SATURATION AND INVERSE THEOREMS FOR SPLINE APPROXIMATION

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The purpose of this note is to examine the connections between the smoothness of a function and its degree of approximation by algebraic polynomial splines of a fixed degree. Results of this type are known, usually in the form of an estimate for the degree of approximation for a certain method of spline approximation in terms of the smoothness of the function. Estimates like this are customarily called direct theorems of approximation. Our main interest lies in the opposite direction, i.e., what inferences can be made about the smoothness of a function when its degree of approximation is known.

We say S is a spline of degree $k - 1$ if there are points $0 = x_o < x_1 < \dots < x_m = 1$ such that on each interval $[x_{i-1}, x_i)$, $i = 1,2,\dots,m$, S is an algebraic polynomial of degree at most $k - 1$. The points x_i are called the knots of the spline. For generality, we make no restriction on the continuity of S at the knots.

If $\delta = \{0 = x_o < x_1 < \ldots < x_m = 1\}$, let $S(\delta)$ denote the collection of all splines of degree $k - 1$ with knots contained in δ. Define the error in approximating f by $S(\delta)$ as

$$E_\delta(f) = \inf_{S \in S(\delta)} \|f - s\|,$$

where $\|.\|$ denotes the supremum norm on $[0,1]$.

Now suppose (δ_n) is a sequence of sets of knots, $\delta_n = \{0 = x_o^{(n)} < x_1^{(n)} < \ldots < x_{m_n}^{(n)} = 1\}$. We let

$$\|\delta_n\| = \max_{1 \leq i \leq m_n} |x_i^{(n)} - x_{i-1}^{(n)}|$$

and assume that $\|\delta_n\| \to 0$. This guarantees that $E_{\delta_n}(f) \to 0$ for each $f \in C[0,1]$.

By Δ_t^k we denote the $k^{\underline{th}}$ difference operator

$$\Delta_t^k(f,x) = (-1)^k \sum_{o}^{k} (-1)^j \binom{k}{j} f(x + jt) ,$$

so that the corresponding $k^{\underline{th}}$ order modulus of continuity of f is given by

$$\omega_k(f,h) = \sup_{o<t\leq h} \|\Delta_t^k(f,x)\| [0,1-kt] .$$

The notation $\|.\|[a,b]$ is used to indicate that the norm is taken over $[a,b]$. When $[a,b]$ is omitted, the norm is understood to be over $[0,1]$.

A proof of the following direct estimate for $E_{\delta_n}(f)$ in terms of $\omega_k(f,h)$ can be found in [3] .

THEOREM 1. *Suppose* $0 < \alpha \leq k$, $f \in C[0,1]$, *and* $\omega_k(f,h) = O(h^\alpha)$ *as* $h \to 0$. *Then*

(1) $$E_{\delta_n}(f) = O(\|\delta_n\|^\alpha) \qquad (n \to \infty) .$$

We should note that the estimate (1) can actually be obtained by using splines $S_n \in S(\delta_n)$, with $S_n \in C^{k-2}[0,1]$.

Our main concern is in what sense are the estimates of (1) the best possible? We ask the following two questions: When does $E_{\delta_n}(f) = 0(\|\delta_n\|^{\alpha})$, $(n \to \infty)$ imply that $\omega_k(f,h) = 0(h^{\alpha})$, $(h \to 0)$, i.e., does the inverse theorem to (1) hold? Secondly, is it possible to improve (1) if we assume higher smoothness for f ?

It is not possible to answer these questions without some additional restrictions on the sets of knots. The easiest way to see this is when a fixed point, say $\frac{1}{2}$, appears in each set δ_n. Then any spline S which has a single knot at $\frac{1}{2}$ will satisfy $E_{\delta_n}(S) = 0$, $n = 1,2,\ldots,$ but S need not even be continuous. More generally, the same phenomenon manifests itself when a fixed point only falls in small intervals, in comparison to $\|\delta_n\|$. In order to avoid this, we will require that (δ_n) satisfies the following mixing condition:

(2) *There is a constant* $\rho > 0$ *with the property that whenever* $n > 0$ *and* $1 \leq i \leq m_n - 1$, *there is an* $n' > n$ *such that* $x_j^{(n')} < x_i^{(n)} < x_{j+1}^{(n')}$ *with*

$$\min\,(|x_i^{(n)} - x_j^{(n')}|,\ |x_i^{(n)} - x_{j+1}^{(n')}|) > \rho\|\delta_n\| .$$

It is easy to see that (2) guarantees that the following must hold:

(3) *There is a constant* $\rho > 0$ *with the property that whenever* $n > 0$ *and* $x \in [0,1-\rho\|\delta_n\|]$ *, then there is an* $n' > n$ *such that* $x_j^{(n')} \leq x < x_{j+1}^{(n')}$ *with*

$$|x_{j+1}^{(n')} - x| > \rho\|\delta_n\| .$$

Note that equally spaced knots (i.e., $\delta_n = (\frac{i}{n})_o^n$) satisfy the mixing condition.

If the mixing condition holds then we can show that the estimate (1) is the best possible in the sense we have asked.

THEOREM 2. *Let* (δ_n) *be a sequence of sets of knots which satisfy the mixing condition (2) and* $\|\delta_n\| \downarrow 0$. *If* $0 < \alpha \leq k$ *and* $f \in C[0,1]$ *, then*

$$E_{\delta_n}(f) = O(\|\delta_n\|^{\alpha}) , \quad (n \to \infty) \tag{4}$$

if and only if

$$\omega_k(f,t) = O(t^{\alpha}) , \quad (t \to 0)$$

and

$$E_{\delta_n}(f) = O(\|\delta_n\|^k), \quad (n \to \infty), \tag{5}$$

if and only if f *is a polynomial of degree* $\leq k-1$.

Remark: The theorem holds without the restriction that $\|\delta_n\|$ tends to 0 monotonically but the proof becomes somewhat more cumbersome and hides the essential ideas involved.

The equivalence in (4) is the inverse theorem to (1). The equivalence in (5) establishes the saturation phenomenon for splines and shows that the estimates (1) can not be improved by assuming higher smoothness for the function. Since Theorem 2 is proved with no continuity requirement at the knots, it applies to any spline approximation method provided the mixing condition on the knots holds.

Theorem 2 is already known for approximation by splines with equally spaced knots. K. Scherer [6] has given a proof of this under the additional assumption that the splines are smooth (i.e., in $C^{k-2}[0,1]$). Scherer's proof is based on the general method for obtaining inverse theorems developed by P.L. Butzer and K. Scherer [1]. For the saturation parts of

Theorem 2 with equally spaced knots, independent proofs have been given by D. Gaier [4] (a "o" theorem) and F. Richards [5] ("o" and "0" theorems). Our proof of Theorem 2 is new and quite simple and of course has the additional advantage of handling non-equally spaced knots. Also, our technique can be generalized to give inverse and saturation theorems for Chebyshevian splines (see [2]).

PROOF OF THEOREM 2. Because of Theorem 1, we need only establish the necessity in (4) and (5). We will only consider (4) since the proof for (5) is almost identical. First observe that because of the mixing condition (2) and the assumption that $(\|\delta_n\|)$ is monotone, we must have

(6) $$\|\delta_{n+1}\| \geq \rho\|\delta_n\| , \quad n = 1,2,\ldots$$

Now, suppose that $S_n \in S(\delta_n)$, $0 < \alpha \leq k$, and

(7) $$\|S_n - f\| \leq K\|\delta_n\|^\alpha$$

with K a constant. We want to show that $\omega_k(f,h) = 0(h^\alpha)$, $(h \to 0)$. Choose n so that

$$(8) \qquad k^{-1}\rho\|\delta_{n+1}\| \le h < k^{-1}\rho\|\delta_n\|$$

We will only consider those h 's for which the index n in (8) satisfies $\|\delta_n\| < \frac{1}{4}$. This covers all sufficiently small h.

Let $0 < t \le h$. Then for any $x \in [0,\frac{3}{4}]$, (8) gives that

$$(9) \qquad [x,x+kt] \subseteq [x,x+kh] \subseteq [x,x+\rho\|\delta_n\|] \subseteq [0,1] .$$

Since (3) holds, there is an $n' \ge n$ satisfying

$$(10) \qquad [x,x+\rho\|\delta_n\|] \subseteq [x_i^{(n')},x_{i+1}^{(n')}) .$$

Now, $S_{n'}$ is a polynomial of degree at most $k-1$ on $[x_i^{(n')},x_{i+1}^{(n')})$ and thus $\Delta_t^k(S_{n'},x) = 0$. Therefore,

$$\Delta_t^k(f,x) = \Delta_t^k(f-S_{n'},x) = \sum_{j=o}^{k} (-1)^{k-j}\binom{k}{j}(f-S_{n'})(x+jt) .$$

Using (7) in the last expression, we find that there is a constant K_1 such that

$$(11)\quad |\Delta_t^k(f,x)| \leq K\|\delta_{n'}\|^\alpha \leq K\|\delta_n\|^\alpha \leq K(\rho^{-1}\|\delta_{n+1}\|)^\alpha \leq K_1 h^\alpha ,$$

$$x \in [0,\tfrac{3}{4}] ,\quad 0 \leq t \leq h .$$

where in the second to last inequality we used (6) and in the last inequality we used (8).

To get the inequality (11) for $x \in [\frac{3}{4}, 1-kt]$, we consider the function $g(x) = f(1-x)$. The splines $T_n(x) = S_n(1-x)$ have their knots contained in $\delta'_n = \{1-x_i^{(n)} : x_i^{(n)} \in \delta_n\}$ and satisfy

$$\|T_n - g\| \leq K\|\delta'_n\|^\alpha .$$

The sequence of sets of knots (δ'_n) also satisfies the mixing condition (2). Hence, arguing as we have in obtaining (11), we find that

$$|\Delta_t^k(g,x)| < K_2 h^\alpha ,\quad x \in [0,\tfrac{3}{4}] ,\quad 0 \leq t \leq h .$$

This shows that (11) is valid for $x \in [\frac{3}{4}, 1-kt]$, $0 \leq t \leq h$. Therefore, $\omega_k(f,h) \leq K_2 h^\alpha$ and the necessity in (4) is established.

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