to this in more detail later.) On the other hand, we are imposing more constraints on acceptable approximants, for which we may have to pay a price in the form of an increase of the error of approximation, when compared to more general approximations using the same number of coefficients; see (1.10). Alternatively, we could start with a general nonlinear approximation and add all the coefficients needed to ensure that the total collection of dyadic interval labels constitutes a tree (see Fig. 1). Whereas the approximation error will decrease somewhat, but not significantly in general, this "filling out the tree" will cost a price in the increase of numbers of coefficients to be sent. We shall show in Section 4 that this cost is minimal in a certain sense described below.

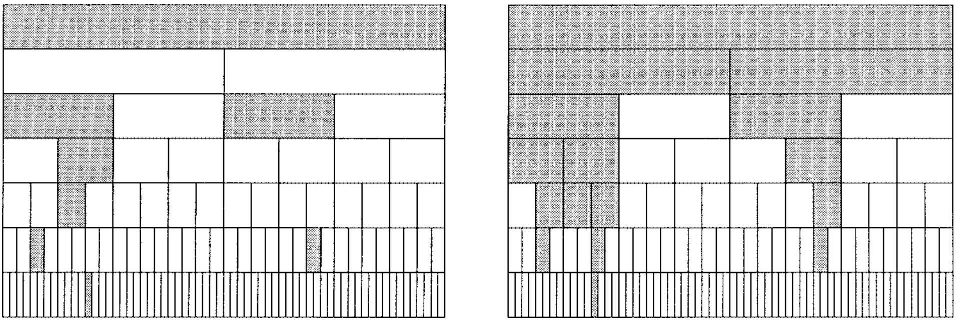

FIG. 1. This figure illustrates the concept of a tree of dyadic intervals in dimension 1. The layers of dyadic intervals are represented in a stack; the top layer corresponds to $\mathcal{D}_0$, the second to $\mathcal{D}_1$, and so on, with the intervals in $\mathcal{D}_j$ all having length 2^{-j}. On the left we have a set of eight highlighted intervals at different scales that do not constitute a tree; on the right we show the smallest tree containing all the intervals on the left.

We start this discussion by explaining the by now standard characterization of Besov spaces through nonlinear approximation results using wavelets. Let $\mathcal{A}^s := \mathcal{A}^s_\infty(L_p(\Omega))$ denote the class of functions $f \in L_p(\Omega)$ such that

$$\sigma_n(f)_p \leq Cn^{-s}, \qquad n = 1, 2, \ldots. \tag{1.11}$$

As we shall describe in more detail in Section 3, it is possible to characterize $\mathcal{A}^s$ in terms of the wavelet coefficients of f. From this characterization, we can deduce the approximation properties of functions in the Besov spaces $B^s_q(L_\tau(\Omega))$. Recall that the functions in this Besov space have smoothness of order s in L_τ, in the sense that their modulus of smoothness $\omega_m(f,t)_{L_\tau} := \sup_{|h|\leq t} \|\Delta_h^m f\|_{L_\tau}$ behaves in $\mathcal{O}(t^s)$ for $m > s$ (where $\Delta_h^m f$ denotes the mth order finite difference; i.e., $\sum_{n=0}^m \binom{m}{n}(-1)^n f(\cdot - nh)$). The parameter q gives a fine tuning of smoothness: by definition, f is in $B^s_q(L_\tau(\Omega))$ if and only if $f \in L_p$ and the sequence $(2^{sj}\omega_m(f, 2^{-j})_{L_\tau})_{j\geq 0}$ is in ℓ^q. In particular for any noninteger $s > 0$, $B^s_{\infty,\infty}$ is identical to the Hölder space $\mathcal{C}^s$. Classical Sobolev spaces also fall in this class: $B^s_{p,p}$ is $W^{s,p}$ for all $s > 0$ if $p = 2$ and for all nonintegers $s > 0$ otherwise.

If $f \in B^s_q(L_\tau(\Omega))$, for some $0 < q \leq \infty$ and $\tau > (s + 1/p)^{-1}$, then $f \in \mathcal{A}^s$. This result is also true if $\tau = (s + 1/p)^{-1}$ and $q \leq \tau$. In fact, in all these cases, n^{-s} is the best possible rate in the following sense. Let $\sigma_N(K)_p := \sup_{f\in K} \sigma_N(f)_p$ where K is any compact subset of $L_p(\Omega)$. Then, when U is the unit ball of $B^s_q(L_\tau(\Omega))$, one has $\sigma_N(U)_p \asymp n^{-s}$. Here $a \asymp b$ means that a can be bounded from above and below by some constant multiple of b uniformly with respect to any parameters on which a and b may depend.

This result has a simple geometrical interpretation given in Fig. 2. We identify with each point (x, y) in the upper right quadrant of the plane the spaces $B^s_q(L_\tau(\Omega))$ with $x = 1/\tau$, $y = s$. Thus a given point has associated to it a family of spaces since we do not distinguish between different values of q. The line $1/\tau = s + 1/p$ is called the *critical line* for nonlinear approximation. Notice that it is also the critical line for the Sobolev embedding theorem. Each of the spaces corresponding to points to the left of the critical line is compactly embedded in $L_p(\Omega)$ (i.e., bounded sets are mapped into compact sets under the identity operator). Points on the critical line may or may not be embedded in $L_p(\Omega)$ depending on the value of q. To be precise, a space $B^s_q(L_\tau(\Omega))$ on the critical line

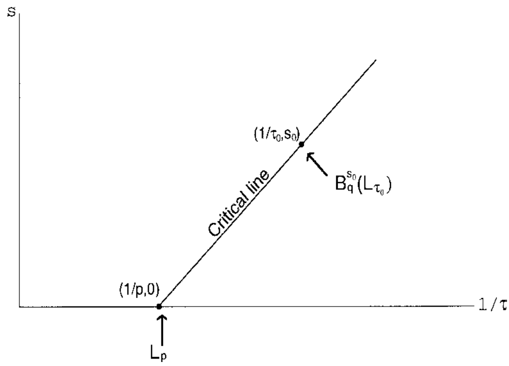

FIG. 2. The critical line for nonlinear approximation in the case $d = 1$, $s = 1/\tau - 1/p$.

is continuously embedded in $L_p(\Omega)$ if $q \leq p$ when $p < \infty$ and if $q \leq 1$ when $p = \infty$; see [15, p. 385]. However, these embeddings are not compact.

We shall show in Section 5 that for each space $B_q^s(L_\tau(\Omega))$ strictly to the left of the critical line, tree approximations are also near optimal; i.e., we have

$$t_n(f)_p \leq C(\tau, s) n^{-s} \|f\|_{B_q^s(L_\tau(\Omega))}. \tag{1.12}$$

Thus, for these spaces, tree approximation has the same performance as n-term approximation. For spaces on the critical line this is no longer true. The construction of good tree approximations can be achieved quite simply from thresholding (see Section 4).

1.2. Kolmogorov Entropy

The inequality (1.12), and its extension to several space dimensions, has many interesting applications. We shall give two of them. The first one concerns the determination of the Kolmogorov entropy of function classes.

Let X be a metric space with distance function ρ. If $f \in X$ and $r > 0$, we let

$$\mathbf{B}(f, r) := \mathbf{B}(f, r)x := \{g \in X : \rho(f, g) < r\}.$$

denote the (open) ball of radius r about f. If $K \subset X$ is *compact*, then for each $\epsilon > 0$, there is a finite collection of balls $B(f_i, \epsilon)$, $i = 1, \ldots, n$, which cover K:

$$K \subset \bigcup_{i=1}^{n} \mathbf{B}(f_i, \epsilon).$$

The *covering number* $N_\epsilon(K) := N_\epsilon(K, X)$ is the smallest integer n for which there is such an ϵ-covering of K. The Kolmogorov ϵ-*entropy* of K is then by definition

$$H_\epsilon(K) := H_\epsilon(K, X) := \log N_\epsilon(K), \qquad \epsilon > 0, \tag{1.13}$$

where log denotes the logarithm to the base two.

We shall restrict our attention in this article to the case $X = L_p(\Omega)$ where Ω is a Lipschitz domain in $\mathbb{R}^d$. We denote by $U(B^s_q(L_\tau(\Omega)))$ the unit ball of the Besov space $B^s_q(L_\tau(\Omega))$. A fundamental result in approximation theory is the following.

THEOREM 1.1. *Let Ω be a Lipschitz domain in $\mathbb{R}^d$ and let $1 \le p \le \infty$, and $s > d/\tau - d/p$. Then,*

$$H_\epsilon(U(B^s_q(L_\tau(\Omega))), L_p(\Omega)) \asymp \epsilon^{-d/s}, \tag{1.14}$$

with constants of equivalency depending only on s and $\delta := s - d/\tau + d/p$.

This theorem is well known when Ω is a cube in $\mathbb{R}^d$ (at least in the case $\tau \ge 1$). We shall be interested in the upper estimate in (1.14). The usual proofs of the upper estimate (see, for example, Chap. 13 of [20]) utilize discretization and finite dimensional geometry. For $\tau \ge 1$ they can also be derived by interpolation arguments from the classical result of Birman and Solomjak on the entropy of Sobolev balls [3]. We shall give a new, more elementary proof of this upper estimate by relating Kolmogorov entropy to deterministic encoding.

To describe deterministic encoding, let $K \subset X$ again be a compact subset of $L_p(\Omega)$. An encoder for K consists of two mappings. The first is a mapping E from K into a set $\mathcal{B}$ of bitstreams. That is, the elements $B \in \mathcal{B}$ are sequences of zeros and ones. Thus, E assigns to each element $f \in K$ element $E(f) \in \mathcal{B}$. The second mapping D associates to each $B \in \mathcal{B}$ an element $f_B \in L_p(\Omega)$. The mapping D decodes B. A codebook describes how a bitstream is converted to f_B.

Generally, $D(E(f)) \neq f$ and $\|f - D(E(f))\|_{L_p(\Omega)}$ measures the error that occurs in the encoding. The *distortion* of the encoding pair (D, E) on K is given by

$$\mathbf{D}(K, E, D) := \sup_{f \in K} \|f - D(E(f))\|_{L_p(\Omega)}. \tag{1.15}$$

Given a compact set K of X, we let $\mathcal{B}_K := \{E(f) : f \in K\}$ denote the set of bitstreams $E(f)$ that arise in encoding the elements of K. We also let

$$R(K, E, D) := \max\{\#(E(f)) : f \in K\} \tag{1.16}$$

denote the largest length of the bitstreams that appear when E encodes the elements of K. The efficiency of encoding is measured by the distortion for a given bit allocation. Thus, the optimal distortion rate is given by

$$\mathbf{D}_n(K) := \inf_{D,E} \mathbf{D}(K, E, D), \tag{1.17}$$

where the infimum is taken over all encoding decoding pairs D, E for which the bit allocation $R(K, E, D)$ is $\le n$.

It is easy to see that the rate distortion theory for optimal encoding is equivalent to determining the Kolmogorov entropy. Indeed, each ϵ covering $\mathbf{B}(f_\ell, \epsilon)$, $\ell = 1, \dots, N$, of K gives an encoding pair E,D in an obvious way. For each f, the encoder E selects an integer $\ell \in \{1, \dots, N\}$ such that $\mathbf{B}(f_\ell, \epsilon)$ contains f and maps f into the binary digits of ℓ. The decoder D maps a bitstream B into the element f_ℓ where ℓ is the integer with bit representation B. This encoding pair has distortion $<\epsilon$. By taking a minimal ϵ-cover of K with $N_\epsilon(K)$ balls, we obtain an encoding of K with distortion $<\epsilon$ using at most $\lceil \log(N_\epsilon(K)) \rceil$ bits; i.e.,

$$R(K, E, D) \leq \lceil H_\epsilon(K, X) \rceil.$$

Conversely, given any encoding pair E,D with distortion $<\epsilon$, the balls $\mathbf{B}(f_B, \epsilon)$, $B \in \mathcal{B}$, give an ϵ cover of K. So any such pair satisfies

$$H_\epsilon(K) \leq R(K, E, D). \tag{1.18}$$

In this sense the two problems of constructing optimal encoders and estimating Kolmogorov entropy are equivalent.

However, in the practice of encoding, one is interested in realizing the mappings E and D by fast algorithms. This is not true in general for the encoder derived from an ϵ-covering since finding f_ℓ might not be a trivial task. Thus a more relevant goal is to design a practical encoder and decoder such that the corresponding distortion $\tilde{\mathbf{D}}_n(K)$ has at least the same asymptotic behavior as the optimal $\mathbf{D}_n(K)$ when n tends to $+\infty$. Moreover it might be desirable that this optimality is achieved not only for a specific K but for many possible classes with the same encoding. For example, we might require that the rate distortion performance is in accordance with Theorem 1.1 above when $K = U(B_q^s(L_\tau(\Omega)))$ for various choices of s, τ, and q. We refer to such an encoder as *universal*.

1.3. Organization of Material

The rest of this article is organized as follows. In Section 2, we introduce wavelet decompositions and recall how the coefficients in these decompositions can be used to describe the Besov spaces. In Section 3, we recall some fundamental results in nonlinear approximation theory. In Section 4, we introduce tree approximation and prove the results (1.12) in the multivariate case. In Sections 5 and 6, we use the tree structure to build a universal encoder for any prescribed L_p metric with $1 \leq p < \infty$ used to measure the error. In Section 7 we employ this encoder in order to prove the upper estimates for encoding (Theorem 7.1) and Kolmogorov entropy (Corollary 7.2) for various Besov balls; the case $p = \infty$, more technical, is handled in Appendix A. It turns out (we are grateful to one of our reviewers for pointing this out to us) that the same result for Besov balls can also be proved using other arguments, not involving trees, and therefore corresponding to a different type of encoder. More details are given in Section 8 (with technical details in Appendix B), where we discuss possible reasons why we expect tree encoding to be nevertheless superior in applications.

It should be pointed out that wavelet decompositions have already been used in [2, 17] for proving upper bounds of Kolmogorov entropies. In [17] the bounds are in the L_2 metric and involve a logarithmic factor since the classes which are considered are slightly different than the $U(B_q^s(L_\tau(\Omega)))$. In [2], wavelets are used to prove upper bounds for Kolmogorov entropies by means of approximation procedures which are also universal with respect to the error metric L_p, $1 \leq p \leq \infty$. However, no explicit encoding strategies are presented there. The specificity of our approach is that the tree-structured encoding technique which is proved to be optimal is essentially close to practical algorithms such as given in [21, 22] and that they are universal for a prescribed L_p metric. Some optimal or near-optimal rate/distortion bounds in the L_2 metric for wavelet-based encoders are also proved in [6, 18]. Note that our approach also provides entropy bounds for the sets that precisely consist of those functions which can be approximated at a certain rate with a tree structure. These sets are larger than the Besov balls that are usually considered, and they are very natural in the context of image compression since they comply with the idea that the important coefficients produced by edges are naturally organized in a tree structure.

2. WAVELET DECOMPOSITIONS AND BESOV CLASSES

In this and the next section, we shall set forth the notation used throughout this article, and we recall some known results on wavelet decompositions and nonlinear approximation which are related to the topics of this article. We assume that the reader is familiar with the basics of wavelet theory (see [11]).

The results of this article hold for quite general wavelet decompositions. However, we shall restrict ourselves to the compactly supported biorthogonal wavelets as introduced by Cohen *et al.* [5]. These are general enough to also include the orthogonal wavelets of compact support as introduced by Daubechies [10]. A good reference for these bases and their properties is Chap. 8 of the monograph of Daubechies [11].

The construction of biorthogonal wavelets begins with two compactly supported univariate scaling functions ϕ and $\tilde{\phi}$ whose shifts are in duality,

$$\int_{\mathbb{R}} \phi(x-k)\tilde{\phi}(x-k')\,dx = \delta(k-k'), \qquad k, k' \in \mathbb{Z},$$

with δ the Kronecker delta. Associated to each of the scaling functions are mother wavelets ψ and $\tilde{\psi}$.

These functions can be used to generate a wavelet basis for the $L_p(\mathbb{R}^d)$ spaces as follows. We define $\psi^0 := \phi$, $\psi^1 := \psi$. Let V' denote the collection of vertices of the unit cube $[0, 1]^d$ and let V denote the nonzero vertices. For each vertex $v = (v_1, \ldots, v_d) \in V'$, we define the multivariate function

$$\psi^v(x_1, \ldots, x_d) := \psi^{v_1}(x_1) \cdots \psi^{v_d}(x_d),$$
$$\tilde{\psi}^v(x_1, \ldots, x_d) := \tilde{\psi}^{v_1}(x_1) \cdots \psi^{v_d}(x_d).$$

The collection of functions

$$\{\psi_I^v,\ I \in \mathcal{D},\ v \in V\},$$

constitutes a Riesz basis for $L_2(\mathbb{R}^d)$ (in the orthogonal case they form a complete orthonormal basis for $L_2(\mathbb{R}^d)$). They are an unconditional basis for $L_p(\mathbb{R}^d)$, $1 < p < \infty$. Each function f which is locally integrable on $\mathbb{R}^d$ has the wavelet expansion

$$f = \sum_{I \in \mathcal{D}} \sum_{v \in V} a_I^v(f)\psi_I^v, \qquad a_I^v(f) := \langle f, \tilde{\psi}_I^v \rangle. \tag{2.1}$$

We can also start the wavelet decomposition at any dyadic level. For example, starting at dyadic level 0, we obtain

$$f = \sum_{I \in \mathcal{D}_0} \sum_{v \in V'} a_I^v(f)\psi_I^v + \sum_{j=1}^{\infty} \sum_{I \in \mathcal{D}_j} \sum_{v \in V} a_I^v(f)\psi_I^v. \tag{2.2}$$

It can be convenient, in the characterizations of Besov spaces, to choose different normalizations for the wavelets and coefficients appearing in the decompositions (2.1), (2.2). In (2.1), (2.2), we have normalized in $L_2(\mathbb{R}^d)$; we can also normalize in $L_p(\mathbb{R}^d)$, $0 < p \leq \infty$, by taking

$$\psi_{I,p}^v := |I|^{-1/p+1/2}\psi_I^v, \qquad I \in \mathcal{D}, \qquad v \in V. \tag{2.3}$$

Then, we can rewrite (2.2) as

$$f = \sum_{I\in\mathcal{D}_0}\sum_{v\in V'} a_{I,p}^v(f)\psi_{I,p}^v + \sum_{j=1}^{\infty}\sum_{I\in\mathcal{D}_j}\sum_{v\in V} a_{I,p}^v(f)\psi_{I,p}^v, \tag{2.4}$$

where

$$a_{I,p}^v(f) := \langle f, \tilde{\psi}_{I,p'}^v\rangle,$$

with $1/p + 1/p' = 1$.

For simplicity of notation, we shall combine all terms associated with a dyadic cube I in one expression:

$$A_I(f) := \begin{cases} \sum_{v\in V'} a_{I,p}^v(f)\psi_{I,p}^v, & I \in \mathcal{D}_0, \\ \sum_{v\in V} a_{I,p}^v(f)\psi_{I,p}^v, & I \in \mathcal{D}_j,\ j \geq 1. \end{cases} \tag{2.5}$$

Note that the definition of $A_I(f)$ does not depend on p and that

$$\|A_I(f)\|_{L_p(\mathbb{R}^d)} \asymp a_{I,p}(f) := \begin{cases} (\sum_{v\in V'} |a_{I,p}^v(f)|^p)^{1/p}, & I \in \mathcal{D}_0, \\ (\sum_{v\in V} |a_{I,p}^v(f)|^p)^{1/p}, & I \in \mathcal{D}_j,\ j \geq 1. \end{cases} \tag{2.6}$$

It is easy to go from one normalization to another. For example, for any $0 < p, q \leq \infty$, we have

$$\psi_{I,p}^v = |I|^{1/q-1/p}\psi_{I,q}^v, \qquad a_{I,p}(f) = |I|^{1/p-1/q}a_{I,q}(f). \tag{2.7}$$

Note that we can compute the $L_p(\mathbb{R}^d)$ norms of single scale wavelet sums $S_j(f) := \sum_{I\in\mathcal{D}_j} A_I(f)$ from a fixed dyadic level j. Namely,

$$\|S_j(f)\|_{L_p(\mathbb{R}^d)} \asymp \|(a_{I,p}(f))_{I\in\mathcal{D}_j}\|_{\ell_p}, \qquad 0 < p \leq \infty, \tag{2.8}$$

with the constants of equivalency depending on p only when p is small.

Many function spaces can be described by wavelet coefficients. In particular, such characterizations hold for the Besov spaces $B_q^s(L_\tau(\mathbb{R}^d))$, $s > 0$, $0 < \tau, q \leq \infty$. We shall need only the case when $B_q^s(L_\tau(\mathbb{R}^d))$ is compactly embedded in $L_1(\mathbb{R}^d)$ which means that $s > d/\tau - d$. We choose a univariate biorthogonal wavelet pair such that ψ has smoothness C^r, and $\tilde{\psi}$ has at least r vanishing moments with $r > s$. The Besov space $B_q^s(L_\tau(\mathbb{R}^d))$ can then be defined as the set of all functions f that are locally in $L_1(\mathbb{R}^d)$ and for which

$$\|f\|_{B_q^s(L_\tau(\mathbb{R}^d))} := \begin{cases} \left(\sum_{j=0}^{\infty} 2^{jsq}\left(\sum_{I\in\mathcal{D}_j} a_{I,\tau}(f)^\tau\right)^{q/\tau}\right)^{1/q}, & 0 < q < \infty, \\ \sup_{j\geq 0} 2^{js}\left(\sum_{I\in\mathcal{D}_j} a_{I,\tau}(f)^\tau\right)^{1/\tau}, & q = \infty, \end{cases} \tag{2.9}$$

is finite and with the usual change if $\tau = \infty$. The quasi-norm in (2.9) is equivalent to the other quasi-norms used to define Besov spaces in terms of moduli of smoothness or Fourier transforms.

In most applications, the functions of interest are not defined on $\mathbb{R}^d$ but rather on a bounded domain $\Omega \subset \mathbb{R}^d$. We shall assume that Ω is a Lipschitz domain (for a definition see, e.g., Adams [1]). The Besov spaces $B^s_q(L_\tau(\Omega))$ for such domains are usually defined by moduli of smoothness but they can also be described by wavelet decompositions similar to (2.9).

To see this, we use the fact that any such function f has an extension $\mathcal{E}f$ to all of $\mathbb{R}^d$ which satisfies (see [16])

$$\|\mathcal{E}f\|_{B^s_q(L_\tau(\mathbb{R}^d))} \le C\|f\|_{B^s_q(L_\tau(\Omega))},$$

with the constant C independent of f. In going further, we simply denote $\mathcal{E}f$ by f.

Since we now have a function f defined on all of $\mathbb{R}^d$, we can apply the characterization (2.9). In (2.9), we only have to include those ψ^v_I which do not vanish identically on Ω. For $j = 0, 1, \ldots,$ we denote by $\mathcal{D}_j(\Omega)$ the collection of all dyadic cubes $I \in \mathcal{D}_j$ such that, for some $v \in V'$, ψ^v_I does not vanish identically on Ω. We further set $\mathcal{D}_+(\Omega) := \cup_{j\ge 0}\mathcal{D}_j(\Omega)$.

In analogy with (2.9), we have the following quasi-norm for the Besov space $B^s_q(L_\tau(\Omega))$:

$$\|f\|_{B^s_q(L_\tau(\Omega))} := \begin{cases} \left(\sum_{j=0}^\infty 2^{jsq}\|(a_{I,\tau}(f))_{I\in\mathcal{D}_j(\Omega)}\|^q_{\ell_\tau}\right)^{1/q}, & 0 < q < \infty, \\ \sup_{j\ge 0} 2^{js}\|(a_{I,\tau}(f))_{I\in\mathcal{D}_j(\Omega)}\|_{\ell_\tau}, & q = \infty. \end{cases} \tag{2.10}$$

For simple domains (e.g., polyhedra, or piecewise smooth domains) one can also directly construct wavelet bases on Ω that satisfy (2.10).

We close this section with the following observation.

Remark 2.1. Let $\tau > (s/d + 1/p)^{-1}$. Then the unit ball $U(B^s_q(L_\tau(\Omega)))$ of $B^s_q(L_\tau(\Omega))$ is a compact subset of $L_p(\Omega)$.

Proof. We define $\mu := \min(p, \tau)$ and introduce the *discrepancy*

$$\delta := s - \frac{d}{\mu} + \frac{d}{p} > 0. \tag{2.11}$$

Then, it follows from (2.10), (2.7), and (2.8) that for each $j = 0, 1, \ldots,$

$$\begin{aligned} 2^{j\delta}\Big\|\sum_{I\in\mathcal{D}_j(\Omega)} A_I(f)\Big\|_{L_p} &\asymp 2^{j\delta}\Big(\sum_{I\in\mathcal{D}_j(\Omega)} a_{I,p}(f)^p\Big)^{1/p} \le 2^{j\delta}\Big(\sum_{I\in\mathcal{D}_j(\Omega)} a_{I,p}(f)^\mu\Big)^{1/\mu} \\ &= 2^{j\delta}\Big(\sum_{I\in\mathcal{D}_j(\Omega)} 2^{-d(\mu/p-1)j} a_{I,\mu}(f)^\mu\Big)^{1/\mu} \\ &= 2^{sj}\Big(\sum_{I\in\mathcal{D}_j(\Omega)} a_{I,\mu}(f)^\mu\Big)^{1/\mu} \\ &\le \|f\|_{B^s_\infty(L_\mu(\Omega))} \le \|f\|_{B^s_q(L_\mu(\Omega))} \le \|f\|_{B^s_q(L_\tau(\Omega))}, \end{aligned} \tag{2.12}$$

for any $0 < q \leq \infty$. This implies the estimate

$$\left\| f - \sum_{j=0}^{J} \sum_{I \in \mathcal{D}_j(\Omega)} A_I(f) \right\|_{L_p} \leq \sum_{j>J} \left\| \sum_{I \in \mathcal{D}_j(\Omega)} A_I(f) \right\|_{L_p} \leq C 2^{-J\delta} \|f\|_{B_q^s(L_\tau(\Omega))}, \quad (2.13)$$

which shows the compactness of $U(B_q^s(L_\tau(\Omega)))$. ■

3. NONLINEAR APPROXIMATION

In this section, we shall recall some facts about nonlinear approximation that will serve as an orientation for the results on tree approximation presented in the next section. A general reference for the results of this section is [12]. We fix a domain Ω and consider the approximation of functions in $L_p(\Omega)$. We begin by describing what is known as *n-term approximation*.

Let Σ_n be defined as the set of all functions S that satisfy the condition

$$S = \sum_{I \in \Lambda} A_I(S), \qquad \#\Lambda \leq n, \quad (3.1)$$

with A_I defined as in (2.5). We shall consider approximation in the space $L_p(\Omega)$ by the elements of Σ_n. Given $f \in L_p(\Omega)$, $1 \leq p \leq \infty$, we define (as in (1.3))

$$\sigma_n(f)_p := \inf_{S \in \Sigma_n} \|f - S\|_{L_p(\Omega)}, \qquad n = 0, 1, \ldots. \quad (3.2)$$

Note that by definition $\sigma_0(f)_p := \|f\|_{L_p(\Omega)}$.

We can describe the functions f for which $\sigma_n(f)_p$ has a prescribed asymptotic behavior as $n \to \infty$. For $1 \leq p \leq \infty$, $0 < q \leq \infty$, and $s > 0$, we define the approximation class $\mathcal{A}_q^s(L_p(\Omega))$ to be the set of all $f \in L_p(\Omega)$ such that

$$\|f\|_{\mathcal{A}_q^s(L_p(\Omega))} := \begin{cases} \left(\sum_{n=0}^{\infty} [(n+1)^s \sigma_n(f)_p]^q \frac{1}{n+1}\right)^{1/q}, & 0 < q < \infty, \\ \sup_{n \geq 0} (n+1)^s \sigma_n(f)_p, & q = \infty, \end{cases} \quad (3.3)$$

is finite. From the monotonicity of $\sigma_n(f)_p$, it follows that (3.3) is equivalent to

$$\|f\|_{\mathcal{A}_q^s(L_p(\Omega))} \asymp \begin{cases} \left(\sum_{\ell \geq -1} [2^{\ell s} \sigma_{2^\ell}(f)_p]^q\right)^{1/q}, & 0 < q < \infty, \\ \sup_{\ell \geq -1} 2^{\ell s} \sigma_{2^\ell}(f)_p, & q = \infty, \end{cases} \quad (3.4)$$

where for the purposes of this formula we define $\sigma_{1/2}(f)_p = \sigma_0(f)_p$.

It is possible to characterize the spaces $\mathcal{A}_q^s(L_p(\Omega))$ in several ways: in terms of interpolation spaces, in terms of wavelet coefficients, and in terms of smoothness spaces (Besov spaces). For spaces X, Y we denote by $(X, Y)_{\theta,q}$ the interpolation spaces generated by the real method of interpolation (K-functional) with parameters $0 < \theta < 1$, $0 < q \leq \infty$. We shall denote by $\ell_{\mu,q}$ the Lorentz space of sequences $(c_I)_{I \in \mathcal{D}_+}$ indexed on dyadic intervals (for a definition of this space see [12, 15]).

THEOREM 3.1 [7, 13]. *Let $1 < p < \infty$ and let $\psi, \tilde{\psi}$ be a biorthogonal wavelet pair where ψ has smoothness of order r and $\tilde{\psi}$ has at least r vanishing moments. Then, the following characterizations of $\mathcal{A}_q^s(L_p(\Omega))$ hold:*

(i) *For each $0 < s < r$, we have that a function f is in $A_q^{s/d}(L_p(\Omega))$ if and only if the sequence $(a_{I,p}(f))_{I\in\mathcal{D}_+(\Omega)}$ defined by (2.6) is in the Lorentz sequence space $\ell_{\tau,q}$ with $1/\tau := s/d + 1/p$ and*

$$\|f\|_{\mathcal{A}_q^{s/d}(L_p(\Omega))} \asymp \|f\|_{L_p(\Omega)} + \|(a_{I,p}(f))_{I\in\mathcal{D}_+(\Omega)}\|_{\ell_{\tau,q}}. \tag{3.5}$$

(ii) *For each $0 < s < r$ and $0 < q \leq \infty$,*

$$\mathcal{A}_q^{s/d}(L_p(\Omega)) = (L_p(\Omega), B_\mu^r(L_\mu(\Omega)))_{s/r,q}, \tag{3.6}$$

with equivalent norms, where $1/\mu = r/d + 1/p$,

(iii) *In the special case $0 < s < r$ and $q = \tau = (s/d + 1/p)^{-1}$, we have*

$$\mathcal{A}_q^{s/d}(L_p(\Omega)) = B_\tau^s(L_\tau(\Omega)), \tag{3.7}$$

where $1/\tau = s/d + 1/p$.

Thus, we have in (i) a characterization of the approximation spaces in terms of the decay of the wavelet coefficients, while in (ii) this space is characterized in terms of interpolation between Besov spaces. Characterization (iii) shows that in the special case $q = \tau = (s/d + 1/p)^{-1}$ the approximation space is identical with a Besov space.

As we already mentioned, a simple and useful geometrical interpretation of this theorem is given in Fig. 2: Theorem 3.1 says that the approximation space $\mathcal{A}_\tau^{s/d}(L_p)$ corresponds to the point $(1/\tau, s)$ of smoothness s on the critical line for nonlinear approximation in $L_p(\Omega)$, or equivalently on the line segment connecting $(1/p, 0)$ (corresponding to $L_p(\Omega)$) to $(1/\mu, r)$ (corresponding to $B_\mu^\tau(L_\mu(\Omega))$).

For any compact subset K of $L_p(\Omega)$, we let

$$\sigma_n(K)_p := \sup_{f\in K} \sigma_n(f)_p \tag{3.8}$$

be the error of the n-term approximation for this class. The main inference we wish to retain from Theorem 3.1 is the following. For any point $(1/\tau, s)$, $0 < s < r$, lying above the critical line of nonlinear approximation and any of the Besov classes $B_q^s(L_\tau(\Omega))$ and its unit ball $U(B_q^s(L_\tau(\Omega)))$, we have

$$\sigma_n(U(B_q^s(L_\tau(\Omega)))) \leq Cn^{-s/d}, \qquad n = 1, 2, \ldots, \tag{3.9}$$

with the constant C independent of n. This inequality also holds for the Besov spaces on the critical line provided $q \leq (s/d + 1/p)^{-1}$.

We have already noted in the Introduction that a way of constructing near best n-term approximations to a function $f \in L_p(\Omega)$ is to retain the n terms in the wavelet expansion of f which have the largest $L_p(\Omega)$ norms. We shall not formulate this result explicitly (see [12]) since we shall not need it. However, we shall need the following closely related result of Temlyakov (see [12, 23]).

THEOREM 3.2. *Let $1 \le p < \infty$ and let $\Lambda \subset \mathcal{D}_+(\Omega)$ be any finite set. If S is a function of the form $S = \sum_{I \in \Lambda} A_{I,p}(S)$, then*

$$\|S\|_{L_p(\Omega)} \le \mathcal{C}_p \max_{I \in \Lambda} a_{I,p}(S)(\#\Lambda)^{1/p}, \tag{3.10}$$

with the constant $\mathcal{C}_p$ depending only on p. Also, for any such set Λ and any $1 < p \le \infty$, we have

$$\mathcal{C}'_p \min_{I \in \Lambda} a_{I,p}(S)(\#\Lambda)^{1/p} \le \|S\|_{L_p(\Omega)}, \tag{3.11}$$

where again the constant depends only on p.

Note that in the case $p = 1$, the inequality (3.10) (as stated and proved in [7]) is given for the Hardy space H_1 in place of L_1 but then (3.10) follows because the L_1 norm can be bounded by the H_1 norm. Similarly, (3.11) holds for the space BMO from which one derives the case $p = \infty$.

4. TREE APPROXIMATION

We turn now to the main topic of this article which is tree approximation. Dyadic cubes I in $\mathbb{R}^d$ have one parent (the smallest dyadic cube which properly contains I) and 2^d children (the largest dyadic cubes strictly contained in I). By a tree $\mathcal{T}$ we shall mean a set of dyadic cubes from $\mathcal{D}_+(\Omega)$ with the following property: if $|I| < 1$ and $I \in \mathcal{T}$, then its parent is also in $\mathcal{T}$. We denote by Σ_n^t the collection of all functions that satisfy

$$S = \sum_{I \in \mathcal{T}} A_I(S), \qquad \#\mathcal{T} \le n, \tag{4.1}$$

with $\mathcal{T}$ a tree. If $f \in L_p(\Omega)$, $1 \le p \le \infty$, then we recall (1.9) and define the error of tree approximation by

$$t_n(f)_p := \inf_{S \in \Sigma_n^t} \|f - S\|_{L_p(\Omega)}. \tag{4.2}$$

More generally for a compact subset $K \subset L_p(\Omega)$, we set

$$t_n(K)_p := \sup_{f \in K} t_n(f)_p. \tag{4.3}$$

In the following we fix the L_p metric in which the error is measured and we shall assume that $p < \infty$. (The case $p = \infty$ is treated in Appendix A). There is a simple and constructive way of generating tree approximations of a given function $f \in L_p(\Omega)$ by thresholding its wavelet coefficients. For each $\eta > 0$, we let

$$\Lambda(f, \eta) := \{I \in \mathcal{D}_+(\Omega) : a_{I,p}(f) \ge \eta\}. \tag{4.4}$$

Defining now $\mathcal{T}(f, \eta)$ as the smallest tree containing $\Lambda(f, \eta)$ we note that

$$\mathcal{T}(f, \eta) \subseteq \mathcal{T}(f, \eta'), \qquad \eta' \le \eta, \tag{4.5}$$

and that these sets depend on p. With each tree $\mathcal{T}(f, \eta)$ we associate now the approximant

$$S(f, \eta) := \sum_{I \in \mathcal{T}(f,\eta)} A_I(f). \tag{4.6}$$

It will be convenient to associate a new family of spaces with this construction, defined by bounding the cardinality of the trees $\mathcal{T}(f, \eta)$ in terms of the threshold η. More precisely, for all $\eta > 0$ we define $\mathcal{B}_\lambda(L_p(\Omega))$ as the set of those $f \in L_p(\Omega)$ for which there exists a constant $C(f)$ such that

$$\#(\mathcal{T}(f, \eta)) \le C(f)\eta^{-\lambda}. \tag{4.7}$$

For general f in L_p, the best possible bound on $\#(\Lambda(f, \eta))$, obtained by applying (3.11) in Theorem 3.2, is

$$\#(\Lambda(f, \eta)) \le \left[C'(f)\|S(f, \eta)\|_{L_p(\Omega)}\right]^p \eta^{-p}.$$

For $f \in \mathcal{B}_\lambda(L_p(\Omega))$ we have

$$\#(\Lambda(f, \eta)) \le \#(\mathcal{T}(f, \eta)) \le C(f)\eta^{-\lambda};$$

for $0 < \lambda < p$ this stronger decay implies that $B_\lambda(L_p(\Omega))$ is a strict subset of $L_p(\Omega)$.

It is easy to see that $B_\lambda(L_p(\Omega))$ constitutes a linear space: if f and g are in $B_\lambda(L_p(\Omega))$, we simply remark that $\mathcal{T}(f + g, \eta) \subset \mathcal{T}(f, \eta/2) \cup \mathcal{T}(g, \eta/2)$ so that we have $\#(\mathcal{T}(f + g, \eta)) \le \#(\mathcal{T}(f, \eta/2)) + \#(\mathcal{T}(g, \eta/2))$. Moreover, we define a quasi-norm $\|f\|_{\mathcal{B}_\lambda(L_p(\Omega))} := C^*(f)^{1/\lambda}$ where $C^*(f)$ is the smallest constant such that (4.7) holds.

The next theorem will examine the approximation properties of $S(f, \eta)$.

THEOREM 4.1. *Let $1 \le p < \infty$, and let $0 < \lambda < p$. Then we have*

$$\|f - S(f, \eta)\|_{L_p(\Omega)} \le c_1 \|f\|^{\lambda/p}_{\mathcal{B}_\lambda(L_p(\Omega))} \eta^{1-\lambda/p}, \tag{4.8}$$

where c_1 depends on λ only if λ is close to p. Moreover, let $(1/\tau, s)$ be a point above the critical line for nonlinear approximation in L_p; i.e., s and τ should satisfy $s > d/\tau - d/p$. Then, $0 < q \le \infty$, $B^s_q(L_\tau(\Omega))$ is continuously embedded in $\mathcal{B}_\lambda(L_p(\Omega))$ with $\lambda := d/(s + d/p)$,

$$\|f\|_{\mathcal{B}_\lambda(L_p(\Omega))} \le c_2 \|f\|_{B^s_q(L_\tau(\Omega))}, \tag{4.9}$$

with c_2 depending on the size of the supports of φ and ψ, and on δ only when δ is close to zero.

Proof. Let $f \in \mathcal{B}_\lambda(L_p(\Omega))$ and let $M := \|f\|_{\mathcal{B}_\lambda(L_p(\Omega))}$. To prove (4.8), we note that for each cube I not in $\mathcal{T}(f, 2^{-\ell}\eta)$, we have $a_{I,p}(f) \le 2^{-\ell}\eta$. Let

$$\Sigma_\ell := \sum_{I \in \mathcal{T}(f,2^{-\ell-1}\eta) \setminus \mathcal{T}(f,2^{-\ell}\eta)} A_I(f).$$

Then, using (3.10) and (4.7), we deduce that

$$\|\Sigma_\ell\|_{L_p(\Omega)} \le C2^{-\ell}\eta[\#(\mathcal{T}(f, 2^{-\ell-1}\eta))]^{1/p} \le C2^{-\ell}\eta[M^\lambda 2^{\ell\lambda}\eta^{-\lambda}]^{1/p}. \tag{4.10}$$

Therefore,

$$\|f - S(f,\eta)\|_{L_p(\Omega)} \le \sum_{\ell=0}^{\infty} \|\Sigma_\ell\|_{L_p(\Omega)} \le CM^{\lambda/p}\eta^{1-\lambda/p} \sum_{\ell=0}^{\infty} 2^{-\ell(1-\lambda/p)}$$
$$\le CM^{\lambda/p}\eta^{1-\lambda/p}, \tag{4.11}$$

where we have used that $\lambda < p$.

In order to prove (4.9), we also define $\Lambda_j(f,\eta) := \Lambda(f,\eta) \cap \mathcal{D}_j(\Omega)$. For $f \in B_q^s(L_\tau(\Omega))$, let $\tilde{M} = \|f\|_{B_q^s(L_\tau(\Omega))}$. Then the estimates (2.12) at the end of Section 2 provide

$$\#(\Lambda_j(f,\eta))\eta^\tau \le \sum_{I\in\mathcal{D}_j(\Omega)} a_{I,p}(f)^\tau \le \tilde{M}^\tau 2^{-j\delta\tau}, \tag{4.12}$$

where δ is defined by (2.11). In order to exploit this estimate for bounding the cardinality of $\mathcal{T}(f,\eta)$ we define $\mathcal{T}_j(f,\eta) := \mathcal{T}(f,\eta) \cap \mathcal{D}_j(\Omega)$. Note next that a cube I is in $\mathcal{T}_j(f,\eta)$, $j \ge 0$, if and only if there is a cube $I' \subseteq I$ such that $I' \in \Lambda(f,\eta)$. In fact, when $I \notin \Lambda_j(f,\eta)$ then I must be an ancestor of an $I' \in \Lambda_k(f,\eta)$ for some $k > j$. Since any such cube I' belongs to at most one cube $I \in \mathcal{T}_j(f,\eta)$, we have, from (4.12),

$$\#\mathcal{T}_j(f,\eta) \le C\min\left(2^{jd}, \tilde{M}^\tau \sum_{k\ge j} 2^{-k\delta\tau}\eta^{-\tau}\right) \le C\min(2^{jd}, \tilde{M}^\tau 2^{-j\delta\tau}\eta^{-\tau}). \tag{4.13}$$

Here we have used the fact that $\mathcal{D}_j(\Omega)$ contains at most $C2^{jd}$ cubes because the wavelets have compact support and Ω is a bounded domain. We temporarily assume $\tau \le \rho$ so that $\delta = s - \frac{d}{\tau} + \frac{d}{p}$. In order to sum (4.13) over all $j \ge 0$, we observe first that the turnover level J, i.e., the smallest integer for which

$$2^{Jd} \ge \tilde{M}^\tau \sum_{k\ge J} 2^{-k\delta\tau}\eta^{-\tau}, \tag{4.14}$$

is given by $J = \lceil(\lambda/d)\log_2(\bar{M}/\eta)\rceil_+$. Thus the sum of (4.13) over all $j \ge 0$ is bounded by $C((\tilde{M}/\eta)^\lambda + (\tilde{M}/\eta)^\tau \sum_{j=J}^{\infty} 2^{-j\delta\tau})$, which yields

$$\#\mathcal{T}(f,\eta) \le C\tilde{M}^\lambda\eta^{-\lambda}. \tag{4.15}$$

This establishes $M \le C\tilde{M}$, confirming (4.9) in the case $\tau \le p$. The case $\tau > p$ follows from the continuous ambedding $\mathcal{B}_p^s(L_\tau) \hookrightarrow \mathcal{B}_q^s(L_p)$. ■

We can use Theorem 4.1 to estimate the error of tree approximation.

COROLLARY 4.2. *Let $1 \le p < \infty$ and $0 < \lambda < p$. Then, for each $n = 1, 2, \ldots$ and each $f \in \mathcal{B}_\lambda(L_p(\Omega))$, we have*

$$t_n(f)_p \le c_1\|f\|_{\mathcal{B}_\lambda(L_p(\Omega))} n^{-(1/\lambda - 1/p)}, \tag{4.16}$$

with c_1 the constant in (4.8). *Moreover, let $(1/\tau, s)$ be a point above the critical line for nonlinear approximation in L_p; i.e., s and τ should satisfy $s > d/\tau - d/p$. Then,* *if* $0 < q \leq \infty$, *for each* $n = 1, 2, \ldots$ *and each* $f \in B^s_q(L_\tau(\Omega))$, *we have*

$$t_n(f)_p \leq c_3 \|f\|_{B^s_q(L_\tau(\Omega))} n^{-s/d}, \tag{4.17}$$

where $c_3 := c_1 c_2$ *and* c_1, c_2 *are the constants in* (4.8), (4.9), *respectively.*

Proof. Let $f \in \mathcal{B}_\lambda(L_p(\Omega))$. Given a positive integer n, we take η such that

$$\|f\|^\lambda_{\mathcal{B}_\lambda(L_p(\Omega))} \eta^{-\lambda} = n. \tag{4.18}$$

Then $S(f, \eta) \in \Sigma^t_n$, and (4.8) yields

$$t_n(f)_p \leq \|f - S(f, \eta)\|_{L_p(\Omega)} \leq c_1 \|f\|^{\lambda/p}_{\mathcal{B}_\lambda(L_p(\Omega))} \eta^{1-\lambda/p}.$$

Writing

$$\|f\|^{\lambda/p}_{\mathcal{B}_\lambda(L_p(\Omega))} \eta^{1-\lambda/p} = \|f\|_{\mathcal{B}_\lambda(L_p(\Omega))} \left(\|f\|^\lambda_{\mathcal{B}_\lambda(L_p(\Omega))} \eta^{-\lambda}\right)^{1/p-1/\lambda} \tag{4.19}$$

provides, in view of (4.18), the first estimate (4.16). Using (4.9) to bound the first factor on the right hand side of (4.19) yields again, on account of (4.18),

$$t_n(f)_p \leq c_1 c_2 \|f\|_{B^s_q(L_\tau(\Omega))} n^{-(1/\lambda - 1/p)}. \tag{4.20}$$

The second estimate (4.17) follows now because $\lambda := d/(s + d/p)$ means $1/\lambda - 1/p = s/d$. ■

5. A TREE BASED WAVELET DECOMPOSITION

The results of the previous section now give rise to a new wavelet decomposition based on trees. For $f \in L_p(\Omega)$, with $1 \leq p < \infty$ and each $k = 0, 1, \ldots,$ we define the trees

$$\mathcal{T}_k(f) := \mathcal{T}(f, 2^{-k}), \tag{5.1}$$

where we adhere to the notation of the previous section and let

$$\Sigma_k(f) := \sum_{I \in \mathcal{T}_k(f)} A_I(f). \tag{5.2}$$

Bearing in mind that, according to (4.5), $\mathcal{T}_{k-1}(f) \subset \mathcal{T}_k(f)$ for all $k \geq 1$, we introduce the layers

$$\mathcal{L}_0(f) := \mathcal{T}_0(f), \qquad \mathcal{L}_k(f) := \mathcal{T}_k(f) \backslash \mathcal{T}_{k-1}(f), \qquad k \in \mathbb{N},$$

corresponding to the wavelet coefficients grouped by size, and set

$$\Delta_0(f) := \Sigma_0(f),$$
$$\Delta_k(f) := \Sigma_k(f) - \Sigma_{k-1}(f) = \sum_{I \in \mathcal{L}_k(f)} A_I(f), \qquad k \geq 1. \tag{5.3}$$

Then, each $f \in L_p(\Omega)$ has the decomposition

$$f = \sum_{k \geq 0} \Delta_k(f). \tag{5.4}$$

When $1 < p < \infty$ the unconditionality of the wavelet basis implies that the series in (5.4) converges in $L_p(\Omega)$ whenever $f \in L_p(\Omega)$. For $p = 1$ the strong convergence of the partial sums in (5.4) for f in any Besov space on the left of the critical line is ensured by Theorem 4.1.

6. A UNIVERSAL ENCODING ALGORITHM

In this section, we shall use the tree decomposition (5.4) to construct encoding pairs for L_p functions, with $1 \leq p < \infty$. The encoder E to be described below assigns to each $f \in L_p(\Omega)$ an *infinite* bitstream which completely determines f. The encoder E is progressive in the following sense. We shall define encoders E_N by associating to each f the first $K_N(f)$ bits of the infinite sequence $E(f)$, where the number $K_N(f)$ wil be made more precise below; the $K_N(f)$ are bounded uniformly in f. Moreover $K_{N+1}(f) \geq K_N(f)$, so that the successive $E_N(f)$ build upon each other: each $E_{N+1}(f)$ starts off by using the bitstream $E_N(f)$ and then gives additional information about f (contained in the next $K_{N+1}(f) - K_N(f)$ bits) that increases the accuracy in approximating f (or, equivalently, decreases the distortion).

The Encoder

To begin the discussion, we fix p with $1 \leq p < \infty$. For $f \in L_p(\Omega)$, the bitstream $E(f)$ will take the form

$$\begin{aligned} &L(f), P_0(f), S_0(f), B_{0,0}(f), P_1(f), S_1(f), B_{0,1}(f), B_{1,0}(f), \ldots, \\ &P_N(f), S_N(f), B_{0,N}(f), B_{1,N-1}(f), \ldots, B_{N,0}(f), \ldots, \end{aligned} \tag{6.1}$$

where

- $L(f)$ is a bitstream that gives the *size* of the largest wavelet coefficient of f;
- each bitstream $P_k(f)$ gives the *positions* of the cubes in $\mathcal{L}_k(f)$ which includes the indices of the wavelet coefficients that exceed 2^{-k} but are smaller than 2^{-k+1};
- each bitstream $S_k(f)$ gives the *signs* of the coefficients $a^v_{I,p}(f)$, $I \in \mathcal{L}_k(f)$;
- each bitstream $B_{k,N-k}(f)$ provides the Nth bit for each of the wavelet coefficients $a^v_{I,p}(f)$ for the $I \in \mathcal{L}_k(f)$, $0 \leq k \leq N$.

The successive $P_k(f)$ are illustrated in Fig. 3. We now describe these bitstreams in more detail.

The normalization bitstream $L(f)$. Let κ be the integer such, that the absolute value of the largest wavelet coefficient $a^v_{I,p}(f)$ of f appearing in (5.4) is in $[2^{-\kappa}, 2^{-\kappa+1})$. The first bit in $L(f)$ is a zero, respectively one, if $\kappa > 0$, respectively $\kappa \leq 0$. It is followed by $|\kappa|$ ones while the last bit is zero, indicating the termination of $L(f)$. Thus in the case $\kappa = 0$, $L(f)$ is the bitstream consisting of two zeros.

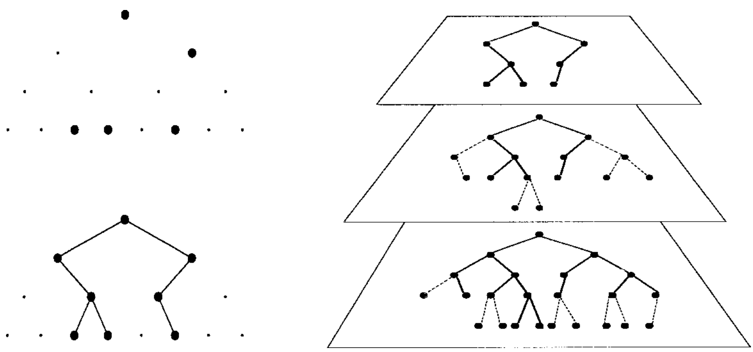

FIG. 3. We illustrate the different trees corresponding to the different precision levels, for the case $d = 1$. For every ℓ, we first determine the wavelet coefficients $a^v_{I,p}(f)$ that exceed $2^{-\ell}$; we then add, if needed, ancestors of their labels I so as to constitute the trees $\mathcal{T}_\ell(f)$, as in Fig. 1. This process is also illustrated here, in the figure on the left, for the special case $\ell = 1$: on top the nodes represent the labels I for which the wavelet coefficient is above the threshold $1/2$, and on the bottom the nodes of the corresponding tree $\mathcal{T}_1(f)$ are shown. The right half of the figure shows how every $\mathcal{T}_\ell(f)$ is obtained from the previous $\mathcal{T}_{\ell-1}(f)$ by adding $\mathcal{L}_\ell(f)$. The top two "sheets" show the tree $\mathcal{T}_1(f)$ (in solid links) and the difference set $\mathcal{L}_2(f)$ (with added dotted links); in the bottom sheet the tree $\mathcal{T}_2(f)$ is in solid links, and the difference set $\mathcal{L}_3(f)$, needed to make up the next tree, is in dotted links. The text explains how the positions of the nodes in each $\mathcal{L}_\ell(f)$ are encoded.

We next discuss the bitstreams $P_k(f)$ which identify the positions of the cubes appearing in $\mathcal{L}_k(f)$. Here essential use will be made of the tree structure to guarantee efficient encoding of these positions.

There is a *natural ordering* of dyadic cubes that will always be referred to below. Each dyadic cube is of the form $2^{-j}(m + [0, 1]^d)$, with $j \in \mathbb{Z}$ and $m \in \mathbb{Z}^d$. The ordering of these cubes is determined by the *lexicographical* ordering of the corresponding $(d + 1)$-tuples (j, k).

We shall use the following general result about encoding trees and their growth. According to the present context a tree is always understood to be a 2^d-tree; i.e., at every node 2^d children are available as possible modes of the tree at the next level down.

LEMMA 6.1. (i) *Given any finite tree $\mathcal{T}$, its positions can be encoded, in their natural order, with a bitstream P consisting of at most $m_0(1 + \#(\mathcal{T} \cap \mathcal{D}_0(\Omega))) + 2^d\#(\mathcal{T})$ bits where $m_0 := \lceil \log M_0 \rceil$ and $M_0 := \#\mathcal{D}_0(\Omega) + 1$.*

(ii) *If $\mathcal{T}$ contains the smaller tree $\mathcal{T}'$ and the positions of $\mathcal{T}'$ are known then the positions of $\mathcal{T} \setminus \mathcal{T}'$ can be encoded, in their natural order, with a bitstream P consisting of at most $m_0(1 + \#((\mathcal{T} \setminus \mathcal{T}') \cap \mathcal{D}_0(\Omega))) + 2^d(\#(\mathcal{T} \setminus \mathcal{T}'))$ bits.*

(iii) *In case* (i) *or* (ii), *if the bitstream P is embedded in a larger bitstream, then whenever the position of the first bit of P is known, we can identify the termination of P (i.e., the position of the last bit of P).*

Proof. We take the natural ordering of the cubes in $\mathcal{D}_0(\Omega)$. The number M_0 of these cubes is part of the codebook known to the decoder. We can identify each cube $I \in \mathcal{D}_0(\Omega)$ with a bitstream consisting of m_0 bits. We do not use the bitstream consisting of all zeros in this identification because this will be used to indicate the termination of this bitstream. The first bits of our encoding will identify the cubes in $\mathcal{D}_0(\Omega) \cap \mathcal{T}$ as follows. If all cubes from $\mathcal{D}_0(\Omega)$ are in $\mathcal{T}$ then we send the bitstream consisting of m_0 zeros terminating the encoding of cubes in $\mathcal{D}_0(\Omega)\cap\mathcal{T}$. Otherwise, we send the bitstreams associated with each of the cubes $I\in\mathcal{T}\cap\mathcal{D}_0(\Omega)$, in their natural order, and terminate with the bitstream consisting of m_0 zeros.

We next identify the cubes in $\mathcal{T}\cap\mathcal{D}_1(\Omega)$. Each such cube is a child of a cube from $\mathcal{T}\cap\mathcal{D}_0(\Omega)$. If $I\in\mathcal{T}\cap\mathcal{D}_0(\Omega)$, then to each of its children we assign a zero if the child is not in $\mathcal{T}$ and a one if the child is in $\mathcal{T}$. We arrange these bits according to the natural ordering of $\mathcal{T}\cap\mathcal{D}_0(\Omega)$ and then according to the natural ordering of the children. This bitstream will use $2^d\#(\mathcal{T}_0\cap\mathcal{D}_0(\Omega))$ bits and will identify all cubes in $\mathcal{T}\cap\mathcal{D}_1(\Omega)$. We can repeat this process to identify all of the cubes in $\mathcal{T}\cap\mathcal{D}_2(\Omega)$ by using $2^d\#(\mathcal{T}\cap\mathcal{D}_1(\Omega))$ bits. If we continue in this way we shall eventually encode all cubes in $\mathcal{T}$ and arrive at (i).

The proof of (ii) is almost identical to (i).

Note also that the encoding will terminate with a sequence of 2^d zeros which will also serve to identify the completion of the encoding. Thus, property (iii) also valid. ■

The position bitstreams $P_k(f)$, $k\geq 0$. These bitstreams are given by Lemma 6.1 and identify the positions of the cubes in $\mathcal{L}_k(f)$ for $k\geq 0$ with $\mathcal{T}_{-1}(f):=\emptyset$. Notice that some of these bitstreams may be empty. This occurs when $\kappa>0$. Recall that the value of κ is identified by the lead bits $L(f)$ and so is known. Let $\kappa_0:=\max(\kappa,0)$. From the lemma, we know that each of the $P_k(f)$, $k\geq\kappa_0$, consist of at most $m_0(1+\#((\mathcal{L}_k(f)))\cap\mathcal{D}_0(\Omega))+2^d(\#(\mathcal{L}_k(f)))$ bits.

We next describe the encoding of the signs of the wavelet coefficients. We take the natural ordering of the set V' of vertices of the unit cube. This in turn induces an ordering of the set V of nonzero vertices.

The sign bitstreams $S_k(f)$, $k\geq 0$. These bitstreams give the signs of the wavelet coefficients. Let $k\geq 0$ and let $I\in\mathcal{L}_k(f)$. If $v\in V$ ($v\in V'$ in the case $I\in\mathcal{D}_0(\Omega)$), we assign the coefficient $a^v_{I,p}(f)$ the bit zero if this coefficient is nonnegative and one if this coefficient is negative. The bitstream $S_\ell(f)$ is this sequence of zeros and ones ordered according to the natural ordering of the cubes $I\in\mathcal{L}_k(f)$ and subsequently the natural ordering of the vertices. Each of the $S_k(f)$, $k\geq 0$, consists of $2^d(\#(\mathcal{L}_k(f)))$ bits.

We next discuss how we encode coefficients. Any real number a has a binary representation

$$\sum_{\ell=-\infty}^{\infty} b_\ell(a)2^{-\ell}$$

with each $b_\ell(a)\in\{0,1\}$. In the case in which a has two representations (i.e., a is a binary rational) we choose the representation with a finite number of ones. First consider the encoding of the coefficients in $\mathcal{T}_0(f)$ which is a little different from the general case of encoding the coefficients in $\mathcal{L}_k(f)$, $k\geq 1$. Recall the integer κ given in the lead bits $L(f)$. $\mathcal{T}_0(f)$ will be nonempty if $\kappa\leq 0$. We know that each coefficient $a=a^v_{I,p}(f)$, $I\in\mathcal{T}_0(f)$, $v\in V'$, satisfies $b_\ell(a)=0$, $\ell\leq\kappa$.

The coefficient bitstreams $B_{0,0}(f)$. In compliance with the natural ordering of the cubes $I\in\mathcal{T}_0(f)$ given by $P_0(f)$, and in compliance with the natural ordering of V', we send the bits $b_\ell(a^v_{I,p}(f))$, $\ell=\kappa,\dots,0$, $I\in\mathcal{T}_0(f)$, $v\in V$ ($v\in V'$ in the case $I\in\mathcal{D}_0(\Omega)$). This bitstream will consist of at most $2^d(|\kappa|+1)\#\mathcal{T}_0(f)$ bits.

We now describe the bitstreams $B_{k,n-k}$, $k = 0, \ldots, n$ for $n \geq 1$. A coefficient $a^v_{I,p}(f)$ corresponding to $I \in \mathcal{L}_n(f)$ and $v \in V$ ($v \in V'$ if $I \in \mathcal{D}_0(\Omega)$) satisfies $|a^v_{I,p}(f)| < 2^{-n+1}$. Thus, $b_\ell(a^v_{I,p}(f)) = 0$, $\ell < n$. Hence sending a single bit $b_n(a^v_{I,p}(f))$ for $I \in \mathcal{L}_n(f)$ reduces the quantization error to 2^{-n} for those cubes. In addition the accuracy of the coefficients $a^v_{I,p}(f)$ for $I \in \mathcal{L}_k(f)$, $k < n$, has to be updated to the level 2^{-n}. By induction this requires a single additional bit $b_n(a^v_{I,p}(f))$ for each such coefficient. Therefore for each $0 \leq k \leq n$ the bitstream $B_{k,n-k}(f)$ consists of the bits $b_n(a^v_{I,p}(f))$ for $I \in \mathcal{L}_k(f)$ ordered according to $P_k(f)$ and the natural ordering of V (respectively V').

The coefficient bitstream $B_{k,\ell}(f)$, $\ell, k \geq 0$, $\ell + k > 0$. In compliance with the natural ordering of the cubes $I \in \mathcal{L}_k(f)$ given by $P_k(f)$, and in compliance with the natural ordering of V ($v \in V'$ in the case $I \in \mathcal{D}_0(\Omega)$), we send the bits $b_{\ell+k}(a^v_I(f))$. This bitstream will consist of at most $2^d(\#(\mathcal{L}_k(f)))$ bits.

This completes the description of the encoder E. For each $N \geq 0$, we define the encoder E_N which assigns to $f \in L_p(\Omega)$ the first portion of the bitstream for $E(f)$:

$$L(f), \ldots, P_N(f), S_N(f), B_{0,N}(f), \ldots, B_{N,0}(f). \tag{6.2}$$

In the special case when the number κ encoded by $L(f)$ exceeds N, the following rule will apply. When successively reading in the bits in $L(f)$, the encoder realizes that $\kappa > N$ when the first bit is a zero and a one appears at position $N + 2$. In this case the encoding terminates and the bitstream $E_N(f)$ consists of one zero followed by $N + 1$ ones. We further define $\mathcal{B}_N$ to be the set of all bitstreams $E_N(f)$, $f \in L_n p$. While each bitstream $E_N(f)$ is finite, the collection $\mathcal{B}_N$ is infinite. However, when we restrict f to come from a compact set U, we will obtain a finite set $\mathcal{B}_N(U)$.

The Decoder

Let us now describe the decoder D_N associated to E_N. Let B be any bitstream from $\mathcal{B}_N$.

Decoding κ

If the first two bits in $L(f)$ are zero then we know that $\kappa = 0$. Otherwise, the first bit of B is zero or one which identifies the sign of the number κ. Next comes a sequence of ones followed by a zero if $\kappa \leq N$. In this case the number of ones determine κ (i.e., $|\kappa|$ is equal to this number of ones). If the first bit is zero and the $N + 2$nd bit is one the decoder knows that $\kappa > N$ and hence that all wavelet coefficients have absolute value below 2^{-N}. The decoder then assigns the approximant 0.

Decoding the Lead Tree

Recall that $\kappa_0 := \max(\kappa, 0)$. Next comes a sequence of zeros and ones which identifies the cubes I in $\mathcal{T}_{\kappa_0}(B)$. Recall from Lemma 6.1 that we know when this sequence terminates. Next comes for each $I \in \mathcal{T}_{\kappa_0}(B)$, in their natural order, a sequence of zeros and ones which gives for each $I \in \mathcal{T}_{\kappa_0}(B)$, $v \in V$ ($v \in V'$, in case $I \in \mathcal{D}_0(\Omega)$) bits $b(\ell, I, v, B)$, where $\ell \leq \kappa_0$ if $\kappa_0 = 0$ and $\ell = \kappa_0$ if $\kappa_0 > 0$.

Progressive Reconstruction of the Trees $\mathcal{T}_k(B)$

Each new subsequent bitstream identifies the new cubes in $\mathcal{L}_k(B)$, sends one additional bit $b_k(a^v_{I,p}(f))$ for each of the old cubes $I \in \mathcal{L}_\ell(B)$, $v \in V$ ($v \in V'$ if $I \in \mathcal{D}_0(\Omega)$), $\ell < k$, and one bit $b_k(a^v_{I,p}(f))$ for the new cubes $I \in \mathcal{L}_k(B)$. In totality, the bitstream B determines a nested sequence of trees $\mathcal{T}_k(B)$, $k = 0, \dots, N$, and for each $I \in \mathcal{T}_N$, $v \in V$ ($v \in V'$, in case $I \in \mathcal{D}_0(\Omega)$), and $\ell \le N$, a number $b(\ell, I, v, B) \in \{0, 1\}$. By definition, these numbers are zero if $\ell < k$ in the case $I \in \mathcal{L}_k(B)$, $k = 1, 2, \dots, N$ ($\ell \le \kappa_0$ in the case $k = 0$). The decoder D_N uses this information to construct an element $S_N(B)$ from $L_p(\Omega)$ as follows.

For each $I \in \mathcal{T}_N(B)$, and each $v \in V$ ($v \in V'$ in case $I \in \mathcal{D}_0(\Omega)$), we define

$$a^v_{I,p,N}(B) := \sum_{\ell \le N} b(\ell, I, v, B) 2^{-\ell} \tag{6.3}$$

and

$$A^N_I(B) := \begin{cases} \sum_{v \in V'} a^v_{I,p,N}(B) \psi^v_{I,p}, & I \in \mathcal{D}_0, \\ \sum_{v \in V} a^v_{I,p,N}(B) \psi^v_{I,p}, & I \in \mathcal{D}_j, \ j \ge 1. \end{cases} \tag{6.4}$$

It follows that in the case $B = E(f)$ one has

$$|a^v_{I,p}(f) - a^v_{I,p,N}(E(f))| \le 2^{-N}. \tag{6.5}$$

We define

$$\Delta^N_k(B) := \sum_{I \in \mathcal{L}_k(B)} A^N_I(B) \tag{6.6}$$

and

$$\tilde{S}_N(B) := \sum_{k=0}^{N} \Delta^N_k(B). \tag{6.7}$$

The decoder D_N maps B into $\tilde{S}_N(B) \in L_p(\Omega)$.

7. PERFORMANCE OF THE ENCODERS E_N ON COMPACT SETS $K \subset L_p(\Omega)$, $1 \le p < \infty$

We next examine the distortion of the encoding E_N, D_N on compact sets which are unit balls of Besov spaces.

THEOREM 7.1. *Let $1 \le p < \infty$, and let $0 < \lambda < p$. If $U := U(\mathcal{B}_\lambda(L_p(\Omega)))$, we have*

$$R(U, E_N, D_N) \le c_4 2^{\lambda N} \tag{7.1}$$

and

$$\mathbf{D}(U, E_N, D_N) \le c_5 2^{-N(1-\lambda/p)}, \tag{7.2}$$

with the constants c_4, c_5 depending only on p and $p - \lambda$.

Moreover, for $0 < q \le \infty$ and $(1/\tau, s)$ above the critical line for nonlinear approximation in L_p, i.e., $\delta := s - d/\tau + d/p > 0$, the same estimate holds for $U := U(B^s_q(L_\tau(\Omega)))$, *with* $\lambda := d/(s + d/p)$ *and the constants* c_4, c_5 *depending only on* p, τ, *and the discrepancy* δ.

Proof. Let $f \in U := U(\mathcal{B}_\lambda(L_p(\Omega)))$. Then, f has all wavelet coefficients ≤ 1 in absolute value. Hence $\kappa \geq 0$. As noted earlier if $\kappa > N$, then $E_N(f)$ consists of $N + 2$ bits. So the number of bits n_L in $L(f)$ satisfies

$$n_L \leq N + 2. \tag{7.3}$$

If $f \in U$, we know from (4.7) that $\#(\mathcal{T}_N(f)) = \#(\mathcal{T}(f, 2^{-N})) \leq 2^{\lambda N}$. This means that the number n_P of bits in all of the bitstreams $P_k(f)$, $k = 0, \ldots, N$, will satisfy

$$n_P \leq m_0 \#(\mathcal{D}_0(\Omega) + 1) + 2^d \# \mathcal{T}_N(f) \leq m_0 \#(\mathcal{D}_0(\Omega) + 1) + 2^d 2^{\lambda N}. \tag{7.4}$$

The total number of bits n_S appearing in the S_k, $k = 0, \ldots, N$, satisfies

$$n_S \leq 2^d \# \mathcal{T}_N(f) \leq 2^d 2^{\lambda N} \tag{7.5}$$

because there are at most 2^d coefficients associated with each $I \in \mathcal{T}_N(f)$. For each coefficient $a^v_{I,p}(f)$, $I \in \mathcal{L}_k(f)$, $k = 0, \ldots, N$, we will send at most $(N - k + 1)$ bits. Hence, using the estimate $\#(\mathcal{T}_k(f)) \leq 2^{k\lambda}$ ensured by (4.7), we find that the total number of bits n_B in all of the sequences $B_{j,k}$, $0 \leq j, k \leq N$ will satisfy

$$n_B \leq \sum_{k=0}^{N} (N + 1 - k) 2^d 2^{k\lambda} \leq C 2^{\lambda N}, \tag{7.6}$$

with C depending only on λ and d. Hence the total number of bits used in $E_N(f)$ does not exceed

$$n_L + n_P + n_S + n_B \leq C(N + 2^{N\lambda}) \leq c_4 2^{N\lambda}. \tag{7.7}$$

This completes the proof of (7.1).

Let $S_N(f) := S(f, 2^{-N})$ be the function of (4.6). We have shown in Theorem 4.1 (see (4.8)) that

$$\|f - S_N(f)\|_{L_p(\Omega)} \leq c_1 2^{-N(1-\lambda/p)}. \tag{7.8}$$

On the other hand, from Temlyakov's inequality (3.10) and (6.5), we have

$$\|S_N(f) - \tilde{S}_N(E(f))\|_{L_p(\Omega)} \leq C 2^{-N} (\# \mathcal{T}_N(f))^{1/p} \leq C 2^{-N} 2^{\lambda N/p}. \tag{7.9}$$

Therefore,

$$\|f - \tilde{S}_N(E(f))\|_{L_p(\Omega)} \leq c_5 2^{-N(1-\lambda/p)},$$

which proves (7.2).

In the case where $U := U(B^s_q(L_\tau(\Omega)))$, similar estimates are obtained by using (4.9) to majorize $\#(\mathcal{T}_N(f))$ and $\|f - S_N(f)\|_{L_p(\Omega)}$ as above (see Corollary 4.2). ■

As a corollary of Theorem 7.1, we obtain upper estimates for the Kolmogorov entropy of the balls U.

COROLLARY 7.2. *Let* $1 \le p < \infty$, $0 < q \le \infty$ *and let the point* $(1/\tau, s)$ *be above the critical line for nonlinear approximation in* L_p. *If* $U := U(B_q^s(L_\tau(\Omega)))$, *then we have*

$$H_\epsilon(U) \le c_6 \epsilon^{-d/s}, \qquad \epsilon > 0, \tag{7.10}$$

with the constant c_6 *depending only on* p, τ, *and the discrepancy* $\delta := s - d/\mu + d/p$ *(see* (3.11)*).*

Proof. Let N be the smallest integer such that $c_5 2^{-N\lambda s/d} < \epsilon$ with c_5 the constant in Theorem 7.1. We have shown in Theorem 7.1 that the encoding pair E_N, D_N has distortion $\le c_5 2^{-N\lambda s/d} < \epsilon$ and $R(U, E_N, D_N) \le c_4 2^{\lambda N} \le c_6 \epsilon^{-d/s}$. Hence the corollary follows from (1.18). ■

Similarly, Theorem 7.1 can be rephrased as giving distortion-rate bounds for the encoder–decoder family (E_N, D_N) applied to functions f in U. Given a bitrate R, find the largest integer N such that $R(E_n, D_N) \le R$; clearly $R \le C_4 2^{\lambda(N+1)}$. Then for $f \in U(\mathcal{B}_\lambda(L_p(\Omega)))$ the corresponding distortion $\mathbf{D} := d(U, E_N, D_N)$ will be bounded by

$$\mathbf{D} \le c_5 2^{-N(1/\lambda/p)} \le c_7 R^{-1/\lambda - 1/p};$$

for $f \in U(B_q^s(L_\tau(\Omega)))$, with $(1/\tau, s)$ above the critical line, we have

$$\mathbf{D} \le c_8 R^{-s/d}.$$

8. DISCUSSION AND CONCLUSION

We have proposed here an encoding strategy based on tree approximation. Our inspiration came from practical encoders like EZW [22] and later refinements such as SPIHT [21]. In these algorithms, the collection of possible coefficient labels is reduced by the elimination of zero-trees, each of which is, in our terminology, given by a dyadic cube and its descendants. Removing such a collection of zero-trees from $\mathcal{D}$ leaves a remainder that is exactly one of our trees $\mathcal{T}$. In particular, if one eliminates those subtrees for which all coefficients of f are below some threshold η (corresponding to "zero-trees" if one considers bitplanes above η), then the remaining tree to be encoded is exactly $\mathcal{T}(f, \eta)$.

In our encoding algorithm, we further separated the coefficients into different bit planes, corresponding to the successive thresholds $\eta_\ell = 2^{-\ell}$, $\ell = \kappa, \kappa + 1, \ldots$. For every ℓ, the information sent up to that point will permit one to reconstruct $\mathcal{T}(f, \eta_\ell)$ and all the corresponding coefficients with precision η_ℓ. This information is sent recursively: given $\mathcal{T}(f, \eta_{\ell-1})$, and the $\ell - 1 - \kappa$ first bits of the corresponding coefficients, one needs to communicate only $\mathcal{L}_\ell(f) = \mathcal{T}(f, \eta_\ell) \backslash \mathcal{T}(f, \eta_{\ell-1})$ as well as one bit for each coefficient in $\mathcal{T}(f, \eta_\ell)$. This separation results in an adaptive quantization of the coefficients in the sense that only large coefficients (which appear rarely) are encoded with many bits, whereas the (much more frequently occurring) small coefficients consume only few bits per coefficient. (Note that this bit allocation is a "rounded-off" version of what an entropy coder would provide for a Laplacian distribution of the coefficients.)

Its interesting to note that by organizing the tree-based data structure appropriately, one can further adapt the encoding to so-called "burn-in" applications [14]. These are situations in which the complete encoded information on a server may be huge (of the order of petabytes, representing, e.g., detailed terrain information for the whole earth). As the client receives coarse-scale information from the progressive data-stream, he or she sends information back to the server, conveying that he or she is interested only in a subregion; this would typically be done repeatedly, zooming in more and more. At every such "burn-in," the server utilizes the associated data structures to determine bit offsets and applies them, either through bit shifts for data in memory or through seeks for data in storage, to effectively mask the data, so that it automatically sends only the information needed by the client, still in a progressive bitstream. The interesting aspect is that if the data structure is constructed appropriately, the server does not have to run through the complete structure, but automatically skips over the huge swaths of data not required by the client. More concretely, if the client requires only $O(M)$ data, with $M \ll N$, where N is the size of the huge data set on the server, then the server needs only $O(M)$ computations and look-ups to send the appropriately pruned data structure to the client, in real time, as the client makes his or her wishes known. For details, see [14].

Our encoding algorithm also results in asymptotical optimal performance when the signal is modeled by the unit Besov balls $U(B^s_q(L_\tau(\Omega)))$ with $1/\tau < s + 1/p$, in the sense that according to the Kolmogorov entropies, one cannot do better. As pointed out to us by one of the reviewers, other addressing strategies, such as runlength coding of appropriate sets of coefficient labels to be retained at each dyadic scale, also result in such optimal performance for these Besov balls. The resulting encoder would still use the bit plane separation described above, but would not rely on tree structures. (We sketch technical details of this argument in Appendix B.) The existence of such non-tree-structured optimal encoding strategies for L_p-approximation of $U(B^s_q(L_\tau))$ with $1/\tau < s + 1/p$ is consistent with the fact that both L_p and $B^s_q(L_\tau)$ have characterizations, in terms of wavelet coefficients, that are invariant under all permutations within each dyadic scale. These permutations need not, of course, respect any tree structure.

There are nevertheless several reasons to stick to trees. First of all, at a very intuitive level, tree structures seem natural for images, especially when we view them as elaborations on piecewise smooth functions, for which the significant wavelet coefficients are naturally organized in trees. The spaces $\mathcal{B}_\lambda(L_p)$ defined in Section 4 capture the importance of such tree-based organization. In contrast to Besov spaces, they are not invariant under permutations of wavelet coefficients within each scale, and runlength based algorithms do not provide, for these spaces, the optimal performance exhibited by our algorithm. One can expect that describing images as elements of appropriate $\mathcal{B}_\lambda(L_p)$-spaces would therefore lead to better distortion-rate bounds than a Besov description. For instance, two-dimensional piecewise smooth functions lie in the space $\mathcal{B}_1(L_2)$, but not in any $B^1_q(L_\tau)$ with $\tau > 1$. Modeling these functions as elements of $\mathcal{B}_1(L_2)$ leads to a distortion-rate bound proportional to $R^{-1/2}$; this cannot be attained by considering them as elements of the spaces $B^1_q(L_\tau)$ with $\tau \leq 1$.

On the other hand, because of the equivalence between eliminating zero-trees as in EZW and defining a tree in the sense of this article, the family of spaces $\mathcal{B}_\lambda(L_p)$, for which our encoding scheme is optimal, is also the natural mathematical setting for algorithms of EZW type. This observation points toward a shortcoming of such algorithms, even though they represent the state-of-the-art at present. The space $\mathcal{B}_1(L_2)$ is still much too big to provide a model for, e.g., the class of piecewise smooth functions. In particular, random changes of the signs of the wavelet coefficients of an image do not affect its norm in $\mathcal{B}_1(L_2)$ or the rate-distortion bound we derived, whereas such changes completely destroy the edge structure and the "naturalness" of an image [19]. It is also known that the class of piecewise smooth two-dimensional functions, with jump discontinuities located on a bounded number of C^1 curves, can be approximated by piecewise linear splines on adapted nonisotropic triangulations with greater accuracy (for the same number of terms) than with wavelets. If a precision of $O(2^{-m})$ is desired, then one can, for instance, bracket each discontinuity curve by two polygonal curves less than 2^{-2m} apart, each containing $O(2^m)$ points, thus dividing the domain into several component sets with polygonal boundaries, separated by very narrow corridors. The L^2-error on the narrow corridors clearly is $O(2^{-m})$. Because the function is C^2 in each component set, classical Kolmogorov entropy bounds ensure that each component function can be encoded to accuracy 2^{-m} using at most $O(2^m)$ bits, for instance, by using splines on an adapted triangulation. Such a spline approximation thus achieves a distortion-rate bound of $O(R^{-1})$ for this small class of functions; this R^{-1} decay can be proved to be optimal. This optimal decay for the distortion-rate bound can also be achieved with other decomposition methods, such as curvelets [4]. This indicates that there is still a lot of room for improvement in the encoding of images, which necessarily will require other types of algorithms, corresponding to better, more narrow function classes than $\mathcal{B}_1(L_2)$.

APPENDIX

A. The Case $p = \infty$

In several sections, the arguments presented are valid only for $p < \infty$. One can adapt them, often at the price of greater technicality, for $p = \infty$. We provide here the details for this adaptation. We shall subdivide this appendix into subsections that refer to the corresponding sections in the main body of the article.

A1. Tree Approximation

The constants in the estimates (4.10) and (4.11) in the proof of Theorem 4.1 depend on p as p tends to infinity (because the inequality (3.10) in Theorem 3.2 was used) and on $p - \lambda$ when λ tends to p. Consequently the constants c_1, c_2 deteriorate as $p - \lambda$ (resp. $\delta = s - d/\tau + d/p$) tends to zero and also when p increases to $+\infty$. In fact, Theorem 4.1 and Corollary 4.2 do not hold for $p = \infty$. To tackle the $p = \infty$ case, we have to overcome several technical difficulties. We have to compensate for the fact that we can no longer resort to Temlyakov's inequality (3.10). The main idea is to change the definition of $\Lambda(f, \eta)$ by introducing a dependence of the threshold on the level of scale. Moreover, recall from (4.13) that complete coarse scales up to a certain turnover level J can be included without spoiling the complexity of the trees. The reader who is not interested in these details should skip the rest of this section.

For f a function in $L_\infty(\Omega)$ and $\eta > 0$, we define, in analogy with (4.14), J to be the smallest positive integer such that

$$2^{Jd} \geq \sum_{j>J} \sum_{l \geq j} \#(\Lambda_l(f, \eta(l - J)^{-2})). \tag{A.1}$$

Whenever such a J exists, we define a modified set $\tilde{\Lambda}(f,\eta)$ by

$$\tilde{\Lambda}(f,\eta) := \{I \in \mathcal{D}_j(\Omega);\ j \le J\} \cup \{I \in \Lambda_j(f, \eta(j-J)^{-2});\ j > J\}. \tag{A.2}$$

If there is no J for which (A.1) holds, we simply set $\tilde{\Lambda}(f,\eta) := \mathcal{D}(\Omega)$. $\tilde{\Lambda}(f,\eta)$ thus includes all cubes with scales $j \le J$, as well as those with scales j larger than J for which $|a_{I,\infty}(f)| \ge \eta(j-J)^{-2}$. As before we define $\tilde{\mathcal{T}}(f,\eta)$ as the smallest tree containing $\tilde{\Lambda}(f,\eta)$ and set

$$\tilde{S}(f,\eta) := \sum_{I \in \tilde{\mathcal{T}}(f,\eta)} A_I(f). \tag{A.3}$$

This construction gives again rise to a class of functions in $L_\infty(\Omega)$ as follows: for $0 < \lambda < \infty$, we say that f is in $\tilde{\mathcal{B}}_\lambda(L_\infty(\Omega))$ if and only if there exists a constant $C(f)$ such that

$$\#(\tilde{\mathcal{T}}(f,\eta)) \le C(f)\eta^{-\lambda}, \tag{A.4}$$

for all $\eta > 0$. For the sake of convenience, we introduce a succinct notation for the r.h.s. of (A.1) in the case where $\tilde{\mathcal{T}}(f,\eta)$ is finite:

$$\Theta(f,\eta,J) := \sum_{j>J} \sum_{l \ge j} \#(\Lambda_l(f, \eta(l-J)^{-2})). \tag{A.5}$$

We will make use of the following simple facts about the quantities $\Theta(f,\eta,J)$.

Remark A.1. If (A.4) holds then for fixed f and η, $\Theta(f,\eta,J)$ decreases as J increases. Moreover, for fixed f and J the quantities $\Theta(f,\eta,J)$ increase as η decreases.

The first statement in Remark A.1 says that (A.4) already implies the existence of a turnover scale J satisfying (A.1) and hence the finiteness of $\tilde{\mathcal{T}}(f,\eta)$ so that $\tilde{S}(f,\eta)$ is well defined. Moreover, (3.11) together with (2.12) implies a geometric decay of the quantities $a_{I,\infty}(f)$ in scale as soon as f belongs to any space $B^s_q(L_\tau(\Omega))$ above the critical line; i.e., $1/\tau < s/d$. Therefore for any $\eta > 0$ the set $\Lambda_l(f, \eta(l-J)^{-2})$ will be empty for l sufficiently large, so that (A.4) will indeed hold anywhere above the critical line for approximation in $L_\infty(\Omega)$.

The second statement in Remark A.1 implies that the turnover scale J increases when η decreases.

We can now formulate the following adaptation of Theorem 4.1.

THEOREM A.2. *For all $\eta > 0$,*

$$\|f - \tilde{S}(f,\eta)\|_{L_\infty(\Omega)} \le \tilde{c}_1 \eta, \tag{A.6}$$

where $\tilde{c}_1$ depends only on the support of φ and ψ. Moreover, let $(1/\tau, s)$ be a point above the critical line for nonlinear approximation in $L_\infty(\Omega)$; i.e., s and τ should satisfy $s > d/\tau$. Then if $0 < q \le \infty$, $B^s_q(L_\tau(\Omega))$ is embedded in $\mathcal{B}_\lambda(L_\infty(\Omega))$ with $\lambda := d/s$, in the sense that any f in $B^s_q(L_\tau(\Omega))$ satisfies (A.4) *with*

$$C(f) \le \tilde{c}_2 \|f\|^\lambda_{B^s_q(L_\tau(\Omega))}, \tag{A.7}$$

where $\tilde{c}_2$ depends only on $s - d/\tau$ when this quantity becomes close to zero.

Proof. The error estimate (A.6) is immediate since we have

$$\begin{aligned}\|f - \tilde{S}(f,\eta)\|_{L_\infty(\Omega)} &\le \Big\| \sum_{I \notin \tilde{\Lambda}(f,\eta)} A_I(f) \Big\|_{L_\infty(\Omega)} \\ &\le \sum_{j>J} \Big\| \sum_{I \in \mathcal{D}_j(\Omega) \setminus \Lambda_\ell(f,\eta(j-J)^{-2})} A_I(f) \Big\|_{L_\infty(\Omega)} \\ &\le \eta \sum_{j>J} (j-J)^{-2} \Big\| \sum_{I \in \mathcal{D}_j(\Omega)} \psi_{I,\infty} \Big\|_{L_\infty(\Omega)} = \tilde{c}_1 \eta.\end{aligned}$$

In order to prove (A.7), we follow the same reasoning as in the proof of Theorem 4.1 (see (4.13), (4.15)) and remark first that

$$\#(\tilde{\mathcal{T}}(f,\eta)) \le 2^{Jd} + \sum_{j>J} \sum_{l \ge j} \#(\Lambda_l(f, \eta(l-J)^{-2})) \le 2^{Jd+1}. \tag{A.8}$$

On the other hand, for $f \in B_q^s(L_\tau(\Omega))$, with $s > d/\tau$, let $\delta = s - d/\tau$ and $\tilde{M} = \|f\|_{B_q^s(L_\tau(\Omega))}$. We then have by the definition of J in (A.1) and by (4.12)

$$\begin{aligned}2^{(J-1)d} &\le \sum_{j \ge J} \sum_{l \ge j} \#(\Lambda_l(f, \eta(l-J+1)^{-2})) \\ &\le \sum_{j \ge J} \sum_{l \ge j} 2^{-l\delta\tau} \tilde{M}^\tau \eta^{-\tau} (l-J+1)^{2\tau} \\ &\le C \tilde{M}^\tau \eta^{-\tau} \sum_{j \ge J} (j-J+1)^{2\tau} 2^{-j\delta\tau} \\ &\le C \tilde{M}^\tau \eta^{-\tau} 2^{-J\delta\tau},\end{aligned}$$

so that we obtain

$$2^{J(d+\delta\tau)} \le C \tilde{M}^\tau \eta^{-\tau}. \tag{A.9}$$

Combining this last estimate with (A.8), we deduce

$$\#(\tilde{\mathcal{T}}(f,\eta)) \le C[\tilde{M}^\tau \eta^{-\tau}]^{d/(d+\delta\tau)} = \tilde{c}_2 \tilde{M}^\lambda \eta^{-\lambda}, \tag{A.10}$$

which concludes the proof. ■

A2. A Tree-Based Wavelet Decomposition

In the case $p = \infty$, we modify the definition of the Δ_k as follows. We wish to employ the modified trees $\tilde{\mathcal{T}}(f,\eta)$ appearing in the definition (A.3) of $\tilde{S}(f,\eta)$. However, since we can no longer guarantee that the trees $\tilde{\mathcal{T}}(f, 2^{-k})$ are nested we note that the union of trees is a tree and set

$$\mathcal{T}_k(f) := \bigcup_{0 \le j \le k} \tilde{\mathcal{T}}(f, 2^{-j}). \tag{A.11}$$

In this case $\mathcal{L}_k(f)$ in (5.3) takes the form

$$\mathcal{L}_k(f) = \mathcal{T}_k(f)\backslash\mathcal{T}_{k-1}(f) = \tilde{\mathcal{T}}(f, 2^{-k})\Bigg\backslash\Bigg(\bigcup_{0\le j<k} \tilde{\mathcal{T}}(f, 2^{-j})\Bigg).$$

The strong convergence of the partial sums $\sum_{0\le k\le K} \Delta_k(f)$ in (5.4), for f in any Besov space on the left of the critical line, is ensured by Theorem A.2.

A3. A Universal Encoding–Decoding Pair and Its Performance

We next describe how to modify the above encoders so as to obtain Theorem 7.1 and Corollary 7.2 also for $p = \infty$.

To this end, we simply use the modified trees $\mathcal{T}_k(f)$ from (A.11). The bitstreams $L(f)$, $P_k(f)$, $S_k(f)$ are then defined in the same way as described in Section 6. The only further modification concerns the bitstreams $B_{k,j}(f)$. The reason is that, since for $p = \infty$ Temlyakov's inequality (3.10) is no longer applicable, a somewhat higher accuracy for the quantization is needed for the estimation of the quantization error. In fact, the main obstruction caused by the L_∞-norm is that locally wavelets from many levels may overlap. Therefore we will exploit the decay of wavelet coefficients required by (A.2). To this end, recall the turnover level J_k for the tree $\mathcal{T}_k(f)$ defined by (A.1). Now from the definition of $\tilde{\mathcal{T}}(f, \eta)$ we know that $I \in \mathcal{D}_j(\Omega) \cap \mathcal{L}_k(f)$ implies that $j > J_{k-1}$ and that

$$|a^v_{I,\infty}(f)| \le 2^{-k+1-\ell(j-J_{k-1})}, \tag{A.12}$$

where ℓ is the function

$$\ell(s) := \lfloor 2\log_2(s)\rfloor.$$

Modified $B_{k,n-k}(f)$. In compliance with the natural ordering of the cubes $I \in \mathcal{L}_k(f)$ and in compliance with the natural ordering of the $v \in V$ ($v \in V'$ if $I \in \mathcal{D}_0(\Omega)$) we send for each $I \in \mathcal{L}_k(f) \cap \mathcal{D}_j(\Omega)$, $k = 1, \dots, n$, $\ell = J_{k-1} + 1, \dots,$ the two bits $b_l(a^v_{I,\infty}(f))$, $l = \ell(j - J_{k-1}) + 2n - k + 1, \ell(j - J_{k-1}) + 2n - k$. Analogous modifications apply to $B_{0,0}(f)$.

Hence, when decoding, (6.3) is replaced for $I \in \mathcal{L}_k(f) \cap \mathcal{D}_j(\Omega)$ by

$$A^v_{I,\infty,N}(B) := \sum_{r\le\ell(j-J_{k-1})+2N-k+1} b(r, I, v, B)2^{-r}, \tag{A.13}$$

so that now for $k = 1, \dots, N$, $I \in \mathcal{L}_k(f) \cap \mathcal{D}_j(\Omega)$,

$$|a^v_{I,\infty}(f) - a^v_{I,\infty,N}(E(f))| \le 2^{-(\ell(j-J_{k-1})+2N-k)}. \tag{A.14}$$

The counterparts, for $p = \infty$, of the results in Section 7 then read as follows.

THEOREM A.3. *Let* $\lambda < \infty$. *If* $U := U(\mathcal{B}_\lambda(L_\infty(\Omega)))$, *we have*

$$R(U, E_N, D_N) \le c_7 2^{\lambda N} \tag{A.15}$$

and

$$d(U, E_N, D_N) \le c_5 2^{-N\lambda s/d}, \tag{A.16}$$

with $s := d/\lambda$ *and the constants* c_4, c_5 *depending only on* λ.

Moreover, for $0 < q \le \infty$ *and* $(1/\tau, s)$ *above the critical line for nonlinear approximation in* L_∞, *i.e.,* $\delta := s - d/\tau > 0$, *the same estimate holds for* $U := U(B_q^s(L_\tau(\Omega)))$, *with* $\lambda := d/s$ *and the constants* c_4, c_5 *depending only on* p, τ, *and the discrepancy* $\delta = s - d/\tau$. *Furthermore, we have*

$$H_\epsilon(U) \le c_6 \epsilon^{-d/s}, \qquad \epsilon > 0, \tag{A.17}$$

with the constant c_6 *depending only on* τ, *and the discrepancy* $\delta := s - d/\tau$.

Proof. We essentially follow the arguments in the proofs of Theorem 7.1 and Corollary 7.2. The estimates (7.4) and (7.5) for n_P and n_S remain the same. The estimate (7.6) is replaced now by

$$n_C \le C \sum_{k=0}^{N} 2(N-k+1)2^d 2^{\lambda k} \le C 2^{\lambda N}, \tag{A.18}$$

so that the total number of bits $n_L + n_P + n_S + n_C$ in the modified $E(f)$ still satisfies (7.7).

The approximation error (7.8) of the form

$$\|f - S_N(f)\|_{l_\infty(\Omega)} \le c_1 2^{-N} \tag{A.19}$$

follows now from Theorem A.2.

Only the estimation of the quantization error requires a different argument because, as mentioned above, Temlyakov's inequality (3.10) is no longer applicable. This will be compensated by the higher accuracy provided by the modification of the bitstreams $B_{k,j}(f)$ and the additional decay of wavelet coefficients required by (A.2) in the definition of the trees $\tilde{\mathcal{T}}(f, \eta)$.

We will use the fact that for each fixed level $j \ge 0$ only a uniformly bounded finite number of terms $A_I(f) - A_I^N(B)$, $I \in \mathcal{D}_j$, are simultaneously nonzero at any given point in Ω. As before let $B := E(f)$ and note that

$$\|S_N(f) - \tilde{S}_N(B)\|_{L_\infty(\Omega)} \le \sum_{I \in \mathcal{T}_0(f)} \|A_I(f) - A_I^N(B)\|_{L_\infty(\Omega)} + \sum_{k=1}^{N} \sum_{j=J_{k-1}+1}^{\infty} \Bigg\| \sum_{I \in \mathcal{L}_k(f) \cap \mathcal{D}_j(\Omega)} \big(A_I(f) - A_I^N(B)\big) \Bigg\|_{L_\infty(\Omega)}. \tag{A.20}$$

By (A.14) the first sum on the right hand side of (A.20) is clearly bounded by $C2^{-N}$. The second sum is, in view of (A.14), bounded by

$$C 2^{-N} \sum_{k=1}^{N} \sum_{j=J_{k-1}+1}^{\infty} 2^{-\ell(j-J_{k-1})-(N-k)} \le C 2^{-N},$$

which provides the desired counterpart to (7.9):

$$\|S_N(f) - \tilde{S}_N(B)\|_{L_\infty(\Omega)} \le C2^{-N}. \tag{A.21}$$

The rest of the proof is the same as before in Section 7. ■

B. A Runlength-Based Coding for Besov Balls Left of the Critical Line

We start by defining the collections of dyadic intervals,

$$\mathcal{C}_p(L) := \Lambda(f, 2^{-L}) = \{I \in \mathcal{D}; a_{I,p}(f) \ge 2^{-L}\}$$
$$\mathcal{C}_{j,p}(L) := \mathcal{C}_p(L) \cap \mathcal{D}_j.$$

We have $\mathcal{C}_p(L_0 - 1) = \emptyset$ for a fixed integer L_0.

The nonlinear approximation $S_L(f) = \sum_{I \in \mathcal{C}_p(L)} A_I(f)$ can be shown (by standard arguments, similar to those in Section 4; see also [12]) to satisfy

$$\|f - S_L(f)\|_{L_p} \le C2^{-Ls/(s+d/p)}, \tag{B.1}$$

where we have used $\|f\|_{B^s_q(L_\tau)} \le 1$. Moreover, we will not use the precise wavelet coefficients $a_{I,p}$ for $I \in \mathcal{C}_p(L)$ but approximate values $\tilde{a}_{I,p}$ such that

$$|a_{I,p} - \tilde{a}_{I,p}| \le 2^{-L}$$

for all $I \in \mathcal{C}_p(L)$. We then find, by combining (B.1) with Theorem 3.2, that $\tilde{S}_L(f) = \sum_{I \in \mathcal{C}_p(L)} \tilde{A}_I(f)$ (where $\tilde{A}_I(f)$ is constructed as in (2.5), but using the $\tilde{a}_{I,p}$ instead of the $a_{I,p}$) satisfies

$$\|f - \tilde{S}_l(f)\|_{L_p} \le C\{2^{-Ls/(s+d/p)} + 2^{-L}\{\#\mathcal{C}_p(L)\}^{1/p}\}. \tag{B.2}$$

This formula will be useful to upper bound the distortion, later in this appendix.

To compute the bitrate, we must find out how to encode the sets $\mathcal{C}_p(L)$ and the first L bits of every $a_{I,p}(f)$ with $I \in \mathcal{C}_p(L)$. We shall do this incrementally, as in the tree case. That is, we shall compute how much additional bitrate is needed to update the information in $[\mathcal{C}_p(m) \cup \{$first m bits of $a_{I,p}$ with $I \in \mathcal{C}_p(m)\}]$ to obtain $[\mathcal{C}_p(m+1) \cup \{$first $m+1$ bits of $a_{I,p}$ with $I \in \mathcal{C}_p(m+1)\}]$. It is clear that this update will consist of two parts:

- information enabling the reconstruction of $\mathcal{C}_p(m+1) \backslash \mathcal{C}_p(m)$,
- one extra bit for each $a_{I,p}$ with $I \in \mathcal{C}_p(m+1)$.

The bitrate for the second part is easy: this will cost us exactly $\#\mathcal{C}_p(m+1)$ bits. The first part is trickier. Let us look at it in each dyadic layer separately. We start by estimating $\#\Delta_{j,p}(m)$ where $\Delta_{j,p}(m) = \mathcal{C}_{j,p}(m+1) \backslash \mathcal{C}_{j,p}(m)$. Using the change in normalization (2.7), we have

$$\Delta_{j,p}(m) = \{I \in \mathcal{D}_j; 2^{-(m+1)-j(s-\delta)} \le a_{I,\tau} < 2^{-m-j(s-\delta)}\},$$

where $\delta = s + d/p - d/\tau$ (assuming $\tau > p$). Because $f \in B_q^s(L_\tau)$, we have

$$\sum_{j=0}^{\infty} 2^{jsq} \left[\sum_{m=0}^{\infty} \#(\Delta_{j,p}(m)) 2^{-\tau(m+j(s-\delta))} \right]^{q/\tau} < \infty,$$

implying, for all j and m,

$$\#\Delta_{j,p}(m) \leq C 2^{-\tau\delta j + \tau m}.$$

Since $\#\Delta_{j,p}(m) \leq 2^{jd}$ as well, we have

$$\#\Delta_{j,p}(m) \leq \min\bigl(2^{jd}, C 2^{-\tau\delta j + \tau m}\bigr) =: \lambda_{j,p}(m). \tag{B.3}$$

It follows that $\Delta_{j,p}(m)$ will be empty if j is larger than some $J(m)$ which can be bounded by $J(m) \leq C + m/\delta$.

We need to encode the subsets $\Delta_{j,p}(m)$ of $\mathcal{D}_j$, for $0 \leq j \leq J(m)$. One simple way is to list, for each j, all the elements of $\mathcal{D}_j$ in a fixed (e.g., lexicographic) order, labeling each with a 1 or a 0 depending on whether it is in $\Delta_{j,p}(m)$ (label 1) or not (label 0), and to do a runlength coding on this sequence of 2^{jd} bits. Let $\nu_{j,m,n}$, $1 \leq n \leq N_0$, be the lengths of consecutive maximal stretches containing only zeros and define $\nu^1_{j,m,n}$, $1 \leq n \leq N_1$, similarly for the ones. Then, $|N_0 - N_1| \leq 1$ and by (B.3), $N_0 \leq \lambda_{j,p}(m)$. Clearly $\sum_{n=1}^{N_0} \nu^0_{j,m,n} + \sum_{n=1}^{N_1} \nu^1_{j,m,n} = 2^{jd}$. A rough estimate for the number of bits needed to encode the $\nu^0_{j,m,n}$, $\nu^1_{j,m,n}$ (which together completely define $\Delta_{j,p}(m)$) is given by

$$\begin{aligned} \sum_{n=1}^{N_0} \log \nu^0_{j,m,n} + \sum_{n=1}^{N_1} \log \nu^1_{j,m,n} &\leq (N_0 + N_1)(jd - \log(N_0 + N_1)) \\ &\leq C\lambda_{j,p}(m)[(jd - \log(\lambda_{j,p}(m)) + C'], \end{aligned}$$

where the first inequality follows from $\max\{\sum_{n=1}^{N} \log \alpha_n;\ \alpha_1, \ldots, \alpha_N > 0,\ \alpha_1 + \cdots + \alpha_N = 1\} = -N \log N$, and the second inequality follows from the fact that $x(jd - \log x)$ is increasing for $1 \leq x \leq 2^{jd-1}$. The number $b_p(m)$ of bits needed to encode $\mathcal{C}_p(m+1) \backslash \mathcal{C}_p(m)$ is given by summing over j, leading to

$$\begin{aligned} b_p(m) &\leq C \sum_{j=0}^{\infty} (jd - \log \lambda_{j,p}(m) + C')\lambda_{j,p}(m) \\ &\leq C' \sum_{j=0}^{j^*} 2^{jd} + \sum_{j > j^*} (jd - \tau(m - \delta j)) 2^{-\tau(\delta j - m)} \\ &\leq C' 2^{m\tau d/(d+\tau\delta)} = C' 2^{md/(s+d/p)}, \end{aligned}$$

where j^* is the last integer j before the minimum in (B.3) switches from 2^{jd} to $2^{\tau(m-\delta j)}$.

Our updating bitstream, to go from level m to level $m+1$ thus costs (taking together the two types of contribution listed above) at most $C 2^{md/(s+d/p)} + \#\mathcal{C}_p(m+1)$ bits in total.

Let us estimate $\#\mathcal{C}_p(\ell)$:

$$\begin{aligned}\#\mathcal{C}_p(\ell) = \sum_{j=0}^{\infty} \#\mathcal{C}_{j,p}(\ell) = \sum_{j=0}^{\infty} \min\left[2^{jd}, \sum_{n=L_0}^{\ell-1} \#\Delta_{j,p}(n)\right] \\ \leq C \sum_{j=0}^{\infty} \min\left[2^{jd}, \sum_{n=L_0}^{\ell-1} 2^{-\tau\delta j} 2^{n\tau}\right] \\ \leq C' \sum_{j=0}^{\infty} \min\left[2^{jd}, 2^{-\tau\delta j} 2^{\ell\tau}\right] \\ \leq C'' 2^{\ell d/(s+d/p)}.\end{aligned}$$

Consequently, our total updating bitstream has at most $C'' 2^{md/(s+d/p)}$ bits. It follows that the total bitrate $R_p(L)$ needed to encode $\mathcal{C}_p(L)$ as well as {first L bits of $a_{I,p}$ with $I \in \mathcal{C}_p(L)$} is bounded by summing this estimate over m:

$$R_p(L) \leq C'' \sum_{m=L_0}^{L-1} 2^{md/(s+d/p)} \leq C''' 2^{Ld/(s+d/p)}.$$

The corresponding distortion is given by (B.2),

$$\mathbf{D}_p(L) \leq C\left[2^{-Ls/(s+d/p)} + 2^{-L} 2^{Ld/(s+d/p)}\right] = C 2^{-Ls/p(s+d/p)},$$

leading to the optimal distortion-rate bound $\mathbf{D}_p \leq C R_p^{-s/d}$.

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