INVERSE THEOREMS FOR APPROXIMATION BY POSITIVE LINEAR OPERATORS

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We are interested in studying the relation between the smoothness of a function and its degree of approximation by means of a sequence (L_n) of positive linear operators defined on ~~a space~~ of continuous functions $C[a,b]$ or $C^*[-\pi,\pi]$. Our main interest is in what inferences can be made about the smoothness of a function f when we assume something about the rate of decrease of $\|f - L_n(f)\|$. Such a result is customarily called an inverse theorem of approximation while a result which estimates $\|f - L_n(f)\|$ in terms of the smoothness of f is called a direct theorem.

Direct theorems are relatively easy to obtain and are known for the classical examples. On the other hand, inverse theorems are much more difficult to prove and indeed may not even hold. The customary way of proving inverse theorems is to use the ideas used by S. Bernstein in his proof of the inverse theorems for approximation by trigonometric polynomials.

Bernstein's technique relies on knowing estimates for suitable derivatives of $L_n(f)$ (the analogue of Bernstein's inequality). This precludes the handling of general sequences (L_n) since $L_n(f)$ need not even be differentiable in the general case.

In this work, we replace the conditions on the derivatives of $L_n(f)$ by suitable conditions on the concentration of "mass" of L_n. For an example, suppose (L_n) is a sequence of positive convolution operators, i.e.

$$L_n(f,x) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x+t)\,d\mu_n(t)$$

with $d\mu_n$ a *non-negative*, *even* Borel measure on $[-\pi,\pi]$ with *unit mass*. Let

$$\phi_n^2 = \int_{-\pi}^{\pi} t^2\,d\mu_n(t) ,$$

then we can show

THEOREM. *Let* $0 < \alpha \leq 2$. *If there is an* $M > 0$ *with*

$$\frac{\phi_n}{\phi_{n+1}} \leq M < +\infty \ , \quad n = 1,2,\ldots \tag{1}$$

and

$$\int_{-\pi}^{\pi} t^4 d\mu_n(t) = O(\phi_n^4) \tag{2}$$

then $f \in \mathrm{Lip}^*\alpha$ *if and only if* $\|f - L_n(f)\| = O(\phi_n^\alpha)$.

The assumption (2) is the restriction on the concentration of mass. The assumption (1) is always needed for general inverse theorems and it essentially guarantees that the sequence (L_n) is not to sparse, i.e. there are sufficiently many L_n .

As an example of this theorem, let $t_n \downarrow 0$ and

$$L_n(f,x) = \frac{1}{2}\,(f(x+t_n) + f(x-t_n)) \ .$$

Each L_n can be written as convolution with the measure $d\mu_n$ which is purely atomic with masses $\frac{\pi}{2}$ at each of the points $-t_n$ and t_n . The theorem shows that if

$$(3) \qquad \frac{t_n}{t_{n+1}} \leq M < +\infty \qquad n = 1,2,\ldots$$

then

$$(4) \qquad \|\Delta^2_{t_n}(f,x)\| = O(t_n^\alpha)$$

implies

$$(5) \qquad \|\Delta^2_t(f,x)\| = O(t^\alpha)$$

where $\Delta^2_t(f,x) = f(x+t) + f(x-t) - 2f(x)$. It can also be shown that (3) is a necessary condition for (4) to imply (5) in the sense that if (3) does not hold then there is a function f which satisfies (4) but not (5).

We can also use our technique to prove inverse theorems for operators that are not given by convolution. In this case, ϕ_n^2 is replaced by

$$\phi_n^2(x) = L_n((t-x)^2,x)$$

and (2) is replaced by

$$L_n((t-x)^4,x) = 0(\phi_n^4(x))$$

This gives, for example, the inverse theorems for Bernstein polynomials which were given by H. Berens and G.G. Lorentz. Namely, a necessary and sufficient condition for f to be in $\mathrm{Lip}^*\alpha$ is that

$$|f(x) - L_n(f,x)| \leq M \left\{\frac{x(1-x)}{n}\right\}^{\alpha/2}$$

for some constant $M > 0$.