A Pointwise "o" Saturation Theorem for Positive Convolution Operators

By

RONALD A. DEVORE

DEPT. OF MATH.
OAKLAND UNIVERSITY
ROCHESTER, MICHIGAN

1. Introduction

We wish to consider the saturation of positive convolution operators in the space $C^*[-\pi, \pi]$ of 2π-periodic and continuous functions. For this purpose, let (L_n) be a sequence of operators given by the convolution formulae

(1) $$L_n(f, x) = (f * d\mu_n)(x) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x+t)\, d\mu_n(t)$$

where each $d\mu_n$ is a non-negative, even, Borel measure on $[-\pi, \pi]$ with $\frac{1}{\pi} \int_{-\pi}^{\pi} d\mu_n(t) = 1$. We also suppose that the Fourier—Stieltjes coefficients $\varrho_{k,n} = \frac{1}{\pi} \int_{-\pi}^{\pi} \cos kt\, d\mu_n(t)$ satisfy

(2) $$\lim_{n\to\infty} \frac{1-\varrho_{k,n}}{1-\varrho_{1,n}} = \psi_k \neq 0. \qquad k = 1, 2, \ldots.$$

The requirement (2) is a standard assumption for saturation theorems. In parrtcular, under these assumptions, we have that for $f \in C^*$

(3) $$\|f - L_n(f)\| = o(1-\varrho_{1,n}), \quad \text{if and only if } f \text{ is constant},$$

where $\|-\|$ denotes the supremum norm. This is the "o" part of the general saturation theorem of SUNOUCHI—WATARI [8] and is easily proved using transforms. Namely, if $\|f - L_n(f)\| = o(1-\varrho_{1,n})$ then it follows by taking transforms that $\hat{f}(k) - \hat{f}(k)\varrho_{k,n} = o(1-\varrho_{1,n})$. Because of (2) we have $\hat{f}(k) = 0$, $k \neq 0$.

The situation becomes more difficult if we seek a characterization of the functions f which satisfy a pointwise "o" condition

(4) $$f(x) - L_n(f, x) = o_x(1-\varrho_{1,n}) \quad \text{for each} \quad x \in [-\pi, \pi].$$

Here, the simplest case occurs when $\psi_k = k^2$. This is equivalent [7] to having for each $S_\varepsilon = [-\pi, \pi] \setminus (-\varepsilon, \varepsilon)$, $\varepsilon > 0$

(5) $$\int_{S_\varepsilon} d\mu_n(t) = o(1-\varrho_{1,n}).$$

ıother equivalent formulation is that the asymptotic formula

$$\lim_{n\to\infty} (1-\varrho_{1,n})^{-1}(L_n(f,x)-f(x)) = f''(x)$$

hold for all functions f with two continuous derivatives. Because of (5), it is e to use the parabola technique of BAJSANSKI and BOJANIC [2] to obtain twise "o" theorem.

sketch of the argument is as follows. If f is a continuous function on $[-\pi, \pi]$ satisfies (4) then subtracting a constant if necessary, we can assume $f(-\pi)=$ $=0$. We also suppose that $f(x_0)>0$ for some $x_0\in(-\pi,\pi)$. So that from Lemma , it follows that there is a point $y\in(-\pi,\pi)$ and a parabola $Q(x)=\alpha(x-y)^2+$ $-y)+f(y)$, with $\alpha<0$, such that $Q(x)\geqq f(x)$ for $x\in[-\pi,\pi]$. Therefore,

$$L_n(f,y)-f(y) = \frac{1}{\pi}\int_{-\pi}^{\pi}[f(x)-f(y)]\,d\mu_n(x-y) \leqq$$

$$\leqq \frac{1}{\pi}\int_{-\pi}^{\pi}(Q(x)-Q(y))\,d\mu_n(x-y) =$$

$$= \frac{\alpha}{\pi}\int_{-\pi}^{\pi}(x-y)^2\,d\mu_n(x-y)+\frac{\beta}{\pi}\int_{-\pi}^{\pi}(x-y)\,d\mu_n(x-y).$$

is easy to show that the second term on the right hand side of (7) is $o(1-\varrho_{1,n})$ ıg (5) and the fact that $d\mu_n$ is even. Therefore,

$$L_n(f,y)-f(y) \leqq \frac{\alpha}{\pi}\int_{-\pi}^{\pi}(x-y)^2\,d\mu_n(x-y)+o(1-\varrho_{1,n}) \leqq$$

$$\leqq \alpha\pi\int_{-\pi}^{\pi}\sin^2\left(\frac{x-y}{2}\right)d\mu_n(x-y)+o(1-\varrho_{1,n}) = \alpha\pi^2(1-\varrho_{1,n})+o(1-\varrho_{1,n}),$$

we have used the inequality $\sin\frac{t}{2}\geqq\frac{t}{\pi}$ for $0\leqq t\leqq\pi$.

nce $\alpha<0$, this contradicts (4) for the point y. Thus $f(x)\leqq 0$ for all $x\in[-\pi,\pi]$. ;ing f by $-f$ in the above argument we conclude that $f(x)\geqq 0$ on $[-\pi,\pi]$ us $f\equiv 0$ on $[-\pi,\pi]$ as desired.

ıe parabola technique cannot be applied directly when (5) does not hold n this case the terms $\int_{-\pi}^{\pi}(x-y)\,d\mu_n(x-y)$ are not negligible. The object of per is to prove a general pointwise "o" theorem with no restrictions on (ψ_k).

2. Main results

THEOREM: *Let (L_n) be a sequence of positive convolution operators of the form* (1) *where the Fourier coefficients of $d\mu_n$ satisfy* (2). *If $f \in C^*$ then*

$$f(x) - L_n(f, x) = o_x(1-\varrho_{1,n}) \quad \text{for each} \quad x \in [-\pi, \pi] \tag{8}$$

if and only if f is constant on $[-\pi, \pi]$.

PROOF: The "if" part of the theorem is obvious. The proof of the "only if" part is based on a trigonometric analogue of the parabola technique of BAJSANSKI BOJANIC. However, we must first prove two lemmas which give some properties of functions which satisfy (8). Of course, ultimately we wish to show that such functions are constant.

If x is a point in $[-\pi, \pi]$, such that, for each neighbourhood I of x we have $\int_I d\mu_n(t) \neq o(1-\varrho_{1,n})$, then we shall say x is an essential point. Otherwise, we say x is a negligible point. Let f be a function which satisfies (8). We set $M = \max_{-\pi \leq t \leq \pi} f(t)$ and when $x_0 \in [-\pi, \pi]$, with $f(x_0) = M$, we let $\mathfrak{M}(x_0) = \{t\colon t$ is an essential point and $f(x_0+t) = M\}$. Also, let $\mathfrak{M} = \bigcap_{x_0} \mathfrak{M}(x_0)$. We consider all points modulo 2π.

LEMMA 1: *If $x \notin \mathfrak{M}$, then x is a negligible point.*

PROOF: Suppose $x \notin \mathfrak{M}(x_0)$ for some x_0 with $f(x_0) = M$. Then either x is negligible or $f(x_0+x) < M$. In the latter case, let $I = \{y\colon f(x_0+y) < \frac{1}{2}[M + f(x_0+x)]\}$. I is a neighbourhood of x and

$$\tfrac{1}{2}[M - f(x_0+x)] \int_I d\mu_n(t) \leq \int_I [f(x_0) - f(x_0+t)]\, d\mu_n(t).$$

However

$$\frac{1}{\pi} \int_I [f(x_0) - f(x_0+t)]\, d\mu_n(t) \leq \frac{1}{\pi} \int_{-\pi}^{\pi} [f(x_0) - f(x_0+t)]\, d\mu_n(t) =$$

$$= f(x_0) - L_n(f, x_0) = o(1-\varrho_{1,n}).$$

This shows that $\int_I d\mu_n(t) = o(1-\varrho_{1,n})$ and thus x is negligible.

LEMMA 2: *If f is not constant, then $\mathfrak{M}$ has only a finite number of points. Also, if x is any point in $\mathfrak{M}$ then $x = 2\pi\alpha$ where α is rational.*

PROOF: Suppose first that (x_n) is a sequence of distinct points each of which is in $\mathfrak{M}$. Choosing a subsequence if necessary we can assume $x_n \to x$ where $0 \leq x < 2\pi$. We write $x_n = 2\pi\alpha_n$ and $x = 2\pi\alpha$.

Let x_0 be any point in $[-\pi, \pi]$, where $f(x_0) = M$. Then, $f(x_0 + x_n) = M$ and $x_n \in \mathfrak{M}(x_0 + x_n)$, so that $f(x_0 + 2x_n) = M$. More generally, for each positive integer

k, we have $f(x_0+kx_n) = M$. By continuity, $f(x_0+kx) = M$ for each positive integer k. If α is irrational then the points $k\alpha$ taken modulo 1 are dense in [0, 1] so that the points kx taken modulo 2π are dense in $[0, 2\pi]$. Thus, in this case, $f=M$ on a set of points which are dense in $[x_0, x_0+2\pi]$ and therefore $f=M$ on $[x_0, x_0+2\pi]$. From periodicity, we conclude that f is constant.

If α is rational then the points $k\alpha_n = k(\alpha+\delta_n)$ where $0\neq\delta_n\to 0$, taken modulo 1, are dense in [0, 1]. Therefore, we again have that f is constant. This shows that $\mathfrak{M}$ has no limit point in $[0, 2\pi]$ and hence must consist of only a finite number of points.

Finally, if a point of the form $x=2\pi\alpha$ with α irrational were in $\mathfrak{M}$ then, as we have mentioned before, $f(x_0+kx) = M$ for each positive integer k so that $f=M$ on a set of points which is dense in $[x_0, x_0+2\pi]$. This again gives that f is constant.

PROOF OF THE THEOREM: Let f be a function which satisfies (8) and suppose f is not constant. By subtracting a constant, if necessary, we can suppose that $f(-\pi)=f(\pi)=0$. Also suppose $M>0$. Then, it follows from Lemma 2 that there is a positive integer m such that $\mathfrak{M} \subseteq \left\{\frac{k\pi}{m}: k=0, \pm 1, \pm 2, \ldots, \pm m\right\}$. Let

$$I = \bigcup_{k=-m}^{m} \left[\frac{k\pi}{m}-\frac{\pi}{8m}, \frac{k\pi}{m}+\frac{\pi}{8m}\right] \cap [-\pi, \pi].$$

Since each point $y\notin\mathfrak{M}$ is negligible we could use a compactness argument to show that for $S=[-\pi, \pi]\setminus I$

$$\int_S d\mu_n(t) = o(1-\varrho_{1,n}). \tag{9}$$

The function $h(x) = -M\sin^2 mx+2M$ is $\geqq f(x)$ on $[-\pi, \pi]$. Let $c = \min_{t\in I}(h(t)-f(t))$. Then, $h(x)-c \geqq f(x)$ on I and for some $y\in I$

$$h(y)-c = f(y).$$

Therefore,

$$\int_I [h(x)-h(y)]\, d\mu_n(x-y) \geqq \int_I [f(x)-f(y)]\, d\mu_n(x-y). \tag{10}$$

But $h(x)-h(y) = -M\cos(2my)\sin^2 m(x-y) - \frac{M}{2}\sin(2my)\sin 2m(x-y)$ and thus

$$\frac{1}{\pi}\int_{-\pi}^{\pi} [h(x)-h(y)]\, d\mu_n(x-y) = -\frac{M}{\pi}\cos(2my)\int_{-\pi}^{\pi}\sin^2 m(x-y)\, d\mu_n(x-y) =$$

$$= -\frac{M}{2}\cos 2my(1-\varrho_{2m,n}).$$

Here, we have used the fact that $d\mu_n$ is even to drop the term involving $\sin 2m(x-y)$ which would normally appear. Because of (9), we have

$$\frac{1}{\pi}\int_I [h(x)-h(y)]\,d\mu_n(x-y) =$$

$$=\frac{1}{\pi}\int_{-\pi}^{\pi}[h(x)-h(y)]\,d\mu_n(x-y)-\frac{1}{\pi}\int_S [h(x)-h(y)]\,d\mu_n(x-y) =$$

$$=-\frac{M}{2}\cos 2my(1-\varrho_{2m,n})+o(1-\varrho_{1,n}).$$

Finally, from (10)

$$L_n(f,y)-f(y)=\frac{1}{\pi}\int_I [f(x)-f(y)]\,d\mu_n(x-y)+o(1-\varrho_{1,n}) \leqq$$

$$\leqq \frac{1}{\pi}\int_I [h(x)-h(y)]\,d\mu_n(x-y)+o(1-\varrho_{1,n}) =$$

$$=-\frac{M}{2}\cos 2my(1-\varrho_{2m,n})+o(1-\varrho_{1,n}) = -\frac{M}{2}\cos 2my(1-\varrho_{1,n})\psi_{2m}+o(1-\varrho_{1,n})$$

where for the last equality we have used (2).

Since $\cos 2my>0$ and $\psi_{2m}>0$, we must have $f(y)-L_n(f,y)\neq o(1-\varrho_{1,n})$ which is the desired contradiction and therefore $M=0$. This shows $f(x)\leqq 0$ on $[-\pi,\pi]$. To see that $f(x)\geqq 0$ on $[-\pi,\pi]$, we merely work with the function $-f$ in place of f in the above argument. This completes the proof of the theorem.

3. Remarks. The first pointwise saturation theorem appears to be the "o" theorem for the Fejér operators (σ_n) which was given by ANDRIENKO [1]. In this case, it is possible to weaken (4) by discarding certain small sets (e.g. countable) and still conclude that f is constant.

This cannot be done in general. For example, if (L_n) is any sequence satisfying (2) with $\psi_k=k^2$ then the function $f(t)=|t|$ satisfies

$$f(x)-L_n(f,x)=o_x(1-\varrho_{1,n}) \qquad (-\pi<x\leqq\pi,\quad x\neq 0).$$

ANDRIENKO has also given local "o" theorems for the Fejér operators. Namely, if

$$f(x)-\sigma_n(f,x)=o_x(n^{-1}) \qquad (a\leqq x\leqq b)$$

then f is constant on $[a,b]$. To see that this cannot be done in general, let

$$\frac{1}{\pi}d\mu_n=\frac{1}{2}\left(1-\frac{1}{2}n^{-2}\right)(d\varrho_{n^{-1}}+d\varrho_{-n^{-1}})+\frac{n^{-2}}{4}(d\varrho_{\pi/2}+d\varrho_{-\pi/2})$$

where $d\varrho_{x_0}$ denotes the Dirac measure at x_0. Then

$$1-\varrho_{1,n} = n^{-2}+o(n^{-2})$$

and

$$\lim_{n\to\infty} \frac{(1-\varrho_{k,n})}{1-\varrho_{1,n}} = \frac{k^2}{2}+\sin^2\frac{k\pi}{4}.$$

If f is any function in C^* which is twice continuously differentiable on $\left(-\frac{\pi}{8}, \frac{\pi}{8}\right)$ then

$$\lim_{n\to\infty} 2n^2(L_n(f, x)-f(x)) = f''(x)+\frac{1}{2}\left[f\left(x+\frac{\pi}{2}\right)+f\left(x-\frac{\pi}{2}\right)-2f(x)\right]$$

for $x\in\left(-\frac{\pi}{8}, \frac{\pi}{8}\right)$. Thus, we can take any such f and define it outside $\left(-\frac{\pi}{8}, \frac{\pi}{8}\right)$ in such a way that

$$f\left(x+\frac{\pi}{2}\right)+f\left(x-\frac{\pi}{2}\right) = 2f(x)-2f''(x) \qquad \left(-\frac{\pi}{8}<x<\frac{\pi}{8}\right),$$

and then f will satisfy

$$f(x)-L_n(f, x) = o_x(1-\varrho_{1,n}) \qquad \left(-\frac{\pi}{8}<x<\frac{\pi}{8}\right).$$

In an unpublished paper, H. Berens has shown the pointwise "o" theorem for the case when there is an $0<\alpha\leqq 2$ such that $\psi_k=k^\alpha$, $k=1, 2, \dots$. Here, the proof is based on knowing that the function $h_\alpha(t)$ whose Fourier coefficients are $k^{-\alpha}$ has the property that

$$\int_a^b h_\alpha(t)\,dt > 0$$

for each $-\pi\leqq a<b\leqq\pi$.

In this same vein, if we let $d\lambda_n(t) = 2(1-\varrho_{1,n})^{-1}\sin^2\frac{t}{2}\,d\mu_n(t)$, then $\frac{1}{\pi}\int|d\lambda_n(t)|=1$. Therefore, there is a subsequence (n_k) and a measure $d\lambda$ such that $d\lambda_{n_j}\to d\lambda$, weak$*$. Our essential points are just the points in the support of $d\lambda$. If the support of $d\lambda$ is all of $[-\pi, \pi]$, the proof of the theorem can be given as we argued in Section 1.

For the Cesàro means [3] and more generally the typical means [4] of the Fourier series, H. Berens has given both pointwise "o" and "O" theorems. When $\psi_k=k^2$, $k=1, 2, \dots$, then there is a companion pointwise "O" theorem which also is due to Berens [5]. However, there is no general pointwise "O" theorem which is companion to our "o" theorem. For such a "O" theorem, it will be necessary to assume the multiplier condition

$$\left(\frac{1-\varrho_{k,n}}{\psi_k(1-\varrho_{1,n})}\right)_{k=0}^{\infty} \in (L_\infty, L_\infty),$$

since this is needed even in the norm case. (See for example DEVORE [6].)

Regarding the form such a "O" would most likely take, we note that when the multiplier condition holds then there is an asymptotic formula for (L_n). If D is the distribution with $\hat{D}(k)=\psi_k$, then there is a subsequence (n_j) such that for each $f\in C^*$ with $f*D$ continuous we have

$$\lim_{n\to\infty} (1-\varrho_{1,n_j})^{-1}(L_{n_j}(f,x)-f(x)) = f*D.$$

This asymptotic condition indicates that the "O" theorem should read: If $g\in L_1$ and

$$\varliminf_{n\to\infty} (1-\varrho_{1,n})^{-1}(L_n(f,x)-f(x)) \leqq g(x) \leqq \varlimsup_{n\to\infty} (1-\varrho_{1,n})^{-1}(L_n(f,x)-f(x)),$$

then $f*D\in L_1$ and $f*D = g$ a.e.

4. Acknowledgement

We would like to thank Professor H. BERENS for many helpful comments.

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