SIAM J. NUMER. ANAL.
Vol. 41, No. 5, pp. 1785–1823

ADAPTIVE WAVELET SCHEMES FOR NONLINEAR VARIATIONAL PROBLEMS*

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Abstract. We develop and analyze wavelet based adaptive schemes for nonlinear variational problems. We derive estimates for convergence rates and corresponding work counts that turn out to be asymptotically optimal. Our approach is based on a new paradigm that has been put forward recently for a class of linear problems. The original problem is transformed first into an equivalent one which is well posed in the Euclidean metric ℓ_2. Then conceptually one seeks iteration schemes for the infinite dimensional problem that exhibits at least a fixed error reduction per step. This iteration is then realized approximately through an adaptive application of the involved operators with suitable dynamically updated accuracy tolerances. The main conceptual ingredients center around nonlinear tree approximation and the sparse evaluation of nonlinear mappings of wavelet expansions. We prove asymptotically optimal complexity for adaptive realizations of first order iterations and of Newton's method.

Key words. variational problems, wavelet representations, semilinear equations, mapping properties, gradient iteration, convergence rates, adaptive application of operators, sparse evaluation of nonlinear mappings of wavelet expansions, tree approximation, Newton's scheme

AMS subject classifications. 65J15, 65N12, 65N15, 35A15, 35A35, 35J60, 41A60, 46A45, 47H17

DOI. 10.1137/S0036142902412269

1. Introduction.

1.1. Background and objectives. Adaptive wavelet schemes for numerically solving a wide class of variational problems have been recently studied in [8, 9] from the perspective of asymptotic estimates for convergence rates and corresponding work counts. The problems covered by that analysis include elliptic boundary integral equations and elliptic boundary value problems but also indefinite problems of elliptic type such as the Stokes problem. Two requirements were essential in this context: (i) the variational problem induces an operator $\mathcal{L}$ that is an isomorphism from some Hilbert space $\mathcal{H}$ onto its dual; (ii) this Hilbert space permits a wavelet characterization; i.e., the $\mathcal{H}$-norm of an element is equivalent to a weighted ℓ_2-norm of its wavelet coefficients. It could then be shown that certain adaptive schemes exhibit an *asymptotically optimal accuracy/work balance* within a certain range of convergence rates depending on the choice of wavelet bases. The precise meaning of this statement is explained in the Meta-Theorem below. To our knowledge for the above range of linear problems such complexity estimates have been established so far only for wavelet methods. Just recently, a similar result was proved for adaptive finite element methods for Laplace's

*Received by the editors July 26, 2002; accepted for publication (in revised form) May 1, 2003; published electronically October 28, 2003. This work was supported in part by the Office of Naval Research contract N0014-91-J1343, the Army Research Office contract DAAD 19-02-1-0028, the IHP-network (N.HPRN-CT-2002-00286) "Breaking Complexity," by the SFB 401, funded by the German Research Foundation, and by the Alexander von Humboldt Foundation.

http://www.siam.org/journals/sinum/41-5/41226.html

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equation in two space dimensions [4].

In this paper we wish to explore the convergence rates and the computational complexity of certain new adaptive wavelet schemes for *nonlinear problems* for which no results of the above type seem to be known so far.

Our primary concern here is *not* to develop a specific algorithm for a concrete application. We are rather interested in developing a numerically realizable *new algorithmic paradigm* in a fairly general context of nonlinear problems and in analyzing its principal complexity features. Therefore, the various algorithmic ingredients will at times not be discussed in full detail but only to an extent that clarifies their principal asymptotic complexity.

The new paradigm is based upon the adaptive evaluation of (linear and nonlinear) operators in the course of an ideal iteration for the *infinite dimensional* problem formulated in the wavelet coordinate domain. Such perturbed iterations will lead to an algorithm **SOLVE** that (with a proper initialization) produces for any target accuracy ϵ a finitely supported vector of coefficients $\bar{\mathbf{u}}(\epsilon)$ that approximates the array of wavelet coefficients of the exact solution (of the underlying variational problem) in ℓ_2 with accuracy ϵ. The choice of wavelet basis will then imply that the corresponding finite expansion approximates the exact solution with accuracy $C\epsilon$ in the energy norm, where C depends only on the wavelet basis. In order to identify the essential mechanisms governing such schemes, we will consider nonlinear variational problems on various levels of generality. The results will be purely asymptotic in nature. They reveal asymptotically optimal work/accuracy balances interrelating the achieved target accuracy with the required computational work and associated adaptively generated number of degrees of freedom. More precisely, we shall prove results of the following type.

META-THEOREM. *If the exact solution can be approximated as a linear combination of N wavelets (subject only to certain tree restrictions on the distribution of active coefficients) to accuracy of order N^{-s} (for a certain range of s), then the support of the output $\bar{\mathbf{u}}(\epsilon)$ of* **SOLVE** *for target accuracy ϵ grows at most as $\epsilon^{-1/s}$, uniformly in ϵ, and the computational complexity also stays proportional to the support size. In this sense, the scheme tracks the exact solution at asymptotically minimal cost.*

Note that the above-mentioned *tree restriction* on the permitted distribution of active coefficients is the analogue of locally refined meshes in the finite element context.

We shall outline now how we approach results of the above type.

1.2. The basic paradigm. The *classical* approach to numerically solving (linear and nonlinear) variational problems is concerned with the following issues:

(c1) well-posedness of the given variational problem;
(c2) discretization of the infinite dimensional problem so as to obtain a finite system of algebraic equations;
(c3) well-posedness of the finite system of equations and error analysis;
(c4) numerical solution of the finite system of equations.

It is important to note that (c1) is often hidden in the analysis and that (c3) is, in general, *not* a direct consequence of (c1). Typical examples even in the linear case are *saddle point problems*. It is well known that, for Galerkin discretizations to be stable, the trial spaces for the different solution components have to satisfy a certain compatibility condition (Ladyšhenskaya–Babŭska–Brezzi (LBB)-condition). For nonlinear problems one can often establish only *local* uniqueness of solutions so that some care is required to ensure that the discrete problems approximate the correct solution branch. Thus the discrete problems do not necessarily inherit the "nice properties" of

the original infinite dimensional problem. Depending on the choice of the discretization, one might introduce "new difficulties." The typical obstructions encountered in (c4) are the *large size* of the discrete systems and possible *ill-conditioning*. The latter issue interferes with the need to resort to iterative solvers, due to the size and sparsity of the systems. Attempts to reduce computational complexity are often based on adaptive and hence possibly economic discretizations. A reliable control of adaptive refinements, however, depends usually in a sensitive way on the particular type of the problem, and rigorous complexity estimates are generally not available yet.

A *new paradigm* has been explored in [9] for *linear variational problems*. It aims at closely intertwining the analysis–discretization–solution process. The basic steps there read as follows:

(n1) well-posedness of the given variational problem;
(n2) transformation of the infinite dimensional problem into an *equivalent* problem in ℓ_2 which is *well posed* in the Euclidean metric;
(n3) the derivation of an iterative scheme for the infinite dimensional ℓ_2-problem that exhibits a fixed error reduction per iteration step;
(n4) numerical realization of the iterative scheme by an *adaptive application* of the involved infinite dimensional operators within some finite dynamically updated accuracy tolerances.

Thus the starting point (n1) is the same as (c1), although it takes a somewhat more exposed and explicit role in the new setting, as will be explained later. The main difference is that one aims at staying as long as possible with the infinite dimensional problem, which one hopes is given in a favorable format. Of course, it remains to see in each concrete case how to exploit (n2) in order to guarantee a fixed error reduction in (n3). We shall present several strategies regarding this task. Only at the very end, when it comes to applying the operators in the ideal iteration scheme (n4), does one enter the finite dimensional realm. However, the finite number of degrees of freedom is determined at each stage by the adaptive application of the operator so that at *no* stage is any specific trial space fixed. Roughly speaking, the "nice properties" of the infinite dimensional problem are preserved through adaptive evaluations. In fact, one can show that thereby compatibility conditions like the LBB-condition indeed become void [9, 13].

The main goal of the present paper is to show how to carry over this paradigm, already existing for linear problems, to the nonlinear setting. On a theoretical level, one then encounters three major issues, namely,

(a) the choice of tolerances in (n4) to ensure that the perturbed iteration converges to the correct solution;
(b) the design of economic approximate application schemes for the possibly nonlinear infinite dimensional operators;
(c) estimating the complexity of the scheme.

Here (a) means that any given *target accuracy* ϵ is achieved after finitely many steps. (b) is the most crucial part and will be discussed in detail in the course of the paper. Clearly (b) is closely related to (c). As in [8, 9, 13], we will measure complexity by the *number of adaptively generated degrees of freedom* $N = N(\epsilon)$ required by the adaptive scheme to achieve the target accuracy ϵ and the corresponding number of floating point operations (which, of course, is aimed at staying proportional to $N(\epsilon)$). Estimating the asymptotic *work/accuracy balance* $N(\epsilon) \leftrightarrow \epsilon$ will be a central theme in the subsequent developments. This part differs significantly from the classical error analysis for finite element methods and relies on concepts from *harmonic analysis* and

nonlinear approximation.

Of course, on a practical level one will encounter in each concrete case further obstacles concerning quantitative information about constants and initial guesses. We shall discuss variational problems on a different level of generality in order to indicate possible strategies of acquiring such information or to identify those issues that require additional work.

Finally, a comment on (n3) is in order. Aiming at a fixed error reduction per iteration step means that one is content with a *first order* scheme. So why not go for faster iteration schemes? The answer to this question is not completely clear. Indeed, a higher order method may not automatically win for the following reason. Usually a higher order method is more costly in function evaluations. In the present context this means, according to (n4), it is more costly in the adaptive application of the full infinite dimensional operators within some dynamically updated accuracy tolerance. Preserving the higher order of the ideal iteration also in its perturbed form in connection with the higher demands of function evaluations may very well increase the cost of each iteration step so as to offset the potential gain of a better error reduction. So with regard to the objective of reaching a target accuracy at possibly low overall computational cost, the role of higher order schemes remains unclear. In fact, it will be seen that *asymptotic optimality* can indeed be achieved already with simple *first order outer iterations*. Nevertheless, we shall show that it is also possible to retain second order convergence of the adaptive version of Newton's scheme so as to arrive at an overall scheme with asymptotically optimal solution complexity, which may offer quantitative advantages over the first order versions.

1.3. Organization of material. The paper is organized as follows. In section 2 we describe (n1), (n2), and (n3) for a general setting that will host all subsequent specifications. In section 3 we distinguish several classes of variational problems to which the subsequent developments will refer frequently, namely, (L) *linear* problems, (SL) semilinear elliptic problems, and (GNL) more general nonlinear problems where we have to assume the existence of locally unique solutions. In section 4 we formulate the prototype of an adaptive perturbed first order iteration which is based on two main ingredients, namely, approximate *residual evaluations* and a certain *coarsening scheme*. In particular, the residual approximations involve the *adaptive* application of linear or nonlinear (infinite dimensional) operators. Assuming at this stage that these ingredients are indeed available, we address for the most general setting first only issue (a) to clarify for which choice of dynamically updated accuracy tolerances is convergence guaranteed. The remaining sections will be devoted to issues (b) and (c) for the problem types (L), (SL), and (GNL).

In section 5 we review briefly concrete realizations of these ingredients for the linear case (L) and indicate the concepts needed for their complexity analysis. This serves two purposes. First, these results will be used in the last section in connection with Newton iterations. Second, they motivate our treatment of the nonlinear case. In section 6 we introduce some new concepts needed to deal with nonlinear problems. They center upon *tree approximation* and related coarsening techniques. This enables us to formulate the notion of s^*-sparsity as the key criterion for controlling the complexity of the adaptive schemes in the nonlinear case. Drawing on several results from [10], we develop in section 7 adaptive evaluation schemes that are proven to be s^*-sparse and thus lead to asymptotically optimal results in the sense of the above Meta-Theorem. To our knowledge these are the first convergence and complexity estimates for adaptive solvers for nonlinear problems. Finally, in section

8 we develop an adaptive Newton scheme and analyze its complexity. It differs in essential ways from the schemes discussed in the previous sections which are based on first order iterations. In particular, we show that the quadratic convergence of the outer iteration can in some sense be preserved in the adaptive context.

2. The setting. We describe now the setting for which the above paradigm will be discussed.

2.1. The general problem format. The variational problems mentioned in step (n1) above will always have the following format. Let $\mathcal{H}$ be a Hilbert space with norm $\|\cdot\|_{\mathcal{H}}$, and let $\mathcal{H}'$ denote its dual endowed with the norm

$$\|v\|_{\mathcal{H}'} := \sup_{w\in\mathcal{H}} \frac{\langle w, v\rangle}{\|w\|_{\mathcal{H}}},$$

where $\langle\cdot,\cdot\rangle$ is the dual pairing between $\mathcal{H}$ and $\mathcal{H}'$ (with respect to L_2 as the pivot space). The Hilbert space $\mathcal{H}$ will always refer to a bounded domain Ω with spatial dimension d. Suppose that $f \in \mathcal{H}'$ and

$$F : \mathcal{H} \to \mathcal{H}' \tag{2.1}$$

is a (possibly nonlinear) mapping. We consider the numerical solution of the problem: Find $u \in \mathcal{H}$ such that

$$\langle v, F(u) - f\rangle =: \langle v, R(u)\rangle = 0 \quad \forall v \in \mathcal{H}. \tag{2.2}$$

The objective in (n1) is the identification of a suitable space $\mathcal{H}$ so that (2.2) is well posed in the following sense. Recall that the Frechét derivative $DR(z) = DF(z)$ is a mapping from $\mathcal{H}$ to $\mathcal{H}'$, defined by the duality

$$\langle v, DR(z)w\rangle = \lim_{h\to 0} \frac{1}{h}\langle v, R(z + hw) - R(z)\rangle. \tag{2.3}$$

The problem (2.2) is called *well posed* if F has the following properties:

- A1. F possesses a continuous Frechét derivative; i.e., $R \in C^1(\mathcal{H}, \mathcal{H}')$ as a mapping $v \mapsto R(v)$.
- A2. There exists a solution $u \in \mathcal{H}$ to (2.2), and in addition to (2.1) the Frechét derivative DF of F at v in some neighborhood $\mathcal{U}$ of u is an isomorphism from $\mathcal{H}$ onto $\mathcal{H}'$; i.e., for $v \in \mathcal{U}$ there exist positive finite constants $c_{v,F}, C_{v,F}$ such that

$$c_{v,F}\|w\|_{\mathcal{H}} \le \|DF(v)w\|_{\mathcal{H}'} \le C_{v,F}\|w\|_{\mathcal{H}} \quad \forall\, w \in \mathcal{H},\ v \in \mathcal{U}. \tag{2.4}$$

Clearly A2 ensures that the solution u is locally unique.

2.2. Wavelet coordinates and an equivalent ℓ_2-problem. The transformations for (n2) will be based on suitable wavelet bases. For a detailed discussion of such bases, we refer the reader to the literature (see, e.g., [5, 6, 14, 11]) and collect here only the relevant facts. A *wavelet basis* $\Psi = \{\psi_\lambda : \lambda \in \mathcal{J}\} \subset \mathcal{H}$ has the following properties: The indices $\lambda \in \mathcal{J}$ encode typical information about the wavelet ψ_λ, namely, its type, its location $k(\lambda)$, and its scale $|\lambda|$.

We shall now explain the meaning of "suitable" in the present context. We will always assume that the wavelets have compact support $S_\lambda := \operatorname{supp}\psi_\lambda$, $\lambda \in \mathcal{J}$, which scales as $\operatorname{diam}(S_\lambda) \sim 2^{-|\lambda|}$.

Furthermore, aside from finitely many functions $\psi_\lambda, \lambda \in \mathcal{J}_\phi \subset \mathcal{J}$, $|\lambda| = j_0$, representing the coarsest scale j_0, the wavelets ψ_λ, $\lambda \in \mathcal{J}\backslash\mathcal{J}_\phi$, have *vanishing moments* of some order $m \in \mathbb{N}$; i.e., these wavelets are orthogonal to all polynomials of order m.

Finally, each $v \in \mathcal{H}$ has a unique expansion $\sum_{\lambda\in\mathcal{J}} v_\lambda\psi_\lambda$ such that

$$c_1\|\mathbf{v}\|_{\ell_2(\mathcal{J})} \le \left\|\sum_{\lambda\in\mathcal{J}} v_\lambda\psi_\lambda\right\|_{\mathcal{H}} \le C_1\|\mathbf{v}\|_{\ell_2(\mathcal{J})} \tag{2.5}$$

holds for some positive constants c_1, C_1; i.e., Ψ forms a *Riesz basis* for $\mathcal{H}$. Note that (unlike the quoted references) we have normalized the wavelets here in the energy space $\mathcal{H}$ associated with the variational problem (2.1), (2.2); i.e., $\|\psi_\lambda\|_{\mathcal{H}} = 1$, $\lambda \in \mathcal{J}$. Again such bases are known whenever $\mathcal{H}$ is a product of Sobolev spaces (or closed subspaces of Sobolev spaces, determined, e.g., by homogeneous boundary conditions or vanishing integral means).

In the following, we will always use boldface notation $\mathbf{v}$ to denote the wavelet coefficients of a given function $v \in \mathcal{H}$ with respect to the basis Ψ (and analogously for $u, w \in \mathcal{H}$).

Next note that by duality (2.5) implies

$$C_1^{-1}\|(\langle w, \psi_\lambda\rangle)_{\lambda\in\mathcal{J}}\|_{\ell_2(\mathcal{J})} \le \|w\|_{\mathcal{H}'} \le c_1^{-1}\|(\langle w, \psi_\lambda\rangle)_{\lambda\in\mathcal{J}}\|_{\ell_2(\mathcal{J})}. \tag{2.6}$$

We can now transform (2.2) into wavelet coordinates. Defining

$$\mathbf{R}(\mathbf{v}) := (\langle\psi_\lambda, R(v)\rangle : \lambda \in \mathcal{J}) \quad \text{whenever} \quad v = \sum_{\lambda\in\mathcal{J}} v_\lambda\psi_\lambda, \tag{2.7}$$

the original problem (2.2) is obviously equivalent to finding $\mathbf{u} \in \ell_2(\mathcal{J})$ so that

$$\mathbf{R}(\mathbf{u}) = \mathbf{0}. \tag{2.8}$$

Now note that the Jacobian $D\mathbf{R}(\mathbf{v}) = D\mathbf{F}(v)$ is given by

$$D\mathbf{R}(\mathbf{v}) = (\langle\psi_\lambda, DR(v)\psi_\nu\rangle)_{\lambda,\nu\in\mathcal{J}}\,, \tag{2.9}$$

where again $DR = DF$ is the Frechét derivative of the mapping R. Combining the norm equivalences (2.5), (2.6) with the mapping property (2.4) shows in what sense (2.8) is now well posed in ℓ_2; see, e.g., [9] for a proof.

Remark 2.1. Under the above assumptions A1 and A2, one has for any $v = \sum_{\lambda\in\mathcal{J}} v_\lambda\psi_\lambda \in \mathcal{U}$

$$c_1^2 c_{v,F}\|\mathbf{w}\|_{\ell_2(\mathcal{J})} \le \|D\mathbf{F}(\mathbf{v})\mathbf{w}\|_{\ell_2(\mathcal{J})} \le C_1^2 C_{v,F}\|\mathbf{w}\|_{\ell_2(\mathcal{J})}, \quad \mathbf{w} \in \ell_2(\mathcal{J}). \tag{2.10}$$

2.3. The basic iteration. According to (n3), we wish to devise an iterative scheme for the problem (2.8) such that each step reduces the current error at least by a fixed rate $\rho < 1$. The schemes we shall consider will have the form

$$\mathbf{u}^{n+1} = \mathbf{u}^n - \mathbf{B}_n\mathbf{R}(\mathbf{u}^n), \tag{2.11}$$

where the (infinite, possibly stage dependent) matrix $\mathbf{B}_n$ is yet to be chosen. For instance, $\mathbf{B}_n = \alpha\mathbf{I}$ corresponds to a fixed point or Richardson iteration, while for $\mathbf{B}_n := D\mathbf{R}(\mathbf{u}^n)^{-1}$ (2.11) becomes Newton's method.

We proceed now to discuss several instances of this setting.

3. The scope of reference problems. We shall address the variational problem (2.2) for the following different levels of generality:

(SL) semilinear elliptic boundary value problems, covering the case (L) of *linear* problems as a special case;

(GNL) general nonlinear problems.

The explicit discussion of (SL) will serve several purposes. First, this class is specific enough to permit a complete complexity analysis for a *globally convergent* adaptive scheme. In particular, we shall be able to obtain in this case concrete bounds on initial guesses or how to find a suitable damping parameter in $\mathbf{B}_n = \alpha\mathbf{I}$. Second, this class is a model representative for the interplay between a linear (diffusion) operator and a nonlinear part. On one hand, it covers linear problems (L) as special cases. Briefly reviewing the essential features of linear problems in this context comes in handy for three reasons. It provides a guideline for the treatment of nonlinear problems. It is a necessary prerequisite for the later discussion of Newton's method. Most importantly, in the presence of a nonlinearity, the complexity analysis for the linear case has to be modified in order to treat problems where both linear and nonlinear operators are involved. It will be instructive to see the conceptual distinctions and how the treatment of nonlinear problems builds on ingredients from the linear case.

Finally, in (GNL) we relax our assumptions on the structure of R to a great extent. We pay for this by making stronger assumptions on initial guesses and being content with *locally convergent first order* iterations on the infinite dimensional level.

We shall exemplify step (n3) for all three cases (L), (SL), and (GNL) in this order. Except for the last section, this will be based on first order iteration schemes for the underlying infinite dimensional problem. It will be seen along the way that it then suffices to employ stationary "preconditioners" $\mathbf{B}_n = \mathbf{B}$ to obtain asymptotically optimal complexity estimates (although more flexible nonstationary choices may well result in quantitative improvements in practical realizations). The use of truly nonstationary $\mathbf{B}_n$ will be necessary only in connection with Newton's method in section 8.

3.1. Semilinear (SL) and linear (L) elliptic problems. Suppose that $a(\cdot,\cdot)$ is a continuous bilinear form on a Hilbert space $\mathcal{H}$ endowed with the norm $\|\cdot\|_{\mathcal{H}}$, which is $\mathcal{H}$-elliptic; i.e., there exist positive constants c, C such that

$$c\|v\|_{\mathcal{H}}^2 \le a(v,v), \quad a(v,w) \le C\|v\|_{\mathcal{H}}\|w\|_{\mathcal{H}} \quad \forall\, v,w \in \mathcal{H}. \tag{3.1}$$

The simplest example is

$$a(v,u) := \langle \nabla v, \nabla u\rangle + \kappa\langle v,u\rangle, \quad \kappa \ge 0, \quad \langle v,w\rangle = \int_\Omega vw, \tag{3.2}$$

and $\mathcal{H} = H_0^1(\Omega)$ (the space of functions with first order weak derivatives in L_2 whose traces vanish on the boundary $\Gamma = \partial\Omega$) endowed with the norm $\|v\|_{\mathcal{H}}^2 := \|\nabla v\|_{L_2(\Omega)}^2 + \kappa\|v\|_{L_2(\Omega)}^2$.

In principle, the subsequent analysis will also cover elliptic integral operators with positive order such as the hypersingular operator.

To introduce a nonlinearity, we suppose that $G : \mathbb{R} \to \mathbb{R}$ is a function with the following property:

P1. The mapping $v \mapsto G(v)$ takes $\mathcal{H}$ into its dual $\mathcal{H}'$ and is *stable* in the sense that

$$\|G(u) - G(v)\|_{\mathcal{H}'} \le C_G(\max\{\|u\|_{\mathcal{H}}, \|v\|_{\mathcal{H}}\})\|u - v\|_{\mathcal{H}}, \quad u, v \in \mathcal{H}, \tag{3.3}$$

where $t \to C_G(t)$ is a nondecreasing function of t.

The problem: Given $f \in \mathcal{H}'$, find $u \in \mathcal{H}$ such that

$$\langle v, F(u)\rangle := a(v, u) + \langle v, G(u)\rangle = \langle v, f\rangle \quad \forall \ \ v \in \mathcal{H} \tag{3.4}$$

is of the form (2.2) with $R(v) = F(v) - f$.

Remark 3.1. If we assume in addition to P1 that G is monotone (as in (3.6)), i.e., $(u - v)(G(u) - G(v)) \ge 0$ for $u, v \in \mathbb{R}$, then (3.4) has for every $f \in \mathcal{H}'$ a unique solution $u \in \mathcal{H}$. Moreover, the problem is well posed in the sense of (2.4) with constants $c_{v,F} := c$, $C_{v,F} := C + C_G(\|v\|_{\mathcal{H}})$, where c, C are the constants from (3.1) and $C_G(s)$ is the constant from (3.3) in P1.

Proof. The argument follows standard lines. Under the above assumptions it is easy to show that the operator F, defined by (3.4), is also monotone and coercive. One can then invoke the Browder–Minty theorem (see, e.g., [22, Theorem 9.45]) to conclude existence, while the strict monotonicity guaranteed by the quadratic part also ensures uniqueness. To confirm the validity of (2.4) with the above constants, let $\mathcal{A}$ be the linear operator defined by $\langle w, \mathcal{A}v\rangle = a(w, v)$, for all $w, v \in \mathcal{H}$, and note that, in view of (3.1),

$$c\|v\|_{\mathcal{H}} \le \|\mathcal{A}v\|_{\mathcal{H}'} \le C\|v\|_{\mathcal{H}}, \quad v \in \mathcal{H}, \tag{3.5}$$

with c, C from (3.1). Since we have $DF(v)w = \mathcal{A}w + G'(v)w$, the assertion follows easily from (3.5), P1, and the monotonicity of G. ∎

Remark 3.2. Alternatively one can argue that, under the above assumptions, G is of potential type so that (3.4) is the Euler equation of a convex minimization problem with a strictly convex functional; see, e.g., [24, Proposition 42.6].

As an example, it is not hard to verify that the weak formulation of the boundary value problem

$$-\Delta u + u^3 = f \quad \text{in} \ \ \Omega, \quad u = 0 \ \ \text{on} \ \ \partial\Omega, \tag{3.6}$$

is of the form (3.4), where for $\mathcal{H} = H_0^1(\Omega)$ the above assumptions hold for $d \le 3$, and that it satisfies the monotonicity assumption of Remark 3.1.

An equivalent ℓ_2-formulation (n2). We turn now to step (n2) in the present setting. In order to rewrite (3.4) in wavelet coordinates, let $\mathbf{A} = (a(\psi_\lambda, \psi_\nu))_{\lambda,\nu\in\mathcal{J}}$ denote the wavelet representation of the operator $\mathcal{A}$, and set $\mathbf{f} = (\langle\psi_\lambda, f\rangle : \lambda \in \mathcal{J})^T$. In addition, define in analogy to (2.7) $\mathbf{G}(\mathbf{v}) := (\langle\psi_\lambda, G(v)\rangle)_{\lambda\in\mathcal{J}}$. Then $u = \sum_{\lambda\in\mathcal{J}} u_\lambda\psi_\lambda$ is the unique solution of (3.4) if and only if $\mathbf{u}$ solves

$$\mathbf{R}(\mathbf{u}) := \mathbf{A}\mathbf{u} + \mathbf{G}(\mathbf{u}) - \mathbf{f} = \mathbf{0}. \tag{3.7}$$

Note that, in view of (2.6), f belongs to $\mathcal{H}'$ if and only if $\mathbf{f} \in \ell_2(\mathcal{J})$. Clearly, by our assumptions on $a(\cdot, \cdot)$, $\mathbf{A}$ is symmetric positive definite. Moreover, it follows from Remarks 2.1 and 3.1 that (2.10) holds for $\mathbf{R}$ in (3.7). In particular, this covers the case $G \equiv 0$, where (2.10) takes the form

$$c_A\|\mathbf{v}\|_{\ell_2(\mathcal{J})} \le \|\mathbf{A}\mathbf{v}\|_{\ell_2(\mathcal{J})} \le C_A\|\mathbf{v}\|_{\ell_2(\mathcal{J})}, \quad \mathbf{v} \in \ell_2(\mathcal{J}), \tag{3.8}$$

with $c_A = c_1^2 c$, $C_A = C_1^2 C$, and c_1, C_1, c, C from (2.5) and (3.1).

We end this section with the simple observation that monotonicity of G carries over into the discrete setting, namely,

$$(\mathbf{u}-\mathbf{v})^T(\mathbf{G}(\mathbf{u})-\mathbf{G}(\mathbf{v}))\geq 0,\quad \mathbf{u},\mathbf{v}\in\ell_2(\mathcal{J}). \tag{3.9}$$

In fact, denoting by $\tilde{\Psi}$ the dual basis to Ψ, we have by definition of $\mathbf{G}(\mathbf{u})$

$$u-v=\sum_{\lambda\in\mathcal{J}}(u_\lambda-v_\lambda)\psi_\lambda,\quad G(u)-G(v)=\sum_{\lambda\in\mathcal{J}}\langle G(u)-G(v),\psi_\lambda\rangle\tilde{\psi}_\lambda.$$

Thus (3.9) follows from monotonicity of G and biorthogonality. Due to (3.1), F is also monotone so that (3.9) holds also for $\mathbf{F}$.

Gradient iterations (n3). We now address (n3) for the above class of semilinear elliptic problems. The simplest option is to take $\mathbf{B}_n=\alpha\mathbf{I}$, which gives the iteration

$$\mathbf{u}^{n+1}=\mathbf{u}^n-\alpha\mathbf{R}(\mathbf{u}^n),\quad n\in\mathbb{N}_0. \tag{3.10}$$

We have to find some $\alpha>0$ for which this iteration converges in $\ell_2(\mathcal{J})$ with a guaranteed error reduction $\rho<1$. To this end, note that, by (2.8), $\mathbf{u}^{n+1}-\mathbf{u}=\mathbf{u}^n-\mathbf{u}-\alpha(\mathbf{R}(\mathbf{u}^n)-\mathbf{R}(\mathbf{u}))$, so that

$$\begin{aligned}\mathbf{u}^{n+1}-\mathbf{u}&=\left(\mathbf{I}-\alpha\int_0^1(\mathbf{A}+D\mathbf{G}(\mathbf{u}+s(\mathbf{u}^n-\mathbf{u})))ds\right)(\mathbf{u}^n-\mathbf{u})\\ &=:(\mathbf{I}-\alpha\mathbf{M}(\mathbf{u}^n,\mathbf{u}))(\mathbf{u}^n-\mathbf{u}).\end{aligned} \tag{3.11}$$

By (3.9) and (3.8), the smallest eigenvalue of the matrix $\mathbf{M}(\mathbf{u}^n,\mathbf{u})$ is bounded from below by c_A. To bound the spectral radius of $\mathbf{M}(\mathbf{u}^n,\mathbf{u})$, note that, by (2.5) and (2.6), one has $\|\mathbf{G}(\mathbf{v})-\mathbf{G}(\mathbf{w})\|_{\ell_2(\mathcal{J})}\leq\hat{C}(\max\{\|\mathbf{v}\|_{\ell_2(\mathcal{J})},\|\mathbf{w}\|_{\ell_2(\mathcal{J})}\})\|\mathbf{v}-\mathbf{w}\|_{\ell_2(\mathcal{J})}$, where $\hat{C}(s):=C_1^2C_G(C_1s)$ and $C_1,C_G(s)$ are the constants from (2.5) and (3.3) for G, respectively. It follows from (3.8) and (3.3) in P1 that $\|\mathbf{M}(\mathbf{u}^n,\mathbf{u})\|_{\ell_2(\mathcal{J})\to\ell_2(\mathcal{J})}\leq C_A+\hat{C}(\|\mathbf{u}-\mathbf{u}^n\|_{\ell_2(\mathcal{J})})$. Given some knowledge about the behavior of the stability constant $C_G(s)$ when s increases, we can estimate $\hat{C}(\|\mathbf{u}-\mathbf{u}^n\|_{\ell_2(\mathcal{J})})$ with the aid of an a priori estimate for $\|\mathbf{u}^n-\mathbf{u}\|_{\ell_2(\mathcal{J})}$. To this end, note that again by (2.8) and (3.7), one has for any $\mathbf{v}\in\ell_2(\mathcal{J})$

$$\begin{aligned}\|\mathbf{u}-\mathbf{v}\|_{\ell_2(\mathcal{J})}\|\mathbf{R}(\mathbf{v})\|_{\ell_2(\mathcal{J})}&\geq(\mathbf{u}-\mathbf{v})^T(\mathbf{R}(\mathbf{v})-\mathbf{R}(\mathbf{u}))=(\mathbf{u}-\mathbf{v})^T\big(\mathbf{A}(\mathbf{u}-\mathbf{v})\\ &\quad+\mathbf{G}(\mathbf{u})-\mathbf{G}(\mathbf{v})\big)\geq c_A\|\mathbf{u}-\mathbf{v}\|^2_{\ell_2(\mathcal{J})},\end{aligned}$$

where we have used (3.8) and (3.9) in the last step. Hence we obtain

$$\|\mathbf{u}-\mathbf{v}\|_{\ell_2(\mathcal{J})}\leq c_A^{-1}\|\mathbf{R}(\mathbf{v})\|_{\ell_2(\mathcal{J})},\quad \|\mathbf{u}\|_{\ell_2(\mathcal{J})}\leq c_A^{-1}\big(\|\mathbf{G}(\mathbf{0})\|_{\ell_2(\mathcal{J})}+\|\mathbf{f}\|_{\ell_2(\mathcal{J})}\big). \tag{3.12}$$

Thus, for any fixed initial guess $\mathbf{v}=\mathbf{u}^0$, we have a computable bound

$$\|\mathbf{u}^0-\mathbf{u}\|_{\ell_2(\mathcal{J})}\leq c_A^{-1}\big(\|\mathbf{R}(\mathbf{u}^0)\|_{\ell_2(\mathcal{J})}\big)=:\delta_0 \tag{3.13}$$

and therefore a bound for $\hat{C}(\delta_0)$.

Remark 3.3. Given $\mathbf{u}^0$ and δ_0 from (3.13), suppose that $\alpha>0$ satisfies

$$0<\alpha<2/(C_A+\hat{C}(\delta_0)) \tag{3.14}$$

for $\hat{C}(\delta_0)$ defined above so that

$$\rho=\rho(\alpha):=\max\{|1-c_A\alpha|,|1-\alpha(C_A+\hat{C}(\delta_0))|\}<1. \tag{3.15}$$

Then, denoting by $B_\delta(\mathbf{u})$ the ball of radius δ with center $\mathbf{u}$, we have

$$\sup_{\mathbf{v}\in B_{\delta_0}(\mathbf{u})} \|\mathbf{I}-\alpha\big(\mathbf{A}+D\mathbf{G}(\mathbf{v})\big)\|_{\ell_2(\mathcal{J})\to\ell_2(\mathcal{J})} =: \rho < 1. \tag{3.16}$$

Hence we obtain $\|\mathbf{I}-\alpha\mathbf{M}(\mathbf{u}^n,\mathbf{u})\|_{\ell_2(\mathcal{J})\to\ell_2(\mathcal{J})} \le \rho$, which, in view of (3.11), yields

$$\|\mathbf{u}-\mathbf{u}^n\|_{\ell_2(\mathcal{J})} \le \rho\|\mathbf{u}-\mathbf{u}^{n-1}\|_{\ell_2(\mathcal{J})}, \quad n\in\mathbb{N}. \tag{3.17}$$

Of course, a better error reduction would result from an optimal stage dependent step size α_n. Keeping Remark 3.2 in mind, one can show that (3.7) are the Euler equations of a strictly convex minimization problem on $\ell_2(\mathcal{J})$. For a given $\mathbf{u}^n$ the residual $\mathbf{r} := \mathbf{f}-\mathbf{A}\mathbf{u}^n-\mathbf{G}(\mathbf{u}^n)$ is the direction of the corresponding steepest descent starting from $\mathbf{u}^n$. The minimum along this direction is given by the zero $\alpha=\alpha_n$ of the function $g(\alpha) = \big(\mathbf{f}-\mathbf{A}(\mathbf{u}^n+\alpha\mathbf{r})-\mathbf{G}(\mathbf{u}^n+\alpha\mathbf{r})\big)^T\mathbf{r} = \mathbf{0}$. We shall later discuss ways of approximately evaluating the terms in $g(\alpha)$. Noting that $g'(\alpha) = -\mathbf{r}^T(\mathbf{A}+D\mathbf{G}(\mathbf{u}^n+\alpha\mathbf{r}))\mathbf{r}$, one could think of using such routines for performing a Newton step with the above initial guess for α to solve approximately $g(\alpha)=0$.

We conclude this section with a remark on the linear case (L), which is to find $u\in\mathcal{H}$ such that

$$a(v,u) = \langle v, f\rangle \quad \forall\, v\in\mathcal{H} \iff \mathbf{A}\mathbf{u}=\mathbf{f}. \tag{3.18}$$

Note that (3.5) follows from (3.1) but may still hold for indefinite problems, which still implies, in view of Remark 2.1, the validity of (3.8). In this case, when $G\equiv 0$, the matrix $\mathbf{A}^T\mathbf{A}$ is symmetric positive definite, and the iteration

$$\mathbf{u}^{n+1} = \mathbf{u}^n - \alpha\mathbf{A}^T(\mathbf{A}\mathbf{u}^n-\mathbf{f}), \quad n=0,1,2,\dots, \tag{3.19}$$

converges with a fixed error reduction $\rho<1$, provided that $0<\alpha<2/C_A^2$; i.e., (3.19) has the form (2.11) with $\mathbf{B}_n := \alpha\mathbf{A}^T$. An analogue for the general nonlinear case (GNL) will be given below.

For saddle point problems there are actually alternatives that avoid squaring the problem (in wavelet coordinates). One option is to employ an Uzawa iteration for applying the Schur complement operator, which conceptually also leads to an iteration of the form (3.10) for the Schur complement [13, 17].

Of course, in either case, step (n4) requires eventually approximating the *weighted residual* $\mathbf{B}_n\mathbf{R}(\mathbf{u}^n)$, which in the above linear case amounts to approximating $\mathbf{f}$ and approximately evaluating the infinite matrix $\mathbf{A}$ (respectively, $\mathbf{A}^T$). We shall address this issue later in some detail.

3.2. The general nonlinear case—locally convergent schemes (GNL). While the assumptions in the previous setting allow us to conclude convergence of the ideal infinite dimensional scheme for *any* initial guess $\mathbf{u}^0$, one often has to be content with weaker assumptions (and correspondingly weaker conclusions). In the literature, variational problems of the type (2.2) are frequently studied under general assumptions on R, such as A1 and A2, that typically guarantee local convergence of an iterative scheme to a locally unique solution provided that a sufficiently good initial guess is known; see, e.g., [21, 23].

Our plan here is to exemplify the above paradigm under assumptions A1 and A2, provided that a sufficiently good initial approximation is known. According to (n2), we consider again the equivalent formulation (2.8) in wavelet coordinates and turn to

devising a suitable iteration of the form (2.11) that converges for a sufficiently good initial guess. To this end, we assume that

$$\mathbf{u}^0 \in B_\delta(\mathbf{u}) := \{\mathbf{v} : \|\mathbf{v}-\mathbf{u}\|_{\ell_2(\mathcal{J})} < \delta\}, \tag{3.20}$$

where δ will be specified below.

As mentioned before, a possible choice for $\mathbf{B}_n$ could involve the Jacobian, which leads to Newton's method. However, under the above weak assumptions on R, we wish to avoid at this point requiring higher order smoothness conditions and consider first the following much simpler option. An analogue to the least squares iteration (3.19) would be $\mathbf{B}_n := D\mathbf{R}(\mathbf{u}^n)^T$. An even simpler alternative, which is presumably less computationally demanding, is to take the *stationary* matrix

$$\mathbf{B} = D\mathbf{R}(\mathbf{u}^0)^T, \tag{3.21}$$

provided that δ is sufficiently small. Let us point out next that for a sufficiently good initial guess $\mathbf{u}^0$

$$\mathbf{W}(\mathbf{v}) := \mathbf{v} - \alpha D\mathbf{R}(\mathbf{u}^0)^T\mathbf{R}(\mathbf{v})$$

is a contraction on $B_\delta(\mathbf{u})$. In fact,

$$\begin{aligned}\mathbf{W}(\mathbf{z}) - \mathbf{W}(\mathbf{v}) &= (\mathbf{z}-\mathbf{v}) - \alpha D\mathbf{R}(\mathbf{u}^0)^T(\mathbf{R}(\mathbf{z})-\mathbf{R}(\mathbf{v}))\\ &= \left(\mathbf{I} - \alpha D\mathbf{R}(\mathbf{u}^0)^T D\mathbf{R}(\mathbf{v})\right)(\mathbf{z}-\mathbf{v}) + \mathrm{o}(\|\mathbf{z}-\mathbf{v}\|_{\ell_2(\mathcal{J})})\\ &= \left(\mathbf{I} - \alpha D\mathbf{R}(\mathbf{u}^0)^T D\mathbf{R}(\mathbf{u}^0)\right)(\mathbf{z}-\mathbf{v}) + \mathrm{o}(\|\mathbf{z}-\mathbf{v}\|_{\ell_2(\mathcal{J})})\\ &\quad + \mathrm{O}(\epsilon(\delta)\|\mathbf{z}-\mathbf{v}\|_{\ell_2(\mathcal{J})}),\end{aligned} \tag{3.22}$$

where we have used assumption A1 and where $\epsilon(\delta)$ tends to zero as $\delta \to 0$. By A1 and A2, $DR(u^0)$ is still an isomorphism from $\mathcal{H}$ onto $\mathcal{H}'$ when δ is sufficiently small. Thus, by Remark 2.1, the positive definite matrix $D\mathbf{R}(\mathbf{u}^0)^T D\mathbf{R}(\mathbf{u}^0)$ is an automorphism on $\ell_2(\mathcal{J})$. Therefore, for $\alpha > 0$ satisfying

$$\alpha\|D\mathbf{R}(\mathbf{u}^0)^T D\mathbf{R}(\mathbf{u}^0)\|_{\ell_2(\mathcal{J})\to\ell_2(\mathcal{J})} < 2, \tag{3.23}$$

$\mathbf{W}$ is a contraction on $B_\delta(\mathbf{u})$. Furthermore, the iterates

$$\mathbf{u}^{n+1} = \mathbf{u}^n - \alpha D\mathbf{R}(\mathbf{u}^0)^T\mathbf{R}(\mathbf{u}^n), \quad n = 0, 1, \ldots, \tag{3.24}$$

stay in $B_\delta(\mathbf{u})$. In fact, as above,

$$\begin{aligned}\mathbf{u}^{n+1} - \mathbf{u} &= \mathbf{u}^n - \mathbf{u} - \alpha D\mathbf{R}(\mathbf{u}^0)^T(\mathbf{R}(\mathbf{u}^n) - \mathbf{R}(\mathbf{u}))\\ &= (\mathbf{I} - \alpha D\mathbf{R}(\mathbf{u}^0)^T D\mathbf{R}(\mathbf{u}^0))(\mathbf{u}^n - \mathbf{u}) + \mathrm{o}(\|\mathbf{u}^n - \mathbf{u}\|_{\ell_2(\mathcal{J})})\\ &\quad + \mathrm{O}(\epsilon(\delta)\|\mathbf{u}^n - \mathbf{u}\|_{\ell_2(\mathcal{J})}).\end{aligned}$$

Hence, for α as above and δ sufficiently small, i.e., $\|\mathbf{I} - \alpha D\mathbf{R}(\mathbf{u}^0)^T D\mathbf{R}(\mathbf{u}^0)\|_{\ell_2(\mathcal{J})\to\ell_2(\mathcal{J})} =: b < 1$ and $\mathrm{o}(1) + \mathrm{O}(\epsilon(\delta)) < 1 - b$, one has $\|\mathbf{u}^{n+1} - \mathbf{u}\|_{\ell_2(\mathcal{J})} < \delta$. We can summarize these observations as follows.

Remark 3.4. Under the above assumptions there exist a $\delta_0 > 0$ and a positive α such that for any $\delta \le \delta_0$ and $\mathbf{u}^0 \in B_\delta(\mathbf{u})$ the iteration (3.24) converges to the locally unique solution $\mathbf{u}$ of (2.8). Moreover, there exists some $\rho < 1$ such that

$$\|\mathbf{u}^n - \mathbf{u}\|_{\ell_2(\mathcal{J})} \le \rho\|\mathbf{u}^{n-1} - \mathbf{u}\|_{\ell_2(\mathcal{J})}, \quad n = 1, 2, \ldots. \tag{3.25}$$

4. A perturbed first order iteration scheme. We shall now turn to step (n4) under the assumption that (2.11) gives rise to a fixed error reduction ρ per iteration step. Recall that by (3.17) and (3.25), this is indeed already the case for (L), (SL), and (GNL) for the corresponding stationary choices of $\mathbf{B}_n = \mathbf{B}$. In order to minimize technicalities we shall consider only this case in connection with such first order schemes. In order to arrive at computable versions of these schemes, we have to *approximate* the weighted residuals $\mathbf{BR}(\mathbf{u}^n)$ in each step. Already in the linear case (L), this requires approximating the application of an infinite matrix to a finitely supported vector and approximating the given data $\mathbf{f}$. In the nonlinear cases (SL), (GNL), the additional difficulty is to approximately evaluate the *nonlinear* expressions $\mathbf{R}(\mathbf{u}^j)$.

Our strategy can be outlined as follows. In the present section we shall address only issue (a) from section 1.2, namely, How accurate must these approximations be to be chosen at a given stage of the iteration so as to guarantee convergence to the correct solution? We shall do so at this point under the *assumption* that a subroutine for approximating the weighted residuals $\mathbf{BR}(\mathbf{v})$ with desired accuracy is at our disposal. Once (a) has been clarified for the general scope of problems, we shall in subsequent sections then narrow down step by step the specific requirements on the basic subroutine, develop concrete realizations for the various problem types (L), (SL), and (GNL), and analyze their complexity.

Thus for the time being we assume now that for $\mathbf{R}(\cdot) = \mathbf{F}(\cdot) - \mathbf{f}$ a routine with the following property is given.

RES $[\eta, \mathbf{B}, \mathbf{F}, \mathbf{f}, \mathbf{v}] \to \mathbf{w}_\eta$ *determines for any positive tolerance* η *and any finitely supported input* $\mathbf{v}$ *a finitely supported* $\mathbf{w}_\eta$ *satisfying*

$$\|\mathbf{BR}(\mathbf{v}) - \mathbf{w}_\eta\|_{\ell_2(\mathcal{J})} \le \eta. \tag{4.1}$$

The need for the following further ingredient is at this point less obvious. It will be applied after a certain finite number of perturbed iterations based on the application of **RES**. It will be seen later that this is crucial for controlling the complexity of the scheme.

CCOARSE $[\eta, \mathbf{v}] \to \mathbf{w}_\eta$ *determines for any positive tolerance* η *and any finitely supported input vector* $\mathbf{v}$ *a finitely supported output vector* $\mathbf{w}_\eta$ *such that*

$$\|\mathbf{v} - \mathbf{w}_\eta\|_{\ell_2(\mathcal{J})} \le \eta, \tag{4.2}$$

while the support of $\mathbf{w}_\eta$ *is minimized subject to certain constraints on the distribution of its entries.*

The constraints mentioned in **CCOARSE** will depend on the particular application and will be specified later. A perturbed iteration based on these ingredients requires specifying a suitable initialization.

Initialization. We distinguish the following three cases for the choice of the initial guess.

(L) In the linear case $\mathbf{R}(\mathbf{v}) = \mathbf{A}\mathbf{v} - \mathbf{f}$, we can set $\mathbf{u}^0 := \mathbf{0}$ so that an initial error bound is given, directly in view of (3.8), by $\|\mathbf{u}-\mathbf{u}^0\|_{\ell_2(\mathcal{J})} = \|\mathbf{u}\|_{\ell_2(\mathcal{J})} \le c_A^{-1}\|\mathbf{f}\|_{\ell_2(\mathcal{J})} =: \epsilon_0$. Moreover, in the positive definite case, any fixed $\alpha < 2/C_A$ in $\mathbf{B} = \alpha\mathbf{I}$ and $\alpha < 2/C_A^2$ in $\mathbf{B} = \alpha\mathbf{A}^T$ for the general least squares formulation ensure that $\mathbf{I} - \mathbf{BA}$ is a contraction on $B = \ell_2(\mathcal{J})$.

(SL) In the case of semilinear elliptic problems (3.7), (3.4), we recall from (3.12) that for $\mathbf{u}^0 = \mathbf{0}$ the initial error is bounded by

$$\|\mathbf{u}\|_{\ell_2(\mathcal{J})} \le c_A^{-1}\big(\|\mathbf{G}(\mathbf{0})\|_{\ell_2(\mathcal{J})} + \|\mathbf{f}\|_{\ell_2(\mathcal{J})}\big) =: \epsilon_0. \tag{4.3}$$

We choose $B = B_{2\epsilon_0}(\mathbf{u})$ and $\mathbf{B} := \alpha\mathbf{I}$ for a fixed $\alpha < 2/(C_A + \hat{C}(2\epsilon_0))$, $\alpha \leq 1$, so that (3.16) holds for $\delta_0 = 2\epsilon_0$ and $\rho = \rho(\alpha) < 1$, defined in (3.15).

(GNL) For the locally convergent scheme, we adhere to the assumptions made in section 3.2. For any fixed $\delta < \delta_0$ (the parameter from Remark 3.4) which satisfies $(1+\alpha)\delta < \delta_0$, where α is the constant from (3.23), we choose $\mathbf{u}^0$ according to (3.20). In this case, we have $\mathbf{B} = \alpha D\mathbf{R}(\mathbf{u}^0)^T$, and $\epsilon_0 := \delta$ is a valid initial error bound which ensures that for $\mathbf{v} \in B := B_\delta(\mathbf{u})$ the matrix $\mathbf{I} - \mathbf{B}D\mathbf{R}(\mathbf{v})$ is a contraction.

Thus in all cases (L), (SL), and (GNL) one has under the above premises

$$\|\mathbf{u} - \mathbf{u}^0\|_{\ell_2(\mathcal{J})} \leq \epsilon_0. \tag{4.4}$$

In order to control the perturbations caused by applications of **RES**, it will be convenient to extract the following fact from the above considerations.

Remark 4.1. For each of the above choices of $\mathbf{B}$ in (L), (SL), and (GNL) and the respective neighborhoods B of the exact solution $\mathbf{u}$ specified in the initialization, one has

$$\|(\mathbf{v} - \mathbf{z}) - \mathbf{B}(\mathbf{R}(\mathbf{v}) - \mathbf{R}(\mathbf{z}))\|_{\ell_2(\mathcal{J})} \leq \rho\|\mathbf{v} - \mathbf{z}\|_{\ell_2(\mathcal{J})}, \quad \mathbf{v}, \mathbf{z} \in B, \tag{4.5}$$

where $\rho < 1$ is the respective error reduction rate in (3.17) for the iteration (2.11).

Proof. The linear case (L) is obvious.

In the case (SL) of the semilinear elliptic problem (3.4), respectively, (3.7), one has for $\mathbf{B} = \alpha\mathbf{I}$ (with α specified in the initialization), by the same reasoning used in (3.11),

$$\mathbf{v} - \mathbf{z} - \alpha(\mathbf{R}(\mathbf{v}) - \mathbf{R}(\mathbf{z})) = \left(\mathbf{I} - \alpha\left(\mathbf{A} + \int_0^1 D\mathbf{G}(\mathbf{z} + s(\mathbf{v} - \mathbf{z}))ds\right)\right)(\mathbf{v} - \mathbf{z}).$$

The assertion follows then from (3.16).

Finally, for (GNL) (see section 3.2), the claim follows from (3.22) for $\mathbf{B} = \alpha D\mathbf{R}(\mathbf{u}^0)^T$ and α satisfying (3.23) with $B = B_\delta(\mathbf{u})$. ∎

The following last prerequisite will allow us to control the number of calls of **RES** before applying a coarsening step.

Remark 4.2. For each of the above choices of $\mathbf{B}$ in (L), (SL), and (GNL) and for the respective neighborhoods B of the exact solution $\mathbf{u}$, there exists a positive finite constant β such that

$$\|\mathbf{u} - \mathbf{v}\|_{\ell_2(\mathcal{J})} \leq \beta\|\mathbf{B}\mathbf{R}(\mathbf{v})\|_{\ell_2(\mathcal{J})}, \quad \mathbf{v} \in B. \tag{4.6}$$

Proof. Of course, in principle, this follows from (4.5) by the triangle inequality. However, β then depends on ρ for which only a poor estimate may be available. For (L) and (SL) one obtains better bounds as follows. From (3.7) and (3.9) we infer that

$$\begin{aligned}\|\mathbf{R}(\mathbf{v})\|_{\ell_2(\mathcal{J})} &= \sup_{\mathbf{z}\neq\mathbf{0}} \frac{\mathbf{z}^T(\mathbf{A}(\mathbf{v} - \mathbf{u}) + \mathbf{G}(\mathbf{v}) - \mathbf{G}(\mathbf{u}))}{\|\mathbf{z}\|_{\ell_2(\mathcal{J})}} \geq \frac{(\mathbf{v} - \mathbf{u})^T\mathbf{A}(\mathbf{v} - \mathbf{u})}{\|\mathbf{v} - \mathbf{u}\|_{\ell_2(\mathcal{J})}} \\ &\geq c_A\|\mathbf{v} - \mathbf{u}\|_{\ell_2(\mathcal{J})}.\end{aligned}$$

Thus (4.6) holds with $\beta = 1/(\alpha c_A)$. Similarly $\beta = 1/(\alpha c_A^2)$ works for the least squares formulation (3.19). For (GNL) we have by our assumptions that $\|\mathbf{B}(\mathbf{R}(\mathbf{v}) - D\mathbf{R}(\mathbf{v}))(\mathbf{v} - \mathbf{u})\|_{\ell_2(\mathcal{J})} \leq o(\|\mathbf{v} - \mathbf{u}\|_{\ell_2(\mathcal{J})})$ so that in a sufficiently small neighborhood of $\mathbf{u}$ (4.6) follows from the well-posedness relation (2.10). ∎

We can now describe our computable analogue of (2.11). For this we choose any fixed summable sequence $(\omega_j)_{j\in\mathbb{N}_0}$, which, for convenience, we arrange to sum to one $\sum_{j=0}^{\infty}\omega_j = 1$, and a fixed constant C^* which depends on the specific realization of the routine **CCOARSE**; see sections 5.2 and 6.2.

SOLVE $[\epsilon, \mathbf{R}, \mathbf{u}^0] \to \bar{\mathbf{u}}(\epsilon)$

(i) Choose some $\bar\rho \in (0,1)$. Set $\bar{\mathbf{u}}^0 = \mathbf{u}^0$ and the corresponding initial bound ϵ_0 according to the above initialization, and define $j = 0$;

(ii) If $\epsilon_j \le \epsilon$, stop and output $\bar{\mathbf{u}}(\epsilon) := \bar{\mathbf{u}}^j$; else set $\mathbf{v}^0 := \bar{\mathbf{u}}^j$ and $k = 0$

(ii.1) Set $\eta_k := \omega_k \bar\rho^k \epsilon_j$ and compute

$$\mathbf{r}^k = \mathbf{RES}\,[\eta_k, \mathbf{B}, \mathbf{F}, \mathbf{f}, \mathbf{v}^k], \qquad \mathbf{v}^{k+1} = \mathbf{v}^k - \mathbf{r}^k.$$

(ii.2) If

$$\beta\big(\eta_k + \|\mathbf{r}^k\|_{\ell_2(\mathcal{J})}\big) \le \epsilon_j/(2(1+2C^*)), \tag{4.7}$$

set $\tilde{\mathbf{v}} := \mathbf{v}^k$ and go to (iii). Else set $k+1 \to k$ and go to (ii.1).

(iii) **CCOARSE** $[\frac{2C^*\epsilon_j}{2(1+2C^*)}, \tilde{\mathbf{v}}] \to \bar{\mathbf{u}}^{j+1}$, $\epsilon_{j+1} = \epsilon_j/2$, $j+1 \to j$, go to (ii).

Let us confirm first that the choice of accuracy tolerances in **SOLVE** implies convergence.

PROPOSITION 4.3. *The iterates $\bar{\mathbf{u}}^j$ produced by the scheme* **SOLVE** *satisfy*

$$\|\mathbf{u} - \bar{\mathbf{u}}^j\|_{\ell_2(\mathcal{J})} \le \epsilon_j \tag{4.8}$$

so that, in particular, $\|\mathbf{u} - \bar{\mathbf{u}}(\epsilon)\|_{\ell_2(\mathcal{J})} \le \epsilon$. By (2.5), this means

$$\left\| u - \sum_{\lambda\in\Lambda(\epsilon)} \bar{u}(\epsilon)_\lambda \psi_\lambda \right\|_{\mathcal{H}} \le C_1\epsilon, \tag{4.9}$$

where C_1 is the constant from (2.5) and $\Lambda(\epsilon) := \operatorname{supp}\mathbf{u}(\epsilon)$.

Moreover, the number of updates in step (ii.1) prior to a coarsening step is uniformly bounded by some fixed $K \in N$, independent of ϵ and the data.

Proof. We assume the above initialization and employ a simple perturbation argument using induction on j. We fix a value of j and let $\mathbf{u}^k := \mathbf{u}^k(\mathbf{v}^0)$ be the exact iterates $\mathbf{u}^{k+1} = \mathbf{u}^k - \mathbf{BR}(\mathbf{u}^k)$ with initial guess $\mathbf{u}^0 = \mathbf{v}^0 = \bar{\mathbf{u}}^j$. Hence

$$\begin{aligned} \mathbf{v}^{k+1} - \mathbf{u}^{k+1} &= \mathbf{v}^k - \mathbf{u}^k - (\mathbf{r}^k - \mathbf{BR}(\mathbf{u}^k)) \\ &= \mathbf{v}^k - \mathbf{u}^k - \mathbf{B}(\mathbf{R}(\mathbf{v}^k) - \mathbf{R}(\mathbf{u}^k)) + (\mathbf{BR}(\mathbf{v}^k) - \mathbf{r}^k). \end{aligned} \tag{4.10}$$

Next we wish to invoke (4.5). To do this we need to make sure that the iterates $\mathbf{v}^k, \mathbf{u}^k$ stay in the neighborhood B mentioned in Remark 4.1. In the linear case (L) there is no constraint, i.e., $B = \ell_2(\mathcal{J})$. Let us look at the semilinear case (SL) next. By the induction assumption we know that $\|\mathbf{u} - \bar{\mathbf{u}}^j\|_{\ell_2(\mathcal{J})} \le \epsilon_j \le \epsilon_0$. Therefore, $\|\mathbf{u} - \mathbf{u}^k\|_{\ell_2(\mathcal{J})} \le \rho^k\|\mathbf{u} - \mathbf{u}^0\|_{\ell_2(\mathcal{J})} \le \rho^k\|\mathbf{u} - \bar{\mathbf{u}}^j\|_{\ell_2(\mathcal{J})} \le \rho^k\epsilon_j$. So $\mathbf{u}^k \in B$ for all $k \le K$. Also $\mathbf{v}^0 = \bar{\mathbf{u}}^j \in B$. Thus suppose that $\mathbf{v}^k$ is in B. We wish to show that then also $\mathbf{v}^{k+1} \in B$. To this end, let ρ_* be the true reduction rate in (2.11) (for which $\rho(\alpha)$ from (3.15) might be a poor estimate) and set $\hat\rho := \max\{\rho_*, \bar\rho\}$. Then we infer from (4.5), (4.10), and the definition of $\mathbf{r}^k$ in step (ii) that

$$\begin{aligned} \|\mathbf{v}^{k+1} - \mathbf{u}^{k+1}\|_{\ell_2(\mathcal{J})} &\le \rho_*\|\mathbf{v}^k - \mathbf{u}^k\|_{\ell_2(\mathcal{J})} + \omega_k\bar\rho^k\epsilon_j \\ &\le \left(\sum_{l=0}^{k} \rho_*^{k-l}\omega_l\bar\rho^l\right)\epsilon_j \le \hat\rho^k\epsilon_j, \end{aligned} \tag{4.11}$$

where we have used that $\mathbf{u}^0 = \mathbf{v}^0$. Moreover, since

$$\|\mathbf{v}^{k+1} - \mathbf{u}\|_{\ell_2(\mathcal{J})} \le \hat{\rho}^k \epsilon_j + \|\mathbf{u}^{k+1} - \mathbf{u}\|_{\ell_2(\mathcal{J})} \le 2\hat{\rho}^k \epsilon_j, \tag{4.12}$$

we see that $\mathbf{v}^{k+1} \in B = B_{2\epsilon_0}(\mathbf{u})$ so that the iteration can be advanced.

For the locally convergent scheme (GNL) with $\mathbf{B} = \alpha \mathbf{R}(\mathbf{u}^0)^T$, the reasoning is analogous. The choice of the initial guess ensures that $(\rho + \alpha)\epsilon_j \le (\rho + \alpha)\epsilon_0 \le (1 + \alpha)\epsilon_0 \le \delta_0$. Then the above arguments for (SL) yield again (4.15) so that all iterates stay in $B = B_{\delta_0}(\mathbf{u})$.

Now note that by (4.12),

$$\|\mathbf{r}^k\|_{\ell_2(\mathcal{J})} \le \|\mathbf{v}^{k+1} - \mathbf{u}\|_{\ell_2(\mathcal{J})} + \|\mathbf{v}^k - \mathbf{u}\|_{\ell_2(\mathcal{J})} \le 4\hat{\rho}^{k-1}\epsilon_j$$

so that

$$\beta(\eta_k + \|\mathbf{r}^k\|_{\ell_2(\mathcal{J})}) \le \beta\epsilon_j(\omega_k\hat{\rho} + 4)\hat{\rho}^{k-1}. \tag{4.13}$$

Hence, choosing

$$K := \min\{k \in \mathbb{N} : \beta(\omega_k\hat{\rho} + 4)\hat{\rho}^{k-1} \le 1/(2(1 + 2C^*))\}, \tag{4.14}$$

we see that (4.7) is met after at most K steps. Moreover, (4.6) says that

$$\|\mathbf{u} - \tilde{\mathbf{v}}\|_{\ell_2(\mathcal{J})} \le \frac{\epsilon_j}{2(1 + 2C^*)}. \tag{4.15}$$

Thus in all cases the estimate (4.8) follows now immediately from step (iii) in **SOLVE**, the definition of **CCOARSE**, and (4.2). Finally, (4.9) is an immediate consequence of the norm equivalence (2.5) and (4.8). □

Thus, for an idealized infinite dimensional scheme of order one in (n3), we know how to choose the tolerances in the routines **RES** and **CCOARSE** so as to guarantee convergence. Moreover, the true error reduction rate need not be known, and one can use any (possibly too optimistic) guess $\bar{\rho}$. Of course, choosing $\bar{\rho}$ too small, the intermediate tolerances get perhaps unnecessarily small.

Our paradigm for solving nonlinear problems is built on the availability of numerical algorithms such as **CCOARSE** and **RES**. The remainder of this paper shows how to construct concrete practical realizations of these algorithms in various settings and then shows how, under suitable controls on the computations in these algorithms, we can give complexity estimates for the entire numerical scheme **SOLVE**. More precisely, we wish to determine its *work/accuracy balance*, i.e., given any target accuracy ϵ, how many degrees of freedom $N = N(\epsilon) := \#\Lambda(\epsilon)$, where $\Lambda(\epsilon) := \operatorname{supp} \bar{\mathbf{u}}(\epsilon)$, are needed to achieve it, and what is the associated (asymptotic) computational work. Of course, one hopes to keep the latter quantity proportional to $N(\epsilon)$ so that the number of degrees of freedom is a reasonable complexity measure. In the following section we shall address these issues first for the linear case (3.18). We review quickly the relevant facts from [8, 9] tailored somewhat to the present situation. On one hand, they will serve as building blocks for the general nonlinear case. On the other hand, they also help to bring out some conceptual distinctions.

5. Realization and complexity analysis in the linear case (L). Recall from (3.10) that in the linear case, $\mathbf{BR}(\mathbf{v}) = \alpha(\mathbf{A}\mathbf{v} - \mathbf{f})$ (or $\alpha\mathbf{A}^T(\mathbf{A}\mathbf{v} - \mathbf{f})$). Thus one part of approximating the residual is to approximate given data, here in the form of the right-hand side $\mathbf{f}$, which, in general, is an infinite sequence.

5.1. Coarsening and best N-term approximation. We will also assume in what follows that all coefficients of $\mathbf{f}$ are *known* and thus in principle accessible. In practice this may require a preprocessing step that computes for some overall target accuracy $\bar\epsilon$ (depending on the desired solution accuracy) an approximation $\mathbf{f}_{\bar\epsilon}$ satisfying $\|\mathbf{f}-\mathbf{f}_{\bar\epsilon}\|_{\ell_2(\mathcal{J})}\le\bar\epsilon$ and then orders the entries by size. Once this has been done, any coarser approximations needed in the course of the iteration process can be produced by the following simplest version of **CCOARSE**, introduced and analyzed in [8].

COARSE $[\eta,\mathbf{v}]\to\mathbf{v}_\eta$ *associates with any finitely supported input* $\mathbf{v}$ *a vector* $\mathbf{v}_\eta$ *such that*

$$\|\mathbf{v}-\mathbf{v}_\eta\|_{\ell_2(\mathcal{J})}\le\eta,\quad \#\operatorname{supp}\mathbf{w}\ge\#\operatorname{supp}\mathbf{v}_\eta,\ \text{whenever}\ \ \|\mathbf{v}-\mathbf{w}\|_{\ell_2(\mathcal{J})}\le\eta. \tag{5.1}$$

Thus **COARSE** determines for a given finitely supported vector a new vector with the smallest possible support deviating no more than a prescribed tolerance from the input. There is no constraint on the distribution of active indices in this case. Ordering the entries of $\mathbf{v}$ sizewise, this can be realized by summing entries in increasing order until the sum of their squares reaches η^2. For a detailed description of this routine, see [8]. In fact, a strict ordering is not necessary. The same effect is realized by collecting the entries in binary bins, which avoids a log factor at the expense of a fixed factor in the accuracy tolerance [1].

The routine **COARSE** can be used to approximate the data $\mathbf{f}$ as follows:

$$\mathbf{RHS}\,[\eta,\mathbf{f}]:=\mathbf{COARSE}\,[\eta-\bar\epsilon,\mathbf{f}_{\bar\epsilon}], \tag{5.2}$$

whenever $\eta>\bar\epsilon$.

Note that **COARSE** is a *nonlinear* process that realizes a given accuracy tolerance at the expense of a minimal number of degrees of freedom. It is therefore a version of *best N-term approximation* in $\ell_2(\mathcal{J})$. In fact, defining

$$\sigma_{N,\ell_2(\mathcal{J})}(\mathbf{u}):=\min_{\#\operatorname{supp}\mathbf{v}\le N}\|\mathbf{u}-\mathbf{v}\|_{\ell_2(\mathcal{J})}, \tag{5.3}$$

one has for any $\mathbf{v}\in\ell_2(\mathcal{J})$

$$\sigma_{N,\ell_2(\mathcal{J})}(\mathbf{v})=\|\mathbf{v}-\mathbf{v}_N\|_{\ell_2(\mathcal{J})}=\left(\sum_{n>N}|v_n^*|^2\right)^{1/2}, \tag{5.4}$$

where $(v_n^*)_{n\in\mathbb{N}}$ is the *nonincreasing rearrangement* of $\mathbf{v}$. Thus $\mathbf{v}_N$ is obtained by retaining the N largest (in modulus) terms of $\mathbf{v}$ and setting all other entries to zero. Depending on the context, $\mathbf{v}_N$ will be viewed as a sequence in $\ell_2(\mathcal{J})$ or a vector in $\mathbb{R}^N$.

The best N-term approximation sets a lower bound for the complexity that could ever be achieved by a scheme like **SOLVE**. In fact, it will serve as our benchmark in the case of linear variational problems of the form (3.18). In order to make this precise, we introduce the corresponding *approximation classes*

$$\mathcal{A}^s:=\{\mathbf{v}\in\ell_2(\mathcal{J}):\sigma_{N,\ell_2(\mathcal{J})}(\mathbf{v})\ \lesssim\ N^{-s}\},$$

which we endow with the quasi norm $\|\mathbf{v}\|_{\mathcal{A}^s}:=\sup_{N\in\mathbb{N}}N^s\sigma_{N,\ell_2(\mathcal{J})}(\mathbf{v})$. Clearly, every $\mathbf{v}$ with finite support belongs to $\mathcal{A}^s$ for any $s>0$. The question is, Does **SOLVE** produce for any target accuracy ϵ an approximate solution within that tolerance at a computational expense that stays bounded by $C\epsilon^{-1/s}$ whenever the exact solution belongs to $\mathcal{A}^s$, at least for some range of $s>0$?

5.2. Adaptive application of compressible matrices. It remains to approximate the action of $\mathbf{A}$ on a finitely supported vector $\mathbf{v}$. While the treatment of the right-hand side data has been already seen to comply with best N-term approximation complexity, the question arises whether $\mathbf{Av}$ can be approximated with a similar efficiency. This has been answered affirmatively in [8], and we briefly recall the relevant facts from there.

Due to the *vanishing moment* property of wavelets, the wavelet representation of many operators turns out to be *quasi-sparse*. The following quantification of sparsity is appropriate [8].

We shall use the notation $(\alpha_j)_{j=1}^{\infty}$ to denote a summable sequence of positive numbers: $\sum_{j=1}^{\infty}\alpha_j<\infty$. A matrix $\mathbf{C}$ is said to be s^*-compressible, $\mathbf{C}\in\mathcal{C}_{s^*}$, if for any $0<s<s^*$ and every $j\in\mathbb{N}$ there exists a matrix $\mathbf{C}_j$ with the following properties: For some summable sequence $(\alpha_j)_{j=1}^{\infty}$, $\mathbf{C}_j$ is obtained by replacing all but the order of $\alpha_j 2^j$ entries per row and column in $\mathbf{C}$ by zero and satisfies

$$\|\mathbf{C}-\mathbf{C}_j\|_{\ell_2(\mathcal{J})\to\ell_2(\mathcal{J})}\le C\alpha_j 2^{-js},\quad j\in\mathbb{N}. \tag{5.5}$$

Specifically, wavelet representations of differential (and also certain singular integral) operators fall into this category. One typically has then estimates of the type

$$|a(\psi_\lambda,\psi_\mu)|\ \lesssim\ 2^{-\sigma||\lambda|-|\mu||}, \tag{5.6}$$

where $\sigma>d/2$ depends on the *regularity* of the wavelets.

In order to describe the essence of an approximate application scheme for compressible matrices, we abbreviate for any finitely supported $\mathbf{v}$ the best 2^j-term approximations by $\mathbf{v}_{[j]}:=\mathbf{v}_{2^j}$ and define

$$\mathbf{w}_j:=\mathbf{A}_j\mathbf{v}_{[0]}+\mathbf{A}_{j-1}(\mathbf{v}_{[1]}-\mathbf{v}_{[0]})+\cdots+\mathbf{A}_0(\mathbf{v}_{[j]}-\mathbf{v}_{[j-1]}) \tag{5.7}$$

as an approximation to $\mathbf{Av}$. Obviously this scheme is *adaptive* in that it exploits directly information on $\mathbf{v}$. In fact, if $\mathbf{A}\in\mathcal{C}_{s^*}$, then the triangle inequality together with the above compression estimates yield for any fixed $s<s^*$

$$\|\mathbf{Av}-\mathbf{w}_j\|_{\ell_2(\mathcal{J})}\le c\left(\underbrace{\|\mathbf{v}-\mathbf{v}_{[j]}\|_{\ell_2(\mathcal{J})}}_{\sigma_{2^j,\ell_2(\mathcal{J})}(\mathbf{v})}+\sum_{l=0}^{j}\alpha_l 2^{-ls}\underbrace{\|\mathbf{v}_{[j-l]}-\mathbf{v}_{[j-l-1]}\|_{\ell_2(\mathcal{J})}}_{\lesssim\ \sigma_{2^{j-l-1},\ell_2(\mathcal{J})}(\mathbf{v})}\right), \tag{5.8}$$

where $\mathbf{v}_{[-1]}:=\mathbf{0}$. One can now exploit the a posteriori information offered by the quantities $\sigma_{2^{j-l-1},\ell_2(\mathcal{J})}(\mathbf{v})$ to choose the smallest j for which the right-hand side of (5.8) is smaller than a given target accuracy η and set $\mathbf{w}_\eta:=\mathbf{w}_j$. Since the sum is finite for each finitely supported input $\mathbf{v}$, such a j does indeed exist. This leads to a concrete multiplication scheme (see [8, 2] for a detailed description, analysis, and implementation) which we summarize as follows.

APPLY $[\eta,\mathbf{A},\mathbf{v}]\to\mathbf{w}_\eta$ *determines for any finitely supported input* $\mathbf{v}$ *a finitely supported output* $\mathbf{w}_\eta$ *such that*

$$\|\mathbf{Av}-\mathbf{w}_\eta\|_{\ell_2(\mathcal{J})}\le\eta. \tag{5.9}$$

The complexity of this scheme would be asymptotically optimal if the size of $\operatorname{supp}(\mathbf{w}_\eta)$ and the corresponding work count remain bounded by $C(N_\eta+\#\operatorname{supp}(\mathbf{v}))$,

where N_η is the smallest N such that $\sigma_{N,\ell_2(\mathcal{J})}(\mathbf{Av}) \le \eta$. We shall see that this is indeed the case for a certain range of decay rates of $\sigma_{N,\ell_2(\mathcal{J})}(\mathbf{v})$.

The main result concerning **APPLY** can be formulated as follows [8].

THEOREM 5.1. *Suppose that* $\mathbf{C} \in \mathcal{C}_{s^*}$ *and that* $0 < s < s^*$. *Then, in addition to* (5.9), *for any input vector* $\mathbf{v}$ *with finite support,* $\mathbf{w}_\eta = \mathbf{APPLY}\,[\eta, \mathbf{C}, \mathbf{v}]$ *satisfies*

(i) $\|\mathbf{w}_\eta\|_{\mathcal{A}^s} \lesssim \|\mathbf{v}\|_{\mathcal{A}^s}$,

(ii) $\#\mathrm{supp}\,\mathbf{w}_\eta \lesssim \|\mathbf{v}\|_{\mathcal{A}^s}^{1/s}\eta^{-1/s}$ *and* $\#\mathrm{flops} \lesssim \#\mathrm{supp}\,\mathbf{v} + \|\mathbf{v}\|_{\mathcal{A}^s}^{1/s}\eta^{-1/s}$,

where the constants in these estimates depend only on s *when* s *is small. Moreover, any* $\mathbf{C} \in \mathcal{C}_{s^*}$ *is bounded on* $\mathcal{A}^s$ *as long as* $s < s^*$.

Thus, when dealing with linear problems (3.18), an approximation within the tolerance $\eta > 0$ to the weighted residual $\mathbf{BR}(\mathbf{v}) = \alpha(\mathbf{Av}-\mathbf{f})$ for any finitely supported input $\mathbf{v}$ can be computed as

$$\mathbf{RES}_{\mathrm{lin}}[\eta, \alpha\mathbf{I}, \mathbf{A}, \mathbf{f}, \mathbf{v}] := \alpha\left(\mathbf{APPLY}\,\left[\frac{\eta}{2\alpha}, \mathbf{A}, \mathbf{v}\right] - \mathbf{RHS}\,\left[\frac{\eta}{2\alpha}, \mathbf{f}\right]\right), \tag{5.10}$$

where **RHS** is given by (5.2). The same ideas can be used in the least squares case (3.19), where again **RHS** can be composed of **COARSE** and **APPLY**; see [9] for details.

Remark 5.1. Since by Theorem 5.1 $\mathbf{f} \in \mathcal{A}^s$, whenever the solution $\mathbf{u}$ belongs to $\mathcal{A}^s$, the above considerations and analogous facts about **COARSE** from [8] show that the output $\mathbf{f}_\eta$ of $\mathbf{RHS}\,[\eta, \mathbf{f}]$ satisfies $\|\mathbf{f}_\eta\|_{\mathcal{A}^s} \lesssim \|\mathbf{u}\|_{\mathcal{A}^s}$ and $\#\mathrm{supp}\,\mathbf{f}_\eta \lesssim \eta^{-1/s}\|\mathbf{u}\|_{\mathcal{A}^s}^{1/s}$.

These observations provide the following result.

PROPOSITION 5.2. *If the sequence of wavelet coefficients* $\mathbf{u}$ *of the exact solution* u *of* (3.18) *belongs to* $\mathcal{A}^s$ *and if* $\mathbf{A}$ *belongs to* $\mathcal{C}_{s^*}$ *with* $s^* > s$, *then, for any finitely supported input* $\mathbf{v}$, *the output* $\mathbf{w}_\eta$ *of the scheme* $\mathbf{RES}_{\mathrm{lin}}\,[\eta, \alpha\mathbf{I}, \mathbf{A}, \mathbf{f}, \mathbf{v}]$ *satisfies*

$$\begin{aligned} \|\mathbf{w}_\eta\|_{\mathcal{A}^s} &\lesssim (\|\mathbf{v}\|_{\mathcal{A}^s} + \|\mathbf{u}\|_{\mathcal{A}^s}), \\ \#\,\mathrm{supp}\,\mathbf{w}_\eta &\lesssim \eta^{-1/s}\left(\|\mathbf{u}\|_{\mathcal{A}^s}^{1/s} + \|\mathbf{v}\|_{\mathcal{A}^s}^{1/s}\right), \quad \eta > 0, \end{aligned} \tag{5.11}$$

where the constants in these estimates depend only on s.

Proposition 5.2 controls the complexity within each iteration block (ii) of perturbed iterations where, however, the constants may build up. To avoid this is exactly the role of step (iii) in **SOLVE**, which is based on the following "coarsening lemma" from [8]. (The following version can be found in [7].)

PROPOSITION 5.3. *Let* a *be some fixed number strictly larger than* 1. *If* $\mathbf{v} \in \mathcal{A}^s$ *and* $\|\mathbf{v} - \mathbf{w}\|_{\ell_2(\mathcal{J})} \le \eta$ *with* $\#\mathrm{supp}\,\mathbf{w} < \infty$, *then* $\bar{\mathbf{w}}_\eta := \mathbf{COARSE}\,[a\eta, \mathbf{w}]$ *satisfies* $\|\mathbf{v} - \bar{\mathbf{w}}_\eta\|_{\ell_2(\mathcal{J})} \le (1+a)\eta$ *and*

$$\#\mathrm{supp}\,\bar{\mathbf{w}}_\eta \lesssim \|\mathbf{v}\|_{\mathcal{A}^s}^{1/s}\eta^{-1/s}, \quad \|\bar{\mathbf{w}}_\eta\|_{\mathcal{A}^s} \lesssim \|\mathbf{v}\|_{\mathcal{A}^s}, \tag{5.12}$$

where the constants in these estimates depend only on s *when* s *becomes small.*

By Proposition 5.3, the coarsening step (iii), with the above algorithm **COARSE** used as **CCOARSE**, pulls a current approximation to the unknown $\mathbf{u}$ toward its best N-term approximation and controls the $\mathcal{A}^s$-norms of the approximations, *independently* of the possible increase of these norms caused by several preceding applications of **RES**, provided that $2C^* > 1$. Thus, in connection with **COARSE**, one can take any fixed $C^* > 1/2$ in (4.7) and step (iii) of **SOLVE**.

Let us denote by $\mathbf{SOLVE}_{\mathrm{lin}}$ the specification of **SOLVE** obtained by using $\mathbf{RES}_{\mathrm{lin}}$ and **COARSE** in place of **RES**, respectively, **CCOARSE**. We emphasize that adaptivity enters the scheme $\mathbf{SOLVE}_{\mathrm{lin}}$ solely through the adaptive application of $\mathbf{A}$ and the residual check (4.7).

Under the above premises, Propositions 5.2 and 5.3 allow one to show that $\mathbf{SOLVE}_{\mathrm{lin}}$ exhibits optimal complexity in the sense that it realizes the Meta-Theorem from section 1.1 with (unconstrained) best N-term approximation as a benchmark.

Remark 5.4. We conclude this section by recalling that $\mathbf{u} \in \mathcal{A}^s$ is, for instance, implied by a certain Besov regularity of u. In fact, when $\mathcal{H} = H^t$ (a Sobolev space of smoothness t), $u \in B_\tau^{t+ds}(L_\tau)$, with $\tau^{-1} = s + 1/2$, implies $\mathbf{u} \in \mathcal{A}^s$. This can be used to identify circumstances under which the adaptive scheme performs asymptotically better than a scheme based on uniform refinements. Recall that $B_\tau^{t+ds}(L_\tau)$ is the "largest" space of smoothness $t + sd$ embedded in H^t.

6. The nonlinear case. In view of the fact that **SOLVE** has the same structure, regardless of whether the involved operators are linear or nonlinear, our strategy will be to follow closely the above lines also when the variational problem (2.2) is nonlinear. In principle, this will prove successful, although some important modifications of the ingredients will be encountered. The main distinction lies in the sparsity measure in that the role of best (unrestricted) N-term approximation will be replaced by *best tree approximation*. This constraint on the distribution of active coefficients arises naturally when analyzing the approximate evaluation of nonlinear expressions $\mathbf{R}(\mathbf{v})$. Moreover, index sets with tree structure are analogous to locally refined meshes in adaptive finite element methods.

6.1. Tree approximation and coarsening. Let us explain first what we mean by a tree structure associated to the set of wavelet indices. In the simplest case of a one dimensional basis $\psi_\lambda = \psi_{j,k} = 2^{j/2}\psi(2^j \cdot -k)$, this structure is obvious: each index (j,k) has two children $(j+1, 2k)$ and $(j+1, 2k+1)$. A similar tree structure can be associated to all available constructions of wavelet basis on a multidimensional domain: to each index λ one can assign $m(\lambda) \geq 2$ children μ such that $|\mu| = |\lambda| + 1$, where $m(\lambda)$ might vary from one index to another but is uniformly bounded by some fixed M. We shall use the notation $\mu \prec \lambda$ ($\mu \preceq \lambda$) in order to express that μ is a descendent of λ (or equals λ) in the tree. We also have the property

$$\mu \prec \lambda \Rightarrow S_\mu \subset S_\lambda, \tag{6.1}$$

where we recall that $S_\lambda := \operatorname{supp} \psi_\lambda$. A set $\mathcal{T} \subset \mathcal{J}$ is called a *tree* if $\lambda \in \mathcal{T}$ implies $\mu \in \mathcal{T}$ whenever $\lambda \prec \mu$.

If the tree $\mathcal{T} \subset \mathcal{J}$ is finite, we define the set $\mathcal{L} = \mathcal{L}(\mathcal{T})$ of *outer leaves* as the set of those indices outside the tree whose parent belongs to the tree

$$\mathcal{L} := \{\lambda \in \mathcal{J} : \lambda \notin \mathcal{T},\ \lambda \prec \mu \implies \mu \in \mathcal{T}\}. \tag{6.2}$$

We shall make use of the easily verifiable relation

$$\#\mathcal{T} \sim \#\mathcal{L}(\mathcal{T}), \tag{6.3}$$

where the constants depend only on the number M of children.

Note that $\mathcal{L}(\mathcal{T})$ plays the role of a (locally refined) mesh. Associating to any sequence $\mathbf{v} = (v_\lambda)$ in $\ell_2(\mathcal{J})$, another sequence $\tilde{\mathbf{v}} = (\tilde{v}_\lambda)$ whose entries are defined by

$$\tilde{v}_\lambda := \left(\sum_{\mu \preceq \lambda} |v_\mu|^2 \right)^{1/2}, \tag{6.4}$$

one readily confirms that $\mu \prec \lambda$ implies $\tilde{v}_\lambda \geq \tilde{v}_\mu$ and

$$\|\mathbf{v} - \mathbf{v}|_{\mathcal{T}}\|^2_{\ell_2(\mathcal{J})} = \sum_{\lambda \in \mathcal{L}(\mathcal{T})} \tilde{v}^2_\lambda. \tag{6.5}$$

Recall that in the linear case we have been able to compare the performance of **SOLVE** with the *best (unconstrained) N-term approximation*. We want now to develop sparsity measures that respect tree structure. Once a suitable measure has been identified, one can follow conceptually the lines of section 3. The counterpart for the spaces $\mathcal{A}^s$ are now the analogous spaces defined via *best tree N-term approximation* where the distribution of active coefficients in an approximant has a tree structure; see [10]. In fact, let

$$\sigma^{\rm tree}_{N,\ell_2(\mathcal{J})}(\mathbf{v}) := \min\{\|\mathbf{v} - \mathbf{w}\|_{\ell_2(\mathcal{J})} : \ \mathcal{T} := \operatorname{supp} \mathbf{w} \ \text{ is a tree and } \ \#\mathcal{T} \leq N\}. \tag{6.6}$$

Any minimizing tree will be called $\mathcal{T}_N(\mathbf{v})$. We define now

$$\mathcal{A}^s_{\rm tree} := \{\mathbf{v} \in \ell_2(\mathcal{J}) : \sigma^{\rm tree}_{N,\ell_2(\mathcal{J})}(\mathbf{v}) \ \lesssim \ N^{-s}\}, \tag{6.7}$$

which again becomes a quasi-normed space under the quasi norm

$$\|\mathbf{v}\|_{\mathcal{A}^s_{\rm tree}} := \sup_{n \in \mathbb{N}} N^s \sigma^{\rm tree}_{N,\ell_2(\mathcal{J})}(\mathbf{v}). \tag{6.8}$$

One can again relate the membership of $\mathbf{v}$ to $\mathcal{A}^s_{\rm tree}$ to the regularity of the corresponding expansion v [10].

Remark 6.1. Let $\mathcal{H} = H^t$ for some $t > 0$. If the wavelet expansion v with coefficient sequence $\mathbf{v}$ belongs to $B^{t+sd}_{\tau'}(L_{\tau'})$ for some τ' satisfying $\tau' > (s + 1/2)^{-1}$, then $\mathbf{v} \in \mathcal{A}^s_{\rm tree}$. Thus, in terms of regularity, $\mathcal{A}^s_{\rm tree}$ differs from $\mathcal{A}^s$ by a little additional regularity imposed on the respective expansions, due to the stronger metric $L_{\tau'}$, $\tau' > \tau$. Thus a tree approximation rate N^{-s} can still be achieved for much larger spaces than H^{t+sd}, which governs the corresponding rate for uniform refinements.

6.2. Tree coarsening. We shall specify next a coarsening routine **CCOARSE** that preserves *tree structures* and, as before, applies to finitely supported sequences. It will be referred to as **TCOARSE**. Its definition requires some preparation. Given $\mathbf{w}$, a tree $\mathcal{T} = \mathcal{T}^*(\eta, \mathbf{w})$ is called η-best for $\mathbf{w}$ if

$$\|\mathbf{w} - \mathbf{w}|_{\mathcal{T}}\|_{\ell_2(\mathcal{J})} \leq \eta \ \text{ and } \ \#\mathcal{T} = \min\{\#\mathcal{T}' : \|\mathbf{w} - \mathbf{w}|_{\mathcal{T}'}\|_{\ell_2(\mathcal{J})} \leq \eta, \ \mathcal{T}' \text{ a tree}\}.$$

Requiring best trees will be too stringent from a practical point of view. Therefore, we shall be content with the following relaxed version. A tree $\mathcal{T} = \mathcal{T}(\eta, \mathbf{w})$ is called (η, C)-*near best* (or briefly near best when the parameters are clear from the context) if

$$\|\mathbf{w} - \mathbf{w}|_{\mathcal{T}}\|_{\ell_2(\mathcal{J})} \leq \eta \ \text{ and } \ \#\mathcal{T} \leq C\#\mathcal{T}^*(\eta/C, \mathbf{w}).$$

The action of **TCOARSE** can now be described as follows.

TCOARSE $[\eta, \mathbf{w}] \to \bar{\mathbf{w}}_\eta$ *determines for a fixed constant $C^* \geq 1$, any finitely supported input* $\mathbf{w}$*, and any tolerance $\eta > 0$ an (η, C^*)-near best tree $\mathcal{T}(\eta, \mathbf{w})$ and sets* $\bar{\mathbf{w}}_\eta := \mathbf{w}|_{\mathcal{T}(\eta,\mathbf{w})}$.

The realization of this routine can be based on either one of the two algorithms for generating near best tree approximations developed in [3]. To apply the results

from [3] in the present situation, the role of the partition P associated in [3] to a tree $\mathcal{T}$ is played here by the set $\mathcal{L}(\mathcal{T})$ of outer leaves; recall (6.5). Moreover, for $\lambda \in \mathcal{L}(\mathcal{T})$, the local error terms for a given $\mathbf{v} \in \ell_2(\mathcal{J})$ are here given by $e(\lambda) := \tilde{v}_\lambda^2$. Obviously, the $e(\lambda)$ are subadditive in the sense of [3]. Hence the results from [3] apply. To use the algorithm from [3], we need to know the values $\tilde{w}_\lambda$, $\lambda \in \mathcal{T}(\operatorname{supp} \mathbf{w})$, the smallest tree containing the support of $\mathbf{w}$. Summing the squares of the entries of $\mathbf{w}$ starting from the leaves of $\mathcal{T}(\operatorname{supp} \mathbf{w})$ and working toward the roots provide these quantities at an expense of $\#\mathcal{T}(\operatorname{supp} \mathbf{w})$ operations. Combining this with Theorem 5.2 from [3] establishes the following fact.

PROPOSITION 6.2. *For any given finitely supported input* $\mathbf{w}$*, the computational cost of the output* $\bar{\mathbf{w}}_\eta$ *produced by* **TCOARSE** $[\eta, \mathbf{w}]$ *remains proportional to* $\#\mathcal{T}(\operatorname{supp} \mathbf{w})$*. The underlying tree* $\mathcal{T}(\eta, \mathbf{w})$ *is* (η, C^*)*-near best, where* C^* *is the constant appearing in the estimate* (5.8) *in Theorem* 5.2 *of* [3].

The routine **TCOARSE** will be used as **CCOARSE** in step (iii) of **SOLVE**. The constant C^* appears in the stopping criterion in step (ii.2) of **SOLVE** and in the coarsening step (iii). In analogy to the linear case, its purpose is to control the $\mathcal{A}^s_{\text{tree}}$-norms of the approximants. This is made precise by the following counterpart to Proposition 5.3.

PROPOSITION 6.3. *If* $\mathbf{v} \in \mathcal{A}^s_{\text{tree}}$ *and* $\|\mathbf{v} - \mathbf{w}\|_{\ell_2(\mathcal{J})} \le \eta$ *with* $\#\operatorname{supp} \mathbf{w} < \infty$*, then* $\bar{\mathbf{w}}_\eta := \mathbf{TCOARSE}\,[2C^*\eta, \mathbf{w}]$ *satisfies*

$$\#\operatorname{supp} \bar{\mathbf{w}}_\eta \;\lesssim\; \|\mathbf{v}\|^{1/s}_{\mathcal{A}^s_{\text{tree}}}\, \eta^{-1/s}, \quad \|\mathbf{v} - \bar{\mathbf{w}}_\eta\|_{\ell_2(\mathcal{J})} \le (1 + 2C^*)\eta, \tag{6.9}$$

and

$$\|\bar{\mathbf{w}}_\eta\|_{\mathcal{A}^s_{\text{tree}}} \;\lesssim\; \|\mathbf{v}\|_{\mathcal{A}^s_{\text{tree}}}, \tag{6.10}$$

where the constants depend only on s *when* $s \to 0$ *and on* C^* *in* **TCOARSE**.

Proof. The second estimate in (6.9) follows from the triangle inequality. As for the first estimate in (6.9), assume that $\mathbf{v} \in \mathcal{A}^s_{\text{tree}}$ and consider the best N-term tree $\mathcal{T}_N(\mathbf{v})$ for $\mathbf{v}$ defined by (6.6). We first note that

$$\begin{aligned} \|\mathbf{w} - \mathbf{w}|_{\mathcal{T}_N(\mathbf{v})}\|_{\ell_2(\mathcal{J})} &\le \|(\mathbf{w} - \mathbf{v})|_{\mathcal{J}\setminus\mathcal{T}_N(\mathbf{v})}\|_{\ell_2(\mathcal{J})} + \|\mathbf{v} - \mathbf{v}|_{\mathcal{T}_N(\mathbf{v})}\|_{\ell_2(\mathcal{J})} \\ &\le \eta + \|\mathbf{v} - \mathbf{v}|_{\mathcal{T}_N(\mathbf{v})}\|_{\ell_2(\mathcal{J})}. \end{aligned} \tag{6.11}$$

According to the definition of the norm $\|\cdot\|_{\mathcal{A}^s_{\text{tree}}}$ by (6.8), we have $\|\mathbf{v} - \mathbf{v}|_{\mathcal{T}_N(\mathbf{v})}\|_{\ell_2(\mathcal{J})} \le \eta$ for some $N \;\lesssim\; \|\mathbf{v}\|^{1/s}_{\mathcal{A}^s_{\text{tree}}}\, \eta^{-1/s}$. Therefore, $\|\mathbf{w} - \mathbf{w}|_{\mathcal{T}_N(\mathbf{v})}\|_{\ell_2(\mathcal{J})} \le 2\eta$ so that by the definition of the $(2C^*\eta, C^*)$-near best tree $\mathcal{T}(2C^*\eta, \mathbf{w})$, we have

$$\#\mathcal{T}(2C^*\eta, \mathbf{w}) \le C^*\#\mathcal{T}^*(2\eta, \mathbf{w}) \le C^*\#\mathcal{T}_N(\mathbf{v}) \le C^*N \;\lesssim\; \|\mathbf{v}\|^{1/s}_{\mathcal{A}^s_{\text{tree}}}\, \eta^{-1/s}, \tag{6.12}$$

which proves the first estimate in (6.9). It remains to prove (6.10). We wish to show that for each $\delta > 0$ there exists a tree $\mathcal{T}_\delta$ such that

$$\|\bar{\mathbf{w}}_\eta - \bar{\mathbf{w}}_\eta|_{\mathcal{T}_\delta}\|_{\ell_2(\mathcal{J})} \le \delta \quad \text{and} \quad \#(\mathcal{T}_\delta) \;\lesssim\; \|\mathbf{v}\|^{1/s}_{\mathcal{A}^s_{\text{tree}}}\, \delta^{-1/s}. \tag{6.13}$$

The existence of such a tree has been already established for $\delta = \eta$. Now consider first the case $\delta \le 2(1 + 2C^*)\eta$. In this case, we take $\mathcal{T}_\delta := \operatorname{supp} \bar{\mathbf{w}}_\eta = \mathcal{T}(2C^*\eta, \mathbf{w})$. In fact, since then $\|\bar{\mathbf{w}}_\eta - \bar{\mathbf{w}}_\eta|_{\mathcal{T}_\delta}\|_{\ell_2(\mathcal{J})} = 0$, the first relation in (6.13) holds trivially. Moreover, since $\#(\mathcal{T}_\delta) = \#(\mathcal{T}(2C^*\eta, \mathbf{w})) \;\lesssim\; \|\mathbf{v}\|^{1/s}_{\mathcal{A}^s_{\text{tree}}}\, \eta^{-1/s} \;\lesssim\; \|\mathbf{v}\|^{1/s}_{\mathcal{A}^s_{\text{tree}}}\, \delta^{-1/s}$, the second estimate in (6.13) is also valid.

Now consider the case $\delta > 2(1+2C^*)\eta$. Let $\mathcal{T}_\delta := \mathcal{T}_N(\mathbf{v})$ be a best N-term tree for $\mathbf{v}$, where N will be chosen in a moment. Note that

$$\begin{aligned}\|\bar{\mathbf{w}}_\eta - \bar{\mathbf{w}}_\eta|_{\mathcal{T}_\delta}\|_{\ell_2(\mathcal{J})} &\le \|(\bar{\mathbf{w}}_\eta - \mathbf{v})|_{\mathcal{J}\setminus\mathcal{T}_\delta}\|_{\ell_2(\mathcal{J})} + \|\mathbf{v} - \mathbf{v}|_{\mathcal{T}_N(\mathbf{v})}\|_{\ell_2(\mathcal{J})} \\ &\le (1+2C^*)\eta + \|\mathbf{v} - \mathbf{v}|_{\mathcal{T}_N(\mathbf{v})}\|_{\ell_2(\mathcal{J})} \le \delta,\end{aligned} \tag{6.14}$$

provided that $\delta - (1+2C^*)\eta \ge \|\mathbf{v}-\mathbf{v}|_{\mathcal{T}_N(\mathbf{v})}\|_{\ell_2(\mathcal{J})}$. This holds for some $N \lesssim \|\mathbf{v}\|^{1/s}_{\mathcal{A}^s_{\rm tree}}(\delta - (1+2C^*)\eta)^{-1/s} \le 2^{1/s}\|\mathbf{v}\|^{1/s}_{\mathcal{A}^s_{\rm tree}}\delta^{-1/s}$, which confirms (6.13) also in the case $\delta > 2(1+2C^*)\eta$. This finishes the proof. ∎

6.3. The key requirement. Up to this point, we have not imposed any conditions on the subroutine **RES**, which is used to approximate the residual at each iteration. In later realizations of the routine **RES**, the support of its output will have tree structure which we will therefore assume to hold from now on without further mention. We will now introduce a condition, called s^*-sparsity, motivated by the analysis of the previous section for the linear case; see Proposition 5.2. We will then show that whenever **RES** is s^*-sparse, the algorithm **SOLVE** is optimal in its rate/complexity for the range of error decay rates $\sigma^{\rm tree}_{N,\ell_2(\mathcal{J})}(\mathbf{u}) \lesssim N^{-s}$ with $s < s^*$. The subsequent section will then show how to construct s^*-sparse routines for nonlinear problems.

We say that the scheme **RES** used to approximate residuals is *s^*-sparse* if, in addition to (4.1), the following property holds.

s^*-sparsity. *Whenever the exact solution* $\mathbf{u}$ *of* (2.8) *belongs to* $\mathcal{A}^s_{\rm tree}$ *for some* $s < s^*$*, then one has for any finitely supported input* $\mathbf{v}$ *and any tolerance* $\eta > 0$ *that the output* $\mathbf{w}_\eta := \mathbf{RES}\,[\eta, \mathbf{B}, \mathbf{F}, \mathbf{f}, \mathbf{v}]$ *satisfies*

$$\begin{aligned}\#\operatorname{supp}\mathbf{w}_\eta &\le C\eta^{-1/s}(\|\mathbf{v}\|^{1/s}_{\mathcal{A}^s_{\rm tree}} + \|\mathbf{u}\|^{1/s}_{\mathcal{A}^s_{\rm tree}} + 1), \\ \|\mathbf{w}_\eta\|_{\mathcal{A}^s_{\rm tree}} &\le C\left(\|\mathbf{v}\|_{\mathcal{A}^s_{\rm tree}} + \|\mathbf{u}\|_{\mathcal{A}^s_{\rm tree}} + 1\right),\end{aligned} \tag{6.15}$$

where C *depends only on* s *when* $s \to s^*$*. Moreover, the number of operations needed to compute* $\mathbf{w}_\eta$ *stays proportional to* $C(\eta^{-1/s}(\|\mathbf{v}\|^{1/s}_{\mathcal{A}^s_{\rm tree}} + \|\mathbf{u}\|^{1/s}_{\mathcal{A}^s_{\rm tree}} + 1) + \#\mathcal{T}(\operatorname{supp}\mathbf{v}))$*, where again* $\mathcal{T}(\operatorname{supp}\mathbf{v})$ *denotes the smallest tree containing* $\operatorname{supp}\mathbf{v}$*.*

The occurrence of $\|\mathbf{u}\|_{\mathcal{A}^s_{\rm tree}}$ in the above estimates is already plausible from the linear case, as explained in Remark 5.1.

It will be understood in what follows that **TCOARSE** is used as **CCOARSE** and that a proper initialization is used that complies if necessary with the requirements on the quality of the initial guess (see section 5) so that, in particular, the respective variant of the iteration (2.11) satisfies (3.17).

Under these premises we now show that s^*-sparsity implies asymptotically optimal complexity of the scheme **SOLVE**.

THEOREM 6.1. *Assume that the scheme* **RES** *is s^*-sparse for some $s^* > 0$. If the exact solution* $\mathbf{u}$ *of* (2.8) *belongs to* $\mathcal{A}^s_{\rm tree}$ *for some $s < s^*$, then the approximations* $\bar{\mathbf{u}}(\epsilon)$ *satisfy for every target accuracy* $\epsilon > 0$

$$\|\mathbf{u} - \bar{\mathbf{u}}(\epsilon)\|_{\ell_2(\mathcal{J})} \le \epsilon, \tag{6.16}$$

while

$$\#\operatorname{supp}\bar{\mathbf{u}}(\epsilon) \le C\epsilon^{-1/s}\|\mathbf{u}\|^{1/s}_{\mathcal{A}^s_{\rm tree}}, \quad \|\bar{\mathbf{u}}(\epsilon)\|_{\mathcal{A}^s_{\rm tree}} \le C\|\mathbf{u}\|_{\mathcal{A}^s_{\rm tree}}, \tag{6.17}$$

where the constant C *depends only on* s *when* $s \to s^*$*. Moreover, the number of operations needed to compute* $\bar{\mathbf{u}}(\epsilon)$ *remains bounded by* $C\epsilon^{-1/s}\|\mathbf{u}\|^{1/s}_{\mathcal{A}^s_{\rm tree}}$*.*

Proof. The first part follows directly from Proposition 4.3. From (4.15) we know that the result $\tilde{\mathbf{v}}$ after at most K perturbed iterations in the $(j+1)$st block of step (ii) in **SOLVE** satisfies $\|\mathbf{u}-\tilde{\mathbf{v}}\|_{\ell_2(\mathcal{J})} \le \epsilon_j/(2(1+2C^*))$. Now Proposition 6.3 ensures that then

$$\|\bar{\mathbf{u}}^{j+1}\|_{\mathcal{A}^s_{\rm tree}} \le C\|\mathbf{u}\|_{\mathcal{A}^s_{\rm tree}}, \quad \#\operatorname{supp}\bar{\mathbf{u}}^{j+1} \le C\epsilon_j^{-1/s}\|\mathbf{u}\|^{1/s}_{\mathcal{A}^s_{\rm tree}}. \tag{6.18}$$

Moreover, the computational work required by the routine **TCOARSE** stays, by Proposition 6.2, proportional to the support size of $\tilde{\mathbf{v}}$, since the support of $\tilde{\mathbf{v}}$, as an output of **RES**, has tree structure. Here it is important to note that the constant C is *independent* of the input $\tilde{\mathbf{v}}$ of **TCOARSE**. Thus for $j>0$ the input of the first application of **RES** in step (ii) of **SOLVE** satisfies (6.18). Since there are only at most a uniformly bounded number K of applications of **RES** in each iteration block, $\tilde{\mathbf{v}}$ also satisfies (6.18) with a constant depending now on the number of updates (ii.1) bounded by K; see (4.14). (Here we have tacitly assumed that the initial guess has been subjected to a **TCOARSE** so that it also satisfies (6.18). Otherwise, we would have to add $\#\operatorname{supp}\mathbf{u}^0$ to the above estimates.) The estimate in (6.17) now follows from these estimates for the terminal value of j. This also shows that the number of operations remains proportional to $\#\operatorname{supp}\bar{\mathbf{u}}(\epsilon)$. ∎

We shall discuss below how to obtain schemes **RES** that are s^*-sparse for certain $s^*>0$ and what limits the value of s^*.

7. Nonlinear evaluation schemes. Just as the efficient application of compressible matrices $\mathbf{A}$ was pivotal for the adaptive solution in the linear case (L), we need efficient evaluation schemes for $\mathbf{F}(\mathbf{v})$ that allow us to realize the residual approximation in **RES**. Such a scheme has been already proposed in [10] for a class of nonlinearities F that will be described next.

7.1. A class of nonlinear mappings. We shall be concerned with nonlinear operators of the form

$$V=(v_1,\dots,v_n)\mapsto w=F(D^{\alpha_1}v_1,\dots,D^{\alpha_n}v_n), \tag{7.1}$$

acting from $\mathcal{H}\times\cdots\times\mathcal{H}$ to the dual $\mathcal{H}'$. (Here $\alpha_i=(\alpha_{i,1},\dots,\alpha_{i,d})$ are multi-indices.) This clearly covers our previous example of a single argument $n=1$ but also further important cases like the nonlinearity appearing in the Navier–Stokes equations. Although we shall not address variational problems involving nonlinearities of several arguments in this paper, we shall present the evaluation schemes in this somewhat greater generality because they are important for such applications and because we shall apply the case of two arguments later in connection with Newton's scheme.

We shall first describe our requirements on F in the wavelet coordinate domain and point out later circumstances under which these requirements are met.

Denoting by $\mathbf{v}_i=(v_{i,\lambda})$ the arrays of the wavelet coefficients of the function v_i, setting $\mathbf{V}=(\mathbf{v}_1,\dots,\mathbf{v}_n)$, and defining the corresponding discrete mapping $\mathbf{F}$ by

$$\mathbf{F}(\mathbf{V}) := (\langle\psi_\lambda, F(D^{\alpha_1}v_1,\dots,D^{\alpha_n}v_n)\rangle)_{\lambda\in\mathcal{J}}, \tag{7.2}$$

we make the following basic assumptions.

ASSUMPTION 1. $\mathbf{F}$ *is a Lipschitz map from* $(\ell_2(\mathcal{J}))^n$ *into* $\ell_2(\mathcal{J})$,

$$\|\mathbf{F}(\mathbf{U})-\mathbf{F}(\mathbf{V})\|_{\ell_2(\mathcal{J})} \le C\sum_{i=1}^n \|\mathbf{u}_i-\mathbf{v}_i\|_{\ell_2(\mathcal{J})}, \tag{7.3}$$

with $C = C(\max_i\{\|\mathbf{u}_i\|_{\ell_2(\mathcal{J})}, \|\mathbf{v}_i\|_{\ell_2(\mathcal{J})}\})$*, where* $x \mapsto C(x)$ *is a positive nondecreasing function.*

Bearing the norm equivalences (2.5), (2.6) in mind, we see that property P1 from section 3.1 is a special case of Assumption 1 for $n = 1$.

ASSUMPTION 2. *There exists a constant* $\gamma > d/2$ *such that, for any finitely supported* $\mathbf{V}$ *(i.e., with all* $\mathbf{v}_i$ *finitely supported) and* $\mathbf{w} = \mathbf{F}(\mathbf{V})$*, we have for* $\lambda \in \mathcal{J}_\psi$ *the estimate*

$$|w_\lambda| \le C \sup_{\mu\,:\, S_\lambda \cap S_\mu \neq \emptyset} \left(\sum_{i=1}^n |v_{i,\mu}|\right) 2^{-\gamma(|\lambda|-|\mu|)}, \tag{7.4}$$

where $C = C(\max_i \|\mathbf{v}_i\|_{\ell_2(\mathcal{J})})$ *and* $x \mapsto C(x)$ *is a positive nondecreasing function.*

The parameter γ plays a similar role as the compressibility range s^* for the wavelet representation of linear operators.

Nonlinear mappings that satisfy the above assumptions are, for instance, those with polynomial growth. In the special case of a single argument, a typical condition reads

$$|F^{(k)}(v)| \;\lesssim\; (1+|v|)^{(p-k)_+}, \quad k = 0, \dots, n^*, \qquad \text{for some } \; n^* \ge 1, \tag{7.5}$$

where $a_+ := \max\{0, a\}$.

Remark 7.1. Suppose that $\mathcal{H} = H^t$ is a Sobolev space of smoothness order $t > 0$ (or a closed subspace determined by, e.g., homogeneous boundary conditions). One can show that, for the special case $n = 1$, (7.5) implies Assumptions 1 and 2 and, in particular, property P1, with no condition on p when $t \ge d/2$ and otherwise provided that

$$1 \le p < p^* := \frac{d+2t}{d-2t}; \tag{7.6}$$

see [10].

In the general case, when the nonlinear map F has the form $F(D^{\alpha_1}u_1, \dots, D^{\alpha_n}u_n)$, we impose the growth condition

$$|D^\beta F(x_1, \dots, x_n)| \le C \prod_{i=1}^n (1+|x_i|)^{[p_i - \beta_i]_+}, \;\; |\beta| = 0, 1, \dots, n^*, \tag{7.7}$$

for some $p_i \ge 0$ and n^* a positive integer. The following fact (covering Remark 7.1) has been proven in [10].

THEOREM 7.1. *Suppose that* $\mathcal{H} = H^t$ *is a closed subspace (determined by, e.g., homogeneous boundary conditions) of* $H^t(\Omega)$ *for some* $t \ge 0$*. Assume that the growth assumptions* (7.7) *hold at least with* $n^* = 0$*. Then* F *maps* $\mathcal{H} \times \cdots \times \mathcal{H}$ *to* $\mathcal{H}'$ *whenever* $t \ge 0$ *satisfies*

$$\left(\frac{1}{2} - \frac{t}{d}\right)_+ + \sum_{i=1}^n p_i \left(\frac{1}{2} - \frac{t}{d} + \frac{|\alpha_i|}{d}\right)_+ < 1. \tag{7.8}$$

If in addition $n^* = 1$*, then we also have under the same restriction*

$$\|F(u) - F(v)\|_{\mathcal{H}'} \le C \sum_{i=1}^n \|u_i - v_i\|_{\mathcal{H}}, \tag{7.9}$$

where $C = C(\max_i\{\|u_i\|_{\mathcal{H}}, \|v_i\|_{\mathcal{H}}\})$ *and* $x \to C(x)$ *is nondecreasing, and therefore, on account of* (2.5)*, Assumption* 1 *holds.*

For the verification of Assumption 2, we treat separately the polynomial case for which we have the growth condition

$$|D^\beta F(x_1, \ldots, x_n)| \le C \prod_{i=1}^{n} (1 + |x_i|)^{p_i - \beta_i}, \quad \beta_i \le p_i, \tag{7.10}$$

and $D^\beta F = 0$ if $\beta_i > p_i$ for some i, where the p_i are positive integers. We recall the following result from [10].

THEOREM 7.2. *Assume that the wavelets belong to* C^m *and have vanishing moments of order* m *(i.e., are orthogonal to* $\mathbb{P}_{m-1}$ *the space of polynomials of total degree at most* $m-1$*) for some positive integer* m*. Then Assumption* 2 *holds for* $\gamma = r + t + d/2$ *with the following values of* r*:*

(i) *If* F *satisfies* (7.7) *with* p *such that* $\sum_{i=1}^n p_i(d/2 - t + |\alpha_i|)_+ < d/2 + t$*, then* $r = \lceil \min\{m, n^*, p^*\} \rceil$*, where* $p^* = \min\{p_i \ : \ i \text{ s.t. } d/2 - t + |\alpha_i| > 0\}$.

(ii) *If* F *satisfies* (7.10) *with* p *such that* $\sum_{i=1}^n p_i(d/2 - t + |\alpha_i|)_+ < d/2 + t$*, then* $r = m$.

7.2. An adaptive evaluation scheme. The schemes for approximating $\mathbf{F}(\mathbf{V})$ consist of two conceptual steps: (i) the prediction of a possibly small set of indices that covers the significant coefficients of $\mathbf{F}(\mathbf{V})$ using only knowledge of the indices of the significant coefficients of the input $\mathbf{V}$; (ii) a sufficiently accurate computation of those coefficients of $\mathbf{F}(\mathbf{V})$ that correspond to the predicted index set. Once the predicted sets are known, one can invoke the techniques developed in [16] to tackle the latter task. A more detailed treatment of this issue will be given elsewhere. Motivated by the results in [16], we shall work in what follows with the following assumption.

ASSUMPTION E. *The entries* $w_\lambda = \mathbf{F}(\mathbf{v})_\lambda$ *can be computed (with sufficient accuracy) on average at unit cost.*

Therefore, we shall concentrate here only on task (i) under the assumption that the computation can be done in linear time; see the discussion of this issue in [16]. We recall now briefly the construction from [10] of good predictions for mappings satisfying Assumptions 1 and 2. For $j = 0, 1, \ldots,$ and each component $\mathbf{v}_i$ of $\mathbf{V}$ we define the near best trees introduced in section 6.2

$$\mathcal{T}_{i,j} := \mathcal{T}\left(\frac{2^j \epsilon}{1+j}, \mathbf{v}_i\right), \tag{7.11}$$

by invoking the thresholding algorithm from [3]. Moreover, it can be shown that for any tree $\mathcal{T}$ there exists an expansion $\tilde{\mathcal{T}}$ such that for some constant C one has $\#\tilde{\mathcal{T}} \le C\#\mathcal{T}$, while for any $\lambda \in \mathcal{L}(\tilde{\mathcal{T}})$ the number of $\mu \in \mathcal{L}(\tilde{\mathcal{T}})$ such that $S_\lambda \cap S_\mu \neq \emptyset$ is bounded by C; see Lemma 3.1 in [10]. We denote these expansions of $\mathcal{T}_{i,j}$ by $\tilde{\mathcal{T}}_{i,j}$ and set

$$\tilde{\mathcal{T}}_j := \bigcup_{i=0}^{n} \tilde{\mathcal{T}}_{i,j} \ \text{ and } \ \Delta_j := \tilde{\mathcal{T}}_j \setminus \tilde{\mathcal{T}}_{j+1}. \tag{7.12}$$

In order to build a tree which will be adapted to $\mathbf{w} = \mathbf{F}(\mathbf{V})$, we introduce

$$\alpha := \frac{2}{2\gamma - d} > 0, \tag{7.13}$$

where γ is the constant in (7.4), and for each $\mu \in \Delta_j$, we define the *influence set*

$$\Lambda_{\epsilon,\mu} := \{\lambda \,:\, S_\lambda \cap S_\mu \neq \emptyset \text{ and } |\lambda| \leq |\mu| + \alpha j\}. \tag{7.14}$$

We then define $\mathcal{T}$ by

$$\mathcal{T}_\epsilon(\mathbf{F}, \mathbf{V}) = \mathcal{T} := \mathcal{J}_\phi \cup \left(\cup_{\mu \in \tilde{\mathcal{T}}_0} \Lambda_{\epsilon,\mu}\right). \tag{7.15}$$

The following fact has been shown in [10, Theorem 5.1].

THEOREM 7.3. *Assume that F satisfies Assumptions 1 and 2. Given any $\mathbf{V} \in (\ell_2(\mathcal{J}))^n$, we have the error estimate*

$$\|\mathbf{F}(\mathbf{V}) - \mathbf{F}(\mathbf{V})|_{\mathcal{T}}\|_{\ell_2(\mathcal{J})} \lesssim \epsilon. \tag{7.16}$$

Moreover, if $\mathbf{V} \in (\mathcal{A}^s_{\rm tree})^n$ for some $s \in \left(0, \frac{2\gamma - d}{2d}\right)$, we have the estimate

$$\#(\mathcal{T}) \lesssim \|\mathbf{V}\|^{1/s}_{(\mathcal{A}^s_{\rm tree})^n} \epsilon^{-1/s} + \#(\mathcal{J}_\phi). \tag{7.17}$$

We therefore have $\mathbf{F}(\mathbf{V}) \in \mathcal{A}^s_{\rm tree}$ and

$$\|\mathbf{F}(\mathbf{V})\|_{\mathcal{A}^s_{\rm tree}} \lesssim 1 + \|\mathbf{V}\|_{(\mathcal{A}^s_{\rm tree})^n}. \tag{7.18}$$

The constants in these above inequalities depend only on $\|\mathbf{V}\|_{\ell_2(\mathcal{J})}$, the space dimension d, and the parameter s.

This suggests the following routine for approximating $\mathbf{F}(\mathbf{V})$ for any finitely supported vector $\mathbf{V}$:

EV $[\epsilon, \mathbf{F}, \mathbf{V}] \to \mathbf{w}_\epsilon$. *Given the inputs $\epsilon > 0$ and $\mathbf{V}$ with finite support do:*

Step 1: *Invoke* **TCOARSE** $[2^j\epsilon/(1+j), \mathbf{v}_i]$, *$i = 1, \ldots, n$, to compute the trees*

$$\mathcal{T}_{i,j} := \mathcal{T}\left(\frac{2^j\epsilon}{C_0(j+1)}, \mathbf{v}_i\right), \tag{7.19}$$

where $C_0 = C_0(\|\mathbf{v}\|)$ is the constant involved in (7.16) for $j = 0, \ldots, J$, and stop for the smallest J such that $\mathcal{T}_J$ is empty (we always have $J \lesssim \log_2(\|\mathbf{V}\|_{\ell_2(\mathcal{J})}/\epsilon)$).

Step 2: *Derive the expanded trees $\tilde{\mathcal{T}}_{i,j}$, the layers Δ_j, and the outcome tree $\mathcal{T} = \mathcal{T}_\epsilon(\mathbf{F}, \mathbf{V})$ according to (7.15).*

Step 3: *Compute the coefficients $\mathbf{F}(\mathbf{V})_\lambda$, $\lambda \in \mathcal{T}$; see* [16].

A more detailed discussion of this scheme can be found in [10].

We can now formulate concrete realizations of the scheme **SOLVE** that are suitable for nonlinear problems. We shall use **TCOARSE** as our version of **CCOARSE**. Moreover, for the semilinear elliptic problem (3.4) we can take

$$\begin{aligned}\mathbf{RES}_{\rm ell}[\eta, \alpha\mathbf{I}, \mathbf{A}, \mathbf{G}, \mathbf{f}, \mathbf{v}] := \alpha\,(&\mathbf{APPLY}\,[\eta/3, \mathbf{A}, \mathbf{v}] \\ &+ \mathbf{EV}\,[\eta/3, \mathbf{G}, \mathbf{v}] - \mathbf{RHS}\,[\eta/3, \mathbf{f}]),\end{aligned} \tag{7.20}$$

where **RHS** is defined here as in (5.2) but with **COARSE** replaced by **TCOARSE**.

For the general nonlinear problem (GNL), we shall now devise a residual approximation for the scheme from section 3.2 with $\mathbf{B}_n := \mathbf{B}$. Suppose that $\|\mathbf{B}\|_{\ell_2(\mathcal{J}) \to \ell_2(\mathcal{J})} \leq C_B$, and set

$$\begin{aligned}\mathbf{RES}_{\rm lc}[\eta, \mathbf{B}, \mathbf{F}, \mathbf{f}, \mathbf{v}] := \mathbf{APPLY}\,\big[&\eta/2, \mathbf{B}, (\mathbf{EV}\,[\eta/4C_B, \mathbf{F}, \mathbf{v}] \\ &-\mathbf{RHS}\,[\eta/4C_B, \mathbf{f}])\big].\end{aligned} \tag{7.21}$$

Of course, we have assumed here that the matrix $\mathbf{B}$ is compressible. In particular, for the stationary choice $\mathbf{B} = D\mathbf{R}(\mathbf{u}^0)$ this might be expected to be the case. We shall return to this issue later.

7.3. Complexity estimates. We have just explained how to obtain concrete realizations of the scheme **SOLVE** in each of the three cases (L), (SL), (GNL). The remainder of this section will be devoted to giving a complexity analysis of these schemes. The following theorem summarizes the properties of Algorithm **EV**; see [10, Theorem 3.4].

THEOREM 7.4. *Given the inputs $\epsilon > 0$, a nonlinear function F such that $\mathbf{F}$ satisfies Assumptions* 1 *and* 2*, and a finitely supported vector $\mathbf{V}$, then the output tree $\mathcal{T}$ has the following properties:*

P1. $\|\mathbf{F}(\mathbf{V}) - \mathbf{F}(\mathbf{V})|_{\mathcal{T}}\| \leq \epsilon$.

P2. *For any $0 < s < \frac{2\gamma - d}{2d}$ (see Theorem* 7.3*),*

$$\#(\mathcal{T}) \leq C\|\mathbf{V}\|^{1/s}_{(\mathcal{A}^s_{\mathrm{tree}})^n}\epsilon^{-1/s} + \#(\mathcal{J}_\phi) =: N_\epsilon \tag{7.22}$$

with C a constant depending only on the constants appearing in Theorem 7.3.

P3. *Moreover, the number of computations needed to find $\mathcal{T}$ is bounded by $C(N_\epsilon + \#\mathcal{T}(\mathbf{V}))$, where N_ϵ is the right-hand side of* (7.22) *and $\mathcal{T}(\mathbf{V})$ is the smallest tree containing* $\operatorname{supp}\mathbf{v}$.

Recall that in the case of semilinear equations, $\mathbf{R}$ involves a linear and a nonlinear operator as in (3.2) or (3.4). Also recall from Theorem 5.1 that when the wavelet representation $\mathbf{A}$ of the linear operator $\mathcal{A}$ defined by

$$\langle w, \mathcal{A}v\rangle = a(v, w), \quad v, w \in \mathcal{H} = H^t,$$

belongs to $\mathcal{C}_{s^*}$, then $\mathbf{A}$ is bounded on $\mathcal{A}^s$ with $s < s^*$. However, when nonlinear operators are also involved, Theorem 6.1 tells us that the spaces $\mathcal{A}^s_{\mathrm{tree}}$ should now play the role of $\mathcal{A}^s$. Thus we shall prove in Proposition 7.4 below the boundedness of $\mathbf{A}$ with respect to this slightly stronger norm.

To prepare for Proposition 7.4, we make some remarks.

Remark 7.2. It has been shown in [12] that when $\mathcal{A}$ is a local operator, i.e., $\langle v, \mathcal{A}w\rangle = 0$ whenever $|\operatorname{supp} v \cap \operatorname{supp} w| = 0$, and when $\mathcal{A}$ is still bounded as a mapping from H^{t+a} to H^{-t+a} for $a \leq m + d/2$, where m is less than or equal to the order of differentiability and vanishing moments of the wavelets ψ_λ, then one has

$$|a(\psi_\lambda, \psi_\nu)| \lesssim 2^{-\sigma||\lambda|-|\nu||}, \quad \sigma = t + m + d/2. \tag{7.23}$$

Moreover, we know from Proposition 3.4 in [8] that in this case $\mathbf{A}$ belongs to $\mathcal{C}_{s^*}$ with

$$s^* = \frac{t+m}{d} = \frac{2\sigma - d}{2d}. \tag{7.24}$$

Note that σ agrees with the value of γ in Assumption 2 or at least depends in an analogous way on the spatial dimension, the order of the operator, and the order of the vanishing moments and the smoothness of the wavelets; see Theorem 7.2 (ii). Moreover, the condition $s < s^*$ with s^* from (7.24) agrees with the constraint on s from Theorem 7.3 formulated in terms of γ.

Remark 7.3. One can show that for piecewise polynomial wavelets, s^* can be chosen larger than in (7.24); see [2].

By definition one has $\mathcal{A}^s_{\mathrm{tree}} \subset \mathcal{A}^s$. We shall need the following refinement of Theorem 5.1.

PROPOSITION 7.4. *Under the assumptions from Remark* 7.2 *on the linear part $\mathcal{A}$, let*

$$\sigma = m + t + d/2. \tag{7.25}$$

Then one has for $s < \frac{2\sigma - d}{2d}$

$$\|\mathbf{A}\mathbf{v}\|_{\mathcal{A}^s_{\rm tree}} \lesssim \|\mathbf{v}\|_{\mathcal{A}^s_{\rm tree}}, \quad \mathbf{v} \in \mathcal{A}^s_{\rm tree}; \tag{7.26}$$

that is, $\mathbf{A}$ *maps* $\mathcal{A}^s_{\rm tree}$ *boundedly into itself.*

Proof. By assumption (3.1), $\mathcal{A}$ is a topological isomorphism from $\mathcal{H}$ onto $\mathcal{H}'$ and obviously satisfies Assumption 1. We need to show that Assumption 2 holds for all $\gamma' < \sigma$, defined by (7.25). To this end, note that (7.23) provides

$$\begin{aligned}
|(\mathbf{A}\mathbf{v})_\lambda| &\le \sum_{\substack{|\nu| \le |\lambda|,\\ S_\nu \cap S_\lambda \neq \emptyset}} |v_\nu| 2^{-\sigma(|\lambda|-|\nu|)} + \sum_{\substack{|\nu| > |\lambda|,\\ S_\nu \cap S_\lambda \neq \emptyset}} |v_\nu| 2^{-\sigma(|\lambda|-|\nu|)} 2^{-2\sigma(|\nu|-|\lambda|)} \\
&\le \sup_{\substack{|\nu| \le |\lambda|,\\ S_\nu \cap S_\lambda \neq \emptyset}} 2^{-\gamma'(|\lambda|-|\nu|)} |v_\nu| \sum_{\substack{|\nu| \le |\lambda|,\\ S_\nu \cap S_\lambda \neq \emptyset}} 2^{-(\sigma-\gamma')(|\lambda|-|\nu|)} \\
&\quad + \sup_{\substack{|\nu| > |\lambda|,\\ S_\nu \cap S_\lambda \neq \emptyset}} |v_\nu| 2^{-\sigma(|\lambda|-|\nu|)} \sum_{\substack{|\nu| > |\lambda|,\\ S_\nu \cap S_\lambda \neq \emptyset}} 2^{-2\sigma(|\nu|-|\lambda|)}.
\end{aligned}$$

We now check that both sums appearing on the right-hand side are bounded independently of λ provided that $\gamma' < \sigma$. Indeed, in the first sum, for any $k > |\lambda|$, there is a bounded number C_0 of indices ν with $|\nu| = k$ such that $S_\nu \cap S_\lambda \neq \emptyset$. Hence this sum is bounded by $C_0 \sum_{j=0}^\infty 2^{-j(\sigma-\gamma')}$. For the second sum, note that for $|\nu| = |\lambda| + k$, there are at most $C_0 2^{kd}$ indices ν for which $S_\nu \cap S_\lambda \neq \emptyset$. Since, by (7.25), $2\sigma > d$, this sum can also be bounded by a geometric series. We have thus verified Assumption 2 for all $\gamma' < \sigma$. The assertion now follows from Theorem 7.3 and the restriction on s given in Theorem 7.3. However, in contrast to the general nonlinear case, we can dispense here with the constant term in the right-hand side of (7.26); see Remark 3.2 in [10]. ∎

Note that the support of the output of the scheme **APPLY** from section 5.2 generally does not have tree structure. In order to ensure that the output of **RES** complies with our previous assumption that its support has tree structure, we shall employ the following modification of **APPLY** while keeping the notation unchanged. First, the original version of **APPLY** is carried out with target accuracy $\eta/2$. We then apply **TCOARSE** to the output with target accuracy $\eta/2$ so that (5.9) is still valid.

We shall make use of the following consequence of Proposition 7.4.

COROLLARY 7.5. *Under the same assumptions as in Proposition* 7.4 *let* $s < s^*$. *Then* $\mathbf{w}_\eta = \mathbf{APPLY}\,[\eta, \mathbf{A}, \mathbf{v}]$ *satisfies*

(i) $\#\text{flops} \lesssim \#\mathcal{T}(\operatorname{supp}\mathbf{v}) + \|\mathbf{v}\|^{1/s}_{\mathcal{A}^s_{\rm tree}} \eta^{-1/s}$, $\#\operatorname{supp}\mathbf{w}_\eta \lesssim \|\mathbf{v}\|^{1/s}_{\mathcal{A}^s_{\rm tree}} \eta^{-1/s}$;

(ii) $\|\mathbf{w}_\eta\|_{\mathcal{A}^s_{\rm tree}} \lesssim \|\mathbf{v}\|_{\mathcal{A}^s_{\rm tree}}$.

Proof. Let $\hat{\mathbf{w}}$ denote the output of the original version of **APPLY** with target accuracy $\eta/2$. The first estimate in (i) for $\hat{\mathbf{w}}$ follows directly from Theorem 5.1 (ii) and the fact that $\|\cdot\|_{\mathcal{A}^s} \lesssim \|\cdot\|_{\mathcal{A}^s_{\rm tree}}$ even with $\#\mathcal{T}(\operatorname{supp}\mathbf{v})$ replaced by $\operatorname{supp}\mathbf{v}$. Now the cost of the subsequent application of **TCOARSE** with target accuracy $\eta/2$, yielding $\mathbf{w}_\eta$, is, by Proposition 6.2, proportional to $\#\mathcal{T}(\operatorname{supp}\mathbf{v})$. This confirms the first estimate in (i). Next note that, since the chunks $(\mathbf{v}_{[j]} - \mathbf{v}_{[j-1]})$ have disjoint supports, for each j the vector $\mathbf{w}_j$, defined by (5.7), can be interpreted as $\mathbf{w}_j = \mathbf{C}^{(j)}\mathbf{v}$, where the matrix $\mathbf{C}^{(j)}$ is a compressed version of $\mathbf{A}$ defined as follows. All columns with indices outside $\operatorname{supp}\mathbf{v}_{[j]}$ are zero. The columns of $\mathbf{C}^{(j)}$ whose indices belong to $\operatorname{supp}(\mathbf{v}_{[k]} - \mathbf{v}_{[k-1]})$, $k \le j$, agree with the corresponding columns in the matrix

$\mathbf{A}_{j-k}$. Therefore, since the $\mathbf{A}_j$ are derived from $\mathbf{A}$ by replacing certain entries by zero, we conclude that $\mathbf{C}^{(j)}$ is obtained by replacing certain entries in $\mathbf{A}$ by zero. Thus the $\mathbf{C}^{(j)}$ still satisfy (7.23) uniformly in j. Assumptions 1 and 2 also remain valid. Thus Proposition 7.4 can be applied to $\mathbf{C}^{(j)}$ with constants independent of j, which provides $\|\hat{\mathbf{w}}\|_{\mathcal{A}^s_{\rm tree}} \lesssim \|\mathbf{v}\|_{\mathcal{A}^s_{\rm tree}}$. Therefore, one also has $\|\mathbf{w}_\eta\|_{\mathcal{A}^s_{\rm tree}} \lesssim \|\mathbf{v}\|_{\mathcal{A}^s_{\rm tree}}$, which is (ii). Since **TCOARSE** produces a near best tree approximation, we conclude that $\#\mathrm{supp}\,\mathbf{w}_\eta \lesssim \|\hat{\mathbf{w}}\|^{1/s}_{\mathcal{A}^s_{\rm tree}}\,\eta^{-1/s}$, which, in view of the previous remark, confirms the second estimate in (i) and finishes the proof. ∎

We have now collected all the ingredients needed to confirm s^*-sparsity of the residual approximations defined before. We start with (7.20) for the case (SL).

COROLLARY 7.6. *Let γ be the parameter given in Theorem* 7.2 *for the respective nonlinear mapping G. Suppose that σ, defined by* (7.25), *satisfies $\sigma \geq \gamma$. Then* $\mathbf{RES}_{\rm ell}$, *defined by* (7.20), *is s^*-sparse with $s^* := \frac{2\gamma - d}{2d}$.*

Proof. We have to verify the validity of (6.15) for $s < s^*$. If $\mathbf{u} \in \mathcal{A}^s_{\rm tree}$ for some $s < s^*$, then Proposition 7.4 and Theorem 7.4 imply that $\mathbf{f} \in \mathcal{A}^s_{\rm tree}$ and $\|\mathbf{f}\|_{\mathcal{A}^s_{\rm tree}} \lesssim \|\mathbf{u}\|_{\mathcal{A}^s_{\rm tree}}$. Hence, since **TCOARSE** satisfies completely analogous properties with regard to $\mathcal{A}^s_{\rm tree}$ as **COARSE** with respect to $\mathcal{A}^s$ (see [8]), we conclude that the output $\mathbf{f}_\eta$ of $\mathbf{RHS}\,[\eta, \mathbf{f}]$ satisfies

$$\#\mathrm{supp}\,\mathbf{f}_\eta \lesssim \eta^{-1/s}\|\mathbf{u}\|^{1/s}_{\mathcal{A}^s_{\rm tree}}, \quad \|\mathbf{f}_\eta\|_{\mathcal{A}^s_{\rm tree}} \lesssim \|\mathbf{u}\|_{\mathcal{A}^s_{\rm tree}}. \tag{7.27}$$

Furthermore, Corollary 7.5 says that the output of **APPLY** remains bounded in $\mathcal{A}^s_{\rm tree}$, while Theorem 7.4 ensures that the same is true for the output of **EV**. Hence, by (7.20), one has for $\mathbf{w}_\eta := \mathbf{RES}_{\rm ell}[\eta, \alpha\mathbf{I}, \mathbf{A}, \mathbf{G}, \mathbf{f}, \mathbf{v}]$

$$\|\mathbf{w}_\eta\|_{\mathcal{A}^s_{\rm tree}} \lesssim \left(\|\mathbf{v}\|_{\mathcal{A}^s_{\rm tree}} + \|\mathbf{u}\|_{\mathcal{A}^s_{\rm tree}} + 1\right), \tag{7.28}$$

which is the second estimate in (6.15). The first estimate in (6.15) follows also from (7.27), Theorem 7.4, and Corollary 7.5 (i). Since the supports of the outputs of $\mathbf{RES}_{\rm ell}$ and **TCOARSE** have tree structure, the required bounds for the operations count follow (under Assumption E) from P3 in Theorem 7.4 and Corollary 7.5 (i). This completes the proof. ∎

Combining Corollary 7.6 with Theorem 6.1 yields the first main result of this paper.

THEOREM 7.5. *Under the same assumptions as in Corollary* 7.6, *suppose that the solution $u = \sum_{\lambda\in\mathcal{J}} u_\lambda\psi_\lambda$ satisfies $\mathbf{u} \in \mathcal{A}^s_{\rm tree}$ for some $s < s^* := (2\gamma - d)/(2d)$. Then the approximate solution $u(\epsilon) = \sum_{\lambda\in\Lambda(\epsilon)} \bar{u}(\epsilon)_\lambda\psi_\lambda$ produced by* **SOLVE** *(with the initialization for the semilinear problem) after finitely many steps satisfies*

$$\|u - u(\epsilon)\|_{\mathcal{H}} \leq C_1\epsilon.$$

Moreover,

$$\#(\text{flops}),\ \#(\Lambda(\epsilon)) \lesssim \epsilon^{-1/s}\big(1 + \|\mathbf{u}\|^{1/s}_{\mathcal{A}^s_{\rm tree}}\big), \tag{7.29}$$

and

$$\|\bar{\mathbf{u}}(\epsilon)\|_{\mathcal{A}^s_{\rm tree}} \lesssim \|\mathbf{u}\|_{\mathcal{A}^s_{\rm tree}}, \tag{7.30}$$

where the constants depend only on $\|\mathbf{u}\|_{\ell_2(\mathcal{J})}$, on the constants in (2.5), (3.8), (3.3), *and on s when $s \to s^*$ or $s \to 0$.*

Let us briefly discuss now the locally convergent scheme from section 2.3. We shall assume that for the general problem (2.2) the nonlinear map in (2.1) satisfies Assumptions 1 and 2. Moreover, we assume that a sufficiently good initial guess $\mathbf{u}^0$ is given so that the error reduction (3.25) holds and that **SOLVE** is initialized accordingly; see section 5.

COROLLARY 7.7. *Let γ be the parameter given in Theorem* 7.2 *for the respective nonlinear mapping F. Moreover, assume that the matrix $\mathbf{B} = \mathbf{B}_n$ appearing in* (2.11) *satisfies decay estimates like* (7.23) *for some $\sigma \geq \gamma$. Then the scheme $\mathbf{RES}_{\mathrm{lc}}$ defined by* (7.21) *is s^*-sparse for $s^* := (2\gamma - d)/(2d)$.*

Proof. The assertion follows again from Proposition 7.4 and Theorem 7.4. ∎

COROLLARY 7.8. *Under the assumptions of Corollary* 7.7, *the assertion of Theorem* 7.5 *remains valid for the locally convergent scheme based on $\mathbf{RES}_{\mathrm{lc}}$.*

We end this section with analyzing the compressibility of the special choice $\mathbf{B} = D\mathbf{R}(\mathbf{u}^0)^T$. We consider $\mathcal{H} = H^t$ and only nonlinear maps of a single argument $n = 1$ and the subcritical case $t < d/2$, $p < p^*$; recall (7.6). Recall from (2.9) that the entries of $D\mathbf{R}(\mathbf{u}^0)^T = D\mathbf{R}(\mathbf{u}^0)$ have the form $w_{\lambda,\nu} := \langle \psi_\lambda, \psi_\nu R'(u^0)\rangle$. Since, in view of the $\mathcal{H} = H^t$-normalization of the ψ_λ, one has $\|\psi_\lambda\|_{L_\infty} \sim 2^{(\frac{d}{2}-t)|\lambda|}$, the same arguments as used in the proof of Theorem 4.2 in [10] yield

$$|w_{\lambda,\nu}| \lesssim \|\psi_\lambda\|_{L_\infty} \inf_{P\in\mathbb{P}_r} \|P - \psi_\nu R'(u^0)\|_{L_1(S_\lambda)} \lesssim 2^{-(r+\frac{d}{2}+t)|\lambda|} |\psi_\nu R'(u^0)|_{W^r(L_\infty(S_\lambda))},$$

where we assume without loss of generality that $|\lambda| \geq |\nu|$. Moreover, we obtain

$$|\psi_\nu R'(u^0)|_{W^r(L_\infty(S_\lambda)} \lesssim \max_{l\leq r} |\psi_\nu|_{W^{r-l}(L_\infty(S_\lambda))} |R'(u^0)|_{W^l(L_\infty(S_\lambda))}.$$

Abbreviating as before $\sigma = r + \frac{d}{2} + t$, we can estimate the first factor by

$$2^{(\frac{d}{2}-t)|\nu|} 2^{(r-l)|\nu|} = 2^{\sigma|\nu|} 2^{-(2t+l)|\nu|}, \tag{7.31}$$

while the second factor can be estimated along the lines of the proof of Theorem 4.2 in [10] as

$$\begin{aligned} |R'(u^0)|_{W^l(L_\infty(S_\lambda))} &\lesssim \max_{k=1,\dots,l} \|\mathbf{u}^0\|^{m+k-1}_{\ell_2(\mathcal{J})} \\ &\quad \times \sup_{\mu: S_\mu \cap S_\lambda \cap S_\nu \neq \emptyset} |u^0_\mu| 2^{(l+(p-1)\epsilon)|\mu|} 2^{(p-1)(\frac{d}{2}-t)|\mu|}, \end{aligned} \tag{7.32}$$

where $\epsilon > 0$, $j_i \in \mathbb{N}$, and

$$m := \begin{cases} (p-l-1)_+ & \text{if} \quad \|\mathbf{u}^0\|_{\ell_2(\mathcal{J})} \geq 1, \\ 0 & \text{if} \quad \|\mathbf{u}^0\|_{\ell_2(\mathcal{J})} < 1. \end{cases}$$

As in [10], we can choose ϵ so that

$$2^{(l+(p-1)\epsilon)|\mu|} 2^{(p-1)(\frac{d}{2}-t)|\mu|} \lesssim 2^{(l+\frac{d}{2}+t)|\mu|} 2^{-\epsilon|\mu|} 2^{-(\frac{d}{2}-t)|\mu|} = 2^{-\epsilon|\mu|} 2^{(l+2t)|\mu|}.$$

Thus, combining (7.31) and (7.32), we obtain

$$|w_{\lambda,\nu}| \lesssim C(\|\mathbf{u}^0\|_{\ell_2(\mathcal{J})}) 2^{-\sigma||\lambda|-|\nu||} \sup_{\mu: S_\mu \cap S_\lambda \cap S_\nu \neq \emptyset} |u^0_\mu| 2^{-\epsilon|\mu|} 2^{(r+2t)(|\mu|-|\nu|)}. \tag{7.33}$$

Note that the first factor $C(\|\mathbf{u}^0\|_{\ell_2(\mathcal{J})}) 2^{-\sigma||\lambda|-|\nu||}$ represents the same scalewise decay of the entries as in the matrix $\mathbf{A}$ in (7.23). This ensures that $D\mathbf{R}(\mathbf{u}^0)$ belongs to $\mathcal{C}_{s^*}$

with s^* given by (7.24). However, the entries are weighted by additional u^0-dependent factors that could, in principle, become rather large when the finite expansion u^0 contains basis functions from high scales overlapping S_λ. Nevertheless, these factors depend only on u^0 (and hence on the accuracy δ of the initial guess) but not on the accuracy by which $D\mathbf{R}(\mathbf{u}^0)$ is applied through the scheme **APPLY**. Therefore, in principle, one obtains asymptotically optimal complexity, however, with possibly poor quantitative behavior.

8. Newton's method. In concrete cases, the error reduction ρ obtained in (3.25) or (3.17) may be so close to one that the number K of necessary updates in the perturbed scheme **SOLVE** may become fairly large. So, in spite of its asymptotic optimality, the quantitative performance may be poor. We shall therefore address Newton's method, corresponding to $\mathbf{B}_n = (D\mathbf{R}(\mathbf{u}^n))^{-1}$ in (2.11), as an example where the ideal scheme permits a faster error decay. The adaptive realization of Newton's method, applied to the infinite dimensional problem (2.2) or, better yet, (2.8), does not quite fit into the format of **SOLVE**, explaining its separate treatment in this section.

Note that, for $\mathbf{B}_n = (D\mathbf{R}(\mathbf{u}^n))^{-1}$, (2.11) can be reformulated as follows. Given an initial guess $\mathbf{u}^n$, the next iterate $\mathbf{u}^{n+1}$ is determined by solving

$$D\mathbf{R}(\mathbf{u}^n)\mathbf{w}^n = -\mathbf{R}(\mathbf{u}^n) \tag{8.1}$$

and setting

$$\mathbf{u}^{n+1} := \mathbf{u}^n + \mathbf{w}^n. \tag{8.2}$$

We are not interested here in the weakest assumptions under which the iterative scheme (8.2) converges to a locally unique solution. We are instead content here with the following setting: Recall that the mapping R in the variational problem (2.2) has the form

$$R(v) = F(v) - f, \tag{8.3}$$

where throughout this section we shall confine the discussion again to nonlinear maps F of a single argument satisfying the growth condition (7.5) for some $n^* \geq 1$. (Of course, F can have a linear part as in (3.4).) Therefore, we have, in particular, that R and F have the same Frechét derivative $DR(v) = DF(v)$. Moreover, we assume that for some open ball $\mathcal{U} \subset \mathcal{H}$ one has the following:

(N1) The Frechét derivative $DR(v) : w \mapsto DR(v)w$ is an isomorphism from $\mathcal{H}$ to $\mathcal{H}'$ (see (2.4)), and there exists $\omega > 0$ such that for all $v \in \mathcal{U}$ and y such that $v + y \in \mathcal{U}$, we have

$$\|(DR(v))^{-1}(DR(v+y) - DR(v))y\|_{\mathcal{H}} \leq \omega \|y\|^2_{\mathcal{H}}. \tag{8.4}$$

(N2) There exists a solution $u \in \mathcal{U}$ and an initial guess u^0 in $\mathcal{U}$ such that

$$\|u - u^0\|_{\mathcal{H}} \leq \delta < 2/\omega \quad \text{and} \quad B_\delta(u) \subseteq \mathcal{U} \tag{8.5}$$

with ω from (N1).

Given the validity of (N1) and (N2), standard arguments can be employed to prove that all iterates

$$u^{n+1} = u^n - DR(u^n)^{-1}R(u^n), \tag{8.6}$$

arising from Newton's method formulated in $\mathcal{H}$, remain in $\mathcal{U}$ and satisfy

$$\|u-u^n\|_{\mathcal{H}}<\delta \quad \text{for} \quad n\in\mathbb{N} \quad \text{and} \quad \lim_{n\to\infty}\|u-u^n\|_{\mathcal{H}}=0. \tag{8.7}$$

In fact, one has quadratic convergence

$$\|u-u^{n+1}\|_{\mathcal{H}}\le\frac{\omega}{2}\|u-u^n\|_{\mathcal{H}}^2, \quad n=0,1,2,\dots; \tag{8.8}$$

see, e.g., [18, 19]. Finally, note that, by (2.5), the corresponding iterates in wavelet coordinates satisfy

$$\|\mathbf{u}-\mathbf{u}^{n+1}\|_{\ell_2(\mathcal{J})}\le\tilde{\omega}\|\mathbf{u}-\mathbf{u}^n\|_{\ell_2(\mathcal{J})}^2, \quad n\in\mathbb{N}_0, \quad \tilde{\omega}:=\frac{C_1^2\omega}{2c_1}. \tag{8.9}$$

Let us check that (N1) holds, for instance, in the semilinear case (SL) when F is defined by (3.4) with monotone (scalar valued) G which satisfies (7.5) with $n^*\ge 2$. We have already seen (see Remark 3.1) that $DR(v)=DF(v)$ is an isomorphism from $\mathcal{H}$ to $\mathcal{H}'$. In order to verify (8.4), it therefore suffices to show that

$$\|(G'(v+y)-G'(v))y\|_{\mathcal{H}'}\le\omega\|y\|_{\mathcal{H}}^2. \tag{8.10}$$

For this, we remark that

$$\|(G'(v+y)-G'(v))y\|_{\mathcal{H}'}\le\max_{t\in[0,1]}\|G''(v+ty)y^2\|_{\mathcal{H}'}=\|y\|_{\mathcal{H}}^2\max_{z\in\mathcal{U},\|w\|_{\mathcal{H}}\le 1}\|T(z,w,w)\|_{\mathcal{H}'}$$

with the mapping T defined by

$$T(v,w,z)=G''(v)wz. \tag{8.11}$$

Now, it is easy to check that if G satisfies (7.5) for some p and $n^*\ge 2$, then T satisfies (7.7) with $p_1=[p-2]_+$, $p_2=p_3=1$, and n^* replaced by n^*-2, and therefore, according to Theorem 7.1,

$$\omega:=\max_{z\in\mathcal{U},\|w\|_{\mathcal{H}}\le 1}\|T(z,w,w)\|<\infty, \tag{8.12}$$

as desired. The purpose of this section is to show how the approximate solution of the linear problem (8.1) can be performed again by an iterative scheme along the lines of [9]. By our assumption (N1), we know that $DR(z)$ is an isomorphism from $\mathcal{H}$ to $\mathcal{H}'$ provided that $z\in\mathcal{U}$ and hence $DR(z)$ satisfies (2.10). Given this mapping property (2.10), the adaptive scheme from [9] can actually be applied under fairly general assumptions on the linear isomorphism. For the sake of simplicity, we shall assume that $D\mathbf{R}(\mathbf{z})$ is symmetric positive definite. This is true, for example, in the case (SL) when G is monotone. Recall from Remark 3.3 that one can then find a parameter $\alpha>0$ such that $\mathbf{I}-\alpha D\mathbf{R}(\mathbf{v})$ is a contraction.

The heart of the envisaged adaptive Newton scheme will be to solve the linear problem (8.1) approximately with the aid of a variant, which will be called $\mathbf{SOLVE}_{\mathrm{N}}$, of the scheme $\mathbf{SOLVE}_{\mathrm{lin}}$ discussed in section 3. Before we describe the ingredients of $\mathbf{SOLVE}_{\mathrm{N}}$, let us point out two issues to be addressed when designing these ingredients and analyzing their complexity.

(a) The first point concerns the application of the Jacobian. Approximating at each stage the Jacobian $D\mathbf{R}(\mathbf{u}^n)$ in order to use the **APPLY** scheme based on

(5.7) might be computationally very expensive. (b) The second point concerns the complexity of approximately solving the linear problem (8.1). Recall from Theorem 6.1 that the logic of complexity estimates is to infer from a certain compressibility (or regularity) of the solution a corresponding convergence rate of the adaptive scheme. In the context of the Newton iteration, such a property will be assumed about the solution $\mathbf{u}$ of the original nonlinear problem (2.8) which, however, does not necessarily imply the same property for the solutions of the subproblems (8.1). So it is initially not clear how to derive complexity estimates for the resolution of these subproblems. It will be seen though that the solutions to these subproblems become increasingly closer to elements having the necessary properties, a fact that, as it turns out, can be exploited as long as the subproblems are not solved too accurately. In particular, the question then arises whether the quadratic convergence of the Newton scheme can be preserved.

We now turn to collecting the ingredients of the adaptive Newton scheme. First, the coarsening will be again done by **TCOARSE** even though the problem is linear. More importantly, the **RES** scheme will be of the form $\mathbf{RES}_{\rm lin}$ from (5.10) but with different schemes playing the roles of **APPLY** and **RHS**.

In view of the above issue (a), we shall pursue here the following approach. Recall from (2.9) that for any $\mathbf{v}, \mathbf{z} \in \ell_2(\mathcal{J})$ and corresponding $v = \sum_{\lambda\in\mathcal{J}} v_\lambda\psi_\lambda$, $z = \sum_{\lambda\in\mathcal{J}} z_\lambda\psi_\lambda \in \mathcal{H}$,

$$D\mathbf{R}(\mathbf{z})\mathbf{v} = (\langle\psi_\lambda, DR(z)v\rangle : \lambda \in \mathcal{J}), \tag{8.13}$$

where $DR(z)$ is the Frechét derivative of R at z. This suggests employing the scheme $\mathbf{EV}[\epsilon, \mathbf{Q}, \mathbf{V}]$ with $\mathbf{V} := (\mathbf{z}, \mathbf{v})$ and $\mathbf{Q}(\mathbf{V}) := D\mathbf{R}(\mathbf{z})\mathbf{v} = D\mathbf{F}(\mathbf{z})\mathbf{v}$. We have seen that this scheme has the optimal complexity provided that $\mathbf{Q}$ fulfills Assumptions 1 and 2.

For F defined by (3.4), the mapping $\mathbf{Q}$ has the form

$$\mathbf{Q}(\mathbf{z}, \mathbf{v}) := (\mathbf{A} + D\mathbf{G}(\mathbf{z}))\mathbf{v}. \tag{8.14}$$

The $\mathbf{A}$ part obviously satisfies Assumption 1. We also have seen that it fulfills Assumption 2 under the hypothesis from Remark 7.2. It remains to verify these assumptions for the part

$$D\mathbf{G}(\mathbf{z})\mathbf{v} = (\langle\psi_\lambda, G'(z)v\rangle : \lambda \in \mathcal{J}). \tag{8.15}$$

For this, we simply remark that if G satisfies (7.5) for some p and $n^* \geq 1$, the mapping $(z, v) \mapsto G'(z)v$ satisfies (7.7) with $p_1 = [p-1]_+$, $p_2 = 1$, and n^* replaced by $n^* - 1$. Hence Theorems 7.1 and 7.2 ensure that the mapping $(\mathbf{z}, \mathbf{v}) \mapsto D\mathbf{G}(\mathbf{z})\mathbf{v}$ satisfies Assumptions 1 and 2 in section 7.1.

This suggests the following routine.

$\mathbf{APPLY}_{\rm N}\,[\eta, D\mathbf{R}(\mathbf{z}), \mathbf{v}] \to \mathbf{w}_\eta$ *determines for any tolerance* $\eta > 0$ *and any finitely supported input vectors* $\mathbf{v}$ *and* $\mathbf{z}$ *a finitely supported output vector* $\mathbf{w}_\eta$ *such that*

$$\|D\mathbf{R}(\mathbf{z})\mathbf{v} - \mathbf{w}_\eta\|_{\ell_2(\mathcal{J})} \leq \eta, \tag{8.16}$$

through

$$\mathbf{APPLY}_{\rm N}\,[\eta, D\mathbf{R}(\mathbf{z}), \mathbf{v}] := \mathbf{EV}\,[\eta, \mathbf{Q}, (\mathbf{z}, \mathbf{v})], \tag{8.17}$$

where the routine **EV** *was introduced in section* 7.2.

It remains to specify the routine **RHS**. Here it is issue (b) that calls for some further preparations. The main point is that if the current right-hand sides in (8.1) are not approximated too accurately, then one actually approximates a nearby right-hand side of a problem whose solution is known to be sufficiently sparse and thus can be approximated efficiently by a linear version of **SOLVE**.

Remark 8.1. Suppose that R is twice Frechét differentiable and that $\mathbf{u}$ is the exact solution of (2.8). Then there exists a constant $\hat{C}$ such that, for any $\mathbf{z}$ such that $z = \sum_{\lambda\in\mathcal{J}} z_\lambda\psi_\lambda \in \mathcal{U}$,

$$\|D\mathbf{R}(\mathbf{z})(\mathbf{u}-\mathbf{z}) + \mathbf{R}(\mathbf{z})\|_{\ell_2(\mathcal{J})} \le \hat{C}\|\mathbf{u}-\mathbf{z}\|^2_{\ell_2(\mathcal{J})}. \tag{8.18}$$

Proof. One has

$$-\mathbf{R}(\mathbf{z}) = \mathbf{R}(\mathbf{u}) - \mathbf{R}(\mathbf{z}) = D\mathbf{R}(\mathbf{z})(\mathbf{u}-\mathbf{z}) + \mathcal{O}\left(\|\mathbf{u}-\mathbf{z}\|^2_{\ell_2(\mathcal{J})}\right),$$

which confirms the claim. ∎

We shall employ the following routine in which $\hat{C}$ is the constant of Remark 8.1.

RHS$_{\rm N}[\eta,\mathbf{R},\mathbf{z}] \to \mathbf{r}_\eta(\mathbf{z})$ *is defined for any finitely supported* $\mathbf{z}$ *with* $z\in\mathcal{U}$ *such that* $\|\mathbf{u}-\mathbf{z}\|_{\ell_2(\mathcal{J})} \le \xi$ *and for any* $\eta/2 > \hat{C}\xi^2$ *by*

$$\mathbf{RHS}_{\rm N}[\eta,\mathbf{R},\mathbf{z}] := -\left(\mathbf{EV}\left[\frac{\eta}{2}-\hat{C}\xi^2,\mathbf{F},\mathbf{z}\right] - \mathbf{RHS}[\eta/2,\mathbf{f}]\right), \tag{8.19}$$

where **RHS** *is defined by* (5.2) *but with* **TCOARSE** *used as* **CCOARSE**.

The role of the above conditions on $\mathbf{z}$ and η will become clear later.

We are now prepared to describe the version of **SOLVE** to be used for the approximate solution of the Newton systems (8.1) as follows.

SOLVE$_{\rm N}[\eta,\mathbf{R},\mathbf{z}] \to \mathbf{w}_\eta$ *determines for a given* $\mathbf{z}\in\ell_2(\mathcal{J})$, *such that* $z\in\mathcal{U}$, *an approximate solution* $\mathbf{w}_\eta$ *of the system* $D\mathbf{R}(\mathbf{z})\mathbf{w} = -\mathbf{R}(\mathbf{z})$ *satisfying*

$$\|\mathbf{w}-\mathbf{w}_\eta\|_{\ell_2(\mathcal{J})} \le \eta, \tag{8.20}$$

by invoking **SOLVE**$_{\rm lin}[\eta, D\mathbf{R}(\mathbf{z}), -\mathbf{R}(\mathbf{z})] \to \mathbf{w}_\eta$, *where, under the above assumptions on* $\mathbf{z}$ *and* η, *in step* (ii) *of* **SOLVE** *the residual approximation*

$$\begin{aligned}&\mathbf{RES}_{\rm N}[\eta,\alpha\mathbf{I},D\mathbf{R}(\mathbf{z}),-\mathbf{R}(\mathbf{z}),\mathbf{v}]\\ &\qquad := \alpha\left(\mathbf{APPLY}_{\rm N}\left[\frac{\eta}{2\alpha},D\mathbf{R}(\mathbf{z}),\mathbf{v}\right] - \mathbf{RHS}_{\rm N}\left[\frac{\eta}{2\alpha},\mathbf{R},\mathbf{z}\right]\right),\end{aligned} \tag{8.21}$$

and in step (iii) **TCOARSE** *is used.*

Note that, in view of (8.3), the evaluation of $\mathbf{R}$ also requires the approximation of the data $\mathbf{f}$ as stated explicitly in (8.19). From Theorems 7.3 and 7.4 and Proposition 6.3 we infer, as in Remark 5.1, that $\mathbf{u}\in\mathcal{A}^s_{\rm tree}$ implies $\mathbf{f}\in\mathcal{A}^s_{\rm tree}$, and its η-accurate tree approximation satisfies estimates of the form $\#{\rm supp}\,\mathbf{f}_\eta \lesssim \eta^{-1/s}\|\mathbf{u}\|^{1/s}_{\mathcal{A}^s_{\rm tree}}$, $\|\mathbf{f}_\eta\|_{\mathcal{A}^s_{\rm tree}} \lesssim \|\mathbf{u}\|_{\mathcal{A}^s_{\rm tree}}$.

Moreover, whenever $\mathbf{Q}(\mathbf{z},\mathbf{v}) := D\mathbf{R}(\mathbf{z})\mathbf{v}$ satisfies Assumptions 1 and 2, we can apply Theorems 7.3 and 7.4 to conclude that the output $\mathbf{w}_\eta$ of $\mathbf{APPLY}_{\rm N}[\eta, D\mathbf{R}(\mathbf{z}),\mathbf{v}]$ satisfies

$$\begin{aligned}\#{\rm supp}\,\mathbf{w}_\eta &\lesssim \eta^{-1/s}\left(1+\|\mathbf{z}\|^{1/s}_{\mathcal{A}^s_{\rm tree}} + \|\mathbf{v}\|^{1/s}_{\mathcal{A}^s_{\rm tree}} + \|\mathbf{u}\|^{1/s}_{\mathcal{A}^s_{\rm tree}}\right),\\ \|\mathbf{w}_\eta\|_{\mathcal{A}^s_{\rm tree}} &\lesssim 1+\|\mathbf{z}\|_{\mathcal{A}^s_{\rm tree}} + \|\mathbf{v}\|_{\mathcal{A}^s_{\rm tree}} + \|\mathbf{u}\|_{\mathcal{A}^s_{\rm tree}}.\end{aligned} \tag{8.22}$$

Likewise, the output $\mathbf{r}_\eta(\mathbf{z})$ of $\mathbf{RHS}_{\mathrm{N}}[\eta, \mathbf{R}, \mathbf{z}]$ satisfies

$$\begin{aligned} \#\mathrm{supp}\,\mathbf{r}_\eta(\mathbf{z}) &\lesssim \eta^{-1/s}\left(1+\|\mathbf{z}\|^{1/s}_{\mathcal{A}^s_{\mathrm{tree}}}\right), \\ \|\mathbf{r}_\eta(\mathbf{z})\|_{\mathcal{A}^s_{\mathrm{tree}}} &\lesssim 1+\|\mathbf{z}\|_{\mathcal{A}^s_{\mathrm{tree}}}. \end{aligned} \tag{8.23}$$

Recalling from (6.15) the definition of s^*-sparsity, we can infer from (8.22) the following consequence.

Remark 8.2. Let $s^* := (2\gamma - d)/(2d)$, where γ is the parameter associated with F by Theorem 7.2. Then the scheme $\mathbf{RES}_{\mathrm{N}}$, defined by (8.21), is s^*-sparse whenever $\|\mathbf{z}\|_{\mathcal{A}^s_{\mathrm{tree}}} \lesssim 1$.

We can now formulate an adaptive Newton iteration as follows.

NEWTON $[\epsilon, R, \bar{\mathbf{u}}^0] \to \bar{\mathbf{u}}(\epsilon)$ *determines, for any finitely supported initial guess* $\bar{\mathbf{u}}^0$ *whose corresponding expansion* u_0 *satisfies* (8.5)*, an approximate solution* $\bar{\mathbf{u}}(\epsilon)$ *satisfying*

$$\|\mathbf{u} - \bar{\mathbf{u}}(\epsilon)\|_{\ell_2(\mathcal{J})} \le \epsilon, \tag{8.24}$$

by the following steps:

(i) *Set* $\epsilon_0 := c_1^{-1}\delta$, $j = 0$.

(ii) *If* $\epsilon_j \le \epsilon$, *stop and output* $\bar{\mathbf{u}}(\epsilon) := \bar{\mathbf{u}}^j$. *Otherwise, choose some* $\eta_j > 0$ *(see* (8.27) *below), and perform*

$$\mathbf{SOLVE}_{\mathrm{N}}[\eta_j, \mathbf{R}, \bar{\mathbf{u}}^j, \mathbf{0}] \to \bar{\mathbf{w}}^j.$$

(iii) *Let (see* (8.9)*)*

$$\hat{\mathbf{u}} := \bar{\mathbf{u}}^j + \bar{\mathbf{w}}^j, \quad \hat{\eta}_j := \tilde{\omega}\epsilon_j^2 + \eta_j, \quad \bar{\mathbf{u}}^{j+1} := \mathbf{TCOARSE}[2C^*\hat{\eta}_j, \hat{\mathbf{u}}]$$

(where C^* *is the constant from section* 6.2*), and set* $\epsilon_{j+1} := (1+2C^*)\hat{\eta}_j$, $j+1 \to j$, *and go to* (ii).

The choice of the dynamic tolerance η_j in step (ii) is yet to be specified. The first requirement is to keep the iterates $\bar{\mathbf{u}}^j$ in the right neighborhood of the solution, which means that the corresponding expansions $\bar{u}^j$ lie in $B_\delta(u)$. For this we shall use the following lemma.

LEMMA 8.1. *Fix a positive number* $\beta < 1$, *and assume that* $\delta > 0$ *is chosen sufficiently small to ensure that, in addition to* (8.5)*,*

$$\delta < \frac{c_1^3}{(1+2C^*)C_1^3\omega} \tag{8.25}$$

and

$$\frac{(1+2C^*)\tilde{\omega}\delta}{c_1} < \beta. \tag{8.26}$$

Then the condition $\eta_j \le \eta_0 < \delta/(2(1+2C^*)C_1)$ *implies that* $\bar{u}_j \in B_\delta(u)$ *for all subsequent approximate iterates. Moreover, if*

$$\eta_j \le \frac{\epsilon_j\left(\beta - (1+2C^*)\tilde{\omega}\epsilon_j\right)}{1+2C^*}, \quad j = 0, 1, \ldots, \tag{8.27}$$

one has for $\hat{\eta}_j$ *defined in step* (iii) *of* **NEWTON**

$$\epsilon_{j+1} = (1+2C^*)\hat{\eta}_j \le \beta\epsilon_j, \quad j = 0, 1, \ldots. \tag{8.28}$$

Proof. Denoting by $\mathbf{u}^1$ the exact solution of $D\mathbf{R}(\bar{\mathbf{u}}^0)\mathbf{u}^1 = -\mathbf{R}(\bar{\mathbf{u}}^0)$ and recalling from step (i) that $\|\mathbf{u} - \bar{\mathbf{u}}^0\|_{\ell_2(\mathcal{J})} \le c_1^{-1}\delta = \epsilon_0$, the vector $\hat{\mathbf{u}}$ produced in steps (ii) and (iii) satisfies, by (8.9) and (8.20),

$$\|\mathbf{u} - \hat{\mathbf{u}}\|_{\ell_2(\mathcal{J})} \le \|\mathbf{u} - \mathbf{u}^1\|_{\ell_2(\mathcal{J})} + \|\mathbf{u}^1 - \hat{\mathbf{u}}\|_{\ell_2(\mathcal{J})} \le \tilde{\omega} c_1^{-2}\delta^2 + \eta_0.$$

Thus, taking the coarsening step into account, we infer from (2.5) and (8.9) that

$$\begin{aligned}\|u - \bar{u}^1\|_{\mathcal{H}} &\le C_1\|\mathbf{u} - \bar{\mathbf{u}}^1\|_{\ell_2(\mathcal{J})} \le (1 + 2C^*)C_1(\tilde{\omega}c_1^{-2}\delta^2 + \eta_0)\\ &= \frac{(1 + 2C^*)C_1^3\omega\delta^2}{2c_1^3} + (1 + 2C^*)C_1\eta_0.\end{aligned}$$

Thus when, e.g., $\frac{(1+2C^*)C_1^3\omega\delta^2}{2c_1^3} < \delta/2$, which is (8.25), it suffices to take $\eta_0 < \delta/(2(1 + 2C^*)C_1)$ at the initial stage $j = 0$ to ensure that

$$\|\bar{u}^1 - u\|_{\mathcal{H}} < \|u - u^0\|_{\mathcal{H}},$$

which verifies that $\bar{u}^1 \in B_\delta(u)$. We can now iterate this result; e.g., using $\bar{u}^1$ in place of $\bar{u}^0$, we obtain that $\bar{u}^2 \in B_\delta(u)$, and so on. Now when (8.26) holds, we have $\beta > (1 + 2C^*)\tilde{\omega}\epsilon_0$ so that the condition (8.27) on η_j is feasible for $j = 0$. Moreover, (8.27) implies that $\epsilon_{j+1} = (1 + 2C^*)\hat{\eta}_j \le \beta\epsilon_j$, which is (8.28). This ensures that the error bounds ϵ_j decay to zero so that (8.27) remains feasible for all j. This completes the proof. ∎

PROPOSITION 8.3. *Assume that δ and η_j satisfy* (8.25), (8.26), *and* (8.27), *respectively. Then the scheme* **NEWTON** *terminates after finitely many steps and produces a finitely supported vector $\bar{\mathbf{u}}(\epsilon)$ satisfying*

$$\|\mathbf{u} - \bar{\mathbf{u}}(\epsilon)\|_{\ell_2(\mathcal{J})} \le \epsilon. \tag{8.29}$$

Thus, by (2.5),

$$\left\| u - \sum_{\lambda \in \operatorname{supp} \bar{\mathbf{u}}(\epsilon)} u(\epsilon)_\lambda \psi_\lambda \right\|_{\mathcal{H}} \le C_1\epsilon.$$

Proof. We employ a simple perturbation argument as in the proof of Lemma 8.1. Let $\mathbf{u}^{j+1}$ denote the exact Newton iteration $\mathbf{u}^{j+1} = \bar{\mathbf{u}}^j - D\mathbf{R}(\bar{\mathbf{u}}^j)^{-1}\mathbf{R}(\bar{\mathbf{u}}^j)$. By step (i) we know that $\|\mathbf{u} - \bar{\mathbf{u}}^0\|_{\ell_2(\mathcal{J})} \le \epsilon_0$. Then, supposing that $\|\mathbf{u} - \bar{\mathbf{u}}^j\|_{\ell_2(\mathcal{J})} \le \epsilon_j$, we infer from (8.9) that

$$\|\mathbf{u} - \mathbf{u}^{j+1}\|_{\ell_2(\mathcal{J})} \le \tilde{\omega}\epsilon_j^2. \tag{8.30}$$

Hence, denoting by $\mathbf{w}^j := \mathbf{u}^{j+1} - \bar{\mathbf{u}}^j$ the exact solution of $D\mathbf{R}(\bar{\mathbf{u}}^j)\mathbf{w} = -\mathbf{R}(\bar{\mathbf{u}}^j)$, we obtain, according to step (iii),

$$\begin{aligned}\|\mathbf{u} - \bar{\mathbf{u}}^{j+1}\|_{\ell_2(\mathcal{J})} &\le \|\mathbf{u} - \mathbf{u}^{j+1}\|_{\ell_2(\mathcal{J})} + \|\mathbf{u}^{j+1} - \hat{\mathbf{u}}\|_{\ell_2(\mathcal{J})} + \|\hat{\mathbf{u}} - \bar{\mathbf{u}}^{j+1}\|_{\ell_2(\mathcal{J})}\\ &\le \tilde{\omega}\epsilon_j^2 + \|\mathbf{w}^j - \bar{\mathbf{w}}^j\|_{\ell_2(\mathcal{J})} + 3\hat{\eta}_j \le \tilde{\omega}\epsilon_j^2 + \eta_j + 2C^*\hat{\eta}_j\\ &= (1 + 2C^*)\hat{\eta}_j = \epsilon_{j+1},\end{aligned} \tag{8.31}$$

which advances the induction assumption. By (8.28), this finishes the proof. ∎

It remains to analyze the work/accuracy rate of **NEWTON**. So far the only condition on the tolerances η_j in step (ii) of **NEWTON** is (8.27), which ensures only that the error bounds ϵ_j decay to zero. This would allow us to keep η_j proportional to ϵ_j, which would result in an overall first order error reduction rate. On the other hand, choosing η_j proportional to ϵ_j^2, the error bounds ϵ_j decay, by step (iii), quadratically. However, according to the earlier discussion of issue (b), the subproblem (8.1) should not be resolved too accurately, as reflected by the above right-hand side scheme $\mathbf{RHS}_{\mathrm{N}}$; see (8.19). The following main result of this section says that within these constraints on the tolerances η_j one can still realize an outer convergence rate, ranging from first to second order, in such a way that the overall scheme exhibits optimal rate/complexity.

THEOREM 8.2. *Suppose that* (N1), (N2), *and the above hypotheses on* F *hold. Assume that* δ *satisfies* (8.25) *and* (8.26) *for some fixed* $\beta < 1$. *Moreover, assume that at the* j*th stage of* **NEWTON** *the tolerance* η_j *is in addition to* (8.27) *subjected to the condition*

$$\eta_j \rho^K \geq \zeta \hat{C} \epsilon_j^2 \quad \text{for some fixed } \zeta > 1, \tag{8.32}$$

where ρ, K *are the constants from* (3.17) *and* (4.14). *Then, for any target accuracy* $\epsilon > 0$, **NEWTON** *outputs after finitely many steps a finitely supported vector* $\bar{\mathbf{u}}(\epsilon)$ *satisfying* $\|\mathbf{u} - \bar{\mathbf{u}}(\epsilon)\|_{\ell_2(\mathcal{J})}$, *and hence*

$$\left\| u - \sum_{\lambda \in \operatorname{supp} \bar{\mathbf{u}}(\epsilon)} \bar{u}(\epsilon)_\lambda \psi_\lambda \right\|_{\mathcal{H}} \leq C_1 \epsilon.$$

Moreover, if $\mathbf{Q}(\mathbf{z}, \mathbf{v}) = D\mathbf{R}(\mathbf{z})\mathbf{v}$ *fulfills Assumptions* 1 *and* 2, *and if the solution* $\mathbf{u}$ *of* (2.8) *belongs to* $\mathcal{A}^s_{\mathrm{tree}}$ *for some* $s < s^* = (2\gamma - d)/(2d)$, *the output* $\mathbf{u}(\epsilon)$ *of* **NEWTON** *has the following properties:*

$$\|\mathbf{u}(\epsilon)\|_{\mathcal{A}^s_{\mathrm{tree}}} \lesssim \|\mathbf{u}\|_{\mathcal{A}^s_{\mathrm{tree}}}, \quad \#\operatorname{supp} \mathbf{u}(\epsilon) \lesssim \|\mathbf{u}\|^{1/s}_{\mathcal{A}^s_{\mathrm{tree}}} \epsilon^{-1/s}. \tag{8.33}$$

Under Assumption E the number of floating point operations needed to compute $\bar{\mathbf{u}}(\epsilon)$ *remains proportional to* $\#\operatorname{supp} \mathbf{u}(\epsilon)$.

It is understood that in the final step η_j is chosen within the above constraints as large as possible so as to attain the target accuracy. Note that, as indicated before, within the above constraints on η_j (see (8.27), (8.32)) the convergence rate of the outer inexact Newton iteration can be chosen to vary between linear and quadratic convergence.

Proof of Theorem 8.2. First, observe that in step (iii) one has at the jth stage, by the first part of (8.31),

$$\|\mathbf{u} - \hat{\mathbf{u}}\|_{\ell_2(\mathcal{J})} \leq \|\mathbf{u} - \mathbf{u}^{j+1}\|_{\ell_2(\mathcal{J})} + \|\mathbf{u}^{j+1} - \hat{\mathbf{u}}\|_{\ell_2(\mathcal{J})} \leq \tilde{\omega} \epsilon_j^2 + \eta_j = \hat{\eta}_j. \tag{8.34}$$

Hence, by Proposition 6.3 and step (iii), we obtain

$$\|\bar{\mathbf{u}}^{j+1}\|_{\mathcal{A}^s_{\mathrm{tree}}} \lesssim \|\mathbf{u}\|_{\mathcal{A}^s_{\mathrm{tree}}}, \quad \#\operatorname{supp} \bar{\mathbf{u}}^{j+1} \lesssim \|\mathbf{u}\|^{1/s} \epsilon_{j+1}^{-1/s}. \tag{8.35}$$

Moreover, by Proposition 6.2, the computational work required by the application of **TCOARSE** remains proportional to $\#\operatorname{supp} \hat{\mathbf{u}}$ since the support of $\hat{\mathbf{u}}$ already has tree structure.

Now let us discuss the computational complexity in step (ii) encountered between the coarsening steps. Here it is important that the Newton updates (8.1) are, in view of (8.32), not computed too accurately. In fact, under this constraint the approximation of $\mathbf{R}(\bar{\mathbf{u}}^j)$, computed in $\mathbf{SOLVE}_{\mathrm{N}}$, is incidentally also a sufficiently accurate approximation to $\mathbf{G}(\bar{\mathbf{u}}^j)$; see Remark 8.1. To explain this, recall (8.19) and set

$$\mathbf{Y}^j := \mathbf{EV}\,[\eta_j - \hat{C}\epsilon_j^2, -\mathbf{R}(\bar{\mathbf{u}}^j)].$$

Then one has, by Remark 8.1,

$$\begin{aligned}\|\mathbf{G}(\bar{\mathbf{u}}^j) - \mathbf{Y}^j\|_{\ell_2(\mathcal{J})} &\le \|\mathbf{G}(\bar{\mathbf{u}}^j) + \mathbf{R}(\bar{\mathbf{u}}^j)\|_{\ell_2(\mathcal{J})} + \|\mathbf{R}(\bar{\mathbf{u}}^j) + \mathbf{Y}^j\|_{\ell_2(\mathcal{J})}\\ &\le \hat{C}\epsilon_j^2 + \eta_j - \hat{C}\epsilon_j^2 = \eta_j.\end{aligned}$$

Thus, within the accuracy range permitted by (8.32), the routine $\mathbf{RHS}_{\mathrm{N}}$ invoked by $\mathbf{SOLVE}_{\mathrm{N}}$ satisfies the accuracy requirements for the perturbed equation $D\mathbf{R}(\bar{\mathbf{u}}^j)\hat{\mathbf{w}} = \mathbf{G}(\bar{\mathbf{u}}^j)$, whose solution is, by definition of $\mathbf{G}(\bar{\mathbf{u}}^j)$, just $\hat{\mathbf{w}}^j = \mathbf{u} - \bar{\mathbf{u}}^j$. Now we infer from Remark 8.2, Theorem 6.1, and (8.35) that

$$\#\mathrm{supp}\,\bar{\mathbf{w}}^j \;\lesssim\; \eta_j^{-1/s}\|\mathbf{u} - \bar{\mathbf{u}}^j\|_{\mathcal{A}^s_{\mathrm{tree}}}^{1/s} \;\lesssim\; \eta_j^{-1/s}\|\mathbf{u}\|_{\mathcal{A}^s_{\mathrm{tree}}}^{1/s},$$

that the computational complexity has the same bound, and that $\|\bar{\mathbf{w}}^j\|_{\mathcal{A}^s_{\mathrm{tree}}} \;\lesssim\; \|\mathbf{u}\|_{\mathcal{A}^s_{\mathrm{tree}}}$. The same bounds hold for $\hat{\mathbf{u}}$. By definition of η_j, the computational complexity of step (ii) in **NEWTON** and, by the previous remarks, also of **TCOARSE** in step (iii) therefore remains bounded by $C\|\mathbf{u}\|^{1/s}\epsilon_{j+1}^{-1/s}$, which completes the proof. ∎

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