AN EXTENSION OF BERNSTEIN'S INEQUALITY

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1. Introduction

Let Π_n denote the class of trigonometric polynomials of degree $\leq n$. If $T\in\Pi_n$, then

(1.1) $$\|T'\|_{L^\infty} \leq n\|T\|_{L^\infty} ,$$

where $\|\cdot\|_{L^\infty}$ is the L^∞-norm on $[-\pi,\pi]$. This, of course, is the classical inequality of S. Bernstein which plays an important role in many areas of analysis, especially in approximation theory. There are several generalizations of Bernstein's inequality. These usually are obtained by either replacing Π_n by another class of functions, (e.g., algebraic polynomials or functions of exponential type) or by replacing L^∞ by other spaces (e.g., L^p, $1\leq p<\infty$).

Here, we are interested in a generalization of Bernstein's inequality obtained in the second manner. We will consider trigonometric polynomials as being members of certain dual spaces. Namely, if ω is a modulus of continuity and $1\leq p<\infty$, let L^p_ω denote the space of all 2π-periodic functions in $L^p[-\pi,\pi]$ for which

(1.2) $$\|f\|_{L^p_\omega} = \max\left(\|f\|_{L^p},\ \sup_{t>0}(\omega(t))^{-1}\|f(x+t)-f(x)\|_{L^p}\right)$$

is finite. Here $\|\cdot\|_{L^p}$ is the L^p-norm on $[-\pi,\pi]$. $(L^p_\omega,\|\cdot\|_{L^p_\omega})$ is then a Banach space and we denote its dual space by $(L^p_\omega)^*$. For $p=\infty$, we use L^∞_ω to denote the space of 2π-periodic continuous functions with $\|\cdot\|_{L^\infty_\omega}$ defined by (1.2), with $p=\infty$.

Although, this is some abuse of notation for $p=\infty$, it will provide simpler statements of our results.

Each trigonometric polynomial T determines in a natural way a functional ℓ_T in $(L^p_\omega)^*$ by

$$\ell_T(f) = \int_{-\pi}^{\pi} f(t)T(t)dt.$$

Our main result is Theorem 1 in Section 2 which states that there is an absolute constant $c>0$ such that for $n\geq 1$, $1\leq p\leq\infty$ and ω a modulus of continuity, we have

$$(1.3) \qquad \|\ell_T\|_{(L^p_\omega)^*} \geq c\omega(n^{-1})\|T\|_{L^q}$$

for all $T\in\Pi_n$, where $q = \frac{p}{p-1}$ is the conjugate index. Thus, (1.3) gives a lower bound for the norm of the functional ℓ_T.

To see the connections between (1.3) and Bernstein's inequality, let us take $p=1$ and $\omega(t) = t$. Then L^1_ω is the collection of all 2π-periodic functions of bounded variation (see Butzer and Nessel [1; p. 181 and p. 367]) and

$$\|f\|_{L^1_\omega} = \max(\|f\|_{L^1}, \int_{-\pi}^{\pi} |df|).$$

Now, let $T\in\Pi_n$, then

$$\|T\|_{L^\infty} = \sup_{\|f'\|_{L^1}\leq 1}\left\{\int_{-\pi}^{\pi} f'(t)T(t)dt\right\} \geq \sup_{\|f\|_{L^1_\omega}\leq 1}\left\{\int_{-\pi}^{\pi} f(t)T'(t)dt\right\}$$

$$= \|\ell_{T'}\|_{(L^1_\omega)^*} \geq \frac{c}{n}\|T'\|_{L^\infty}$$

where the last inequality is (1.3). So, we see that (1.3) retrieves Bernstein's inequality in a non-sharp form, since we don't have $c=1$. In a similar way, we could show that Bernstein's inequality gives (1.3) for $p=1$, $\omega(t)=t$. We should remark that we do not concern ourselves as to what is the largest constant c for which (1.3) holds.

Inequality (1.3) gives a lower estimate for the norm of the convolution operator $L_T(f) = f*T$ considered as a mapping from L^p_ω to L^∞. That is,

$$\|L_T\| \geq c\omega(\tfrac{1}{n})\|T\|_{L^q} .$$

Such lower estimates combined with the Banach-Steinhaus theorem give necessary conditions for a smoothness criteria, like $f \in L^p_\omega$, to guarantee uniform approximations by convolution operators. In Section 3, we illustrate this for the n^{th} partial sums of the Fourier series. We also illustrate how to obtain necessary conditions for a smoothness criteria to guarantee absolute convergence of the Fourier series. While the results of Section 3 are well-known, our approach provides some unification for theorems of this type.

2. Proof of Inequality (1.3)

In what follows n will always denote a positive integer, T a polynomial of degree $\leq n$, $1 \leq p \leq \infty$, and $q = p/(p-1)$, the conjugate index to p. For the proof of inequality (1.3), we will use a representation for trigonometric polynomials which can be found in Zygmund [4, Ch.X].

If k is an integer, let $x_k = (2k\pi)(3n)^{-1}$. Then, T can be represented by

$$(2.1) \qquad T(x) = n^{-1} \sum_{k=0}^{3n-1} T(x_k)V_n(x-x_k)$$

where

$$(2.2) \qquad V_n(x) = \frac{\sin(3nx/2)\,\sin(nx/2)}{3n\,\sin^2(x/2)}$$

Also, there is an absolute constant $c_1 > 1$ such that

$$(2.3) \qquad \begin{aligned} c_1^{-1}\left(n^{-1}\sum_{k=0}^{3n-1}|T(x_k)|^p\right)^{\frac{1}{p}} &\leq \|T\|_{L^p} \leq c_1\left(n^{-1}\sum_{k=0}^{3n-1}|T(x_k)|^p\right)^{\frac{1}{p}}, \quad 1 \leq p < \infty, \\ c_1^{-1}\max_{0\leq k\leq 3n-1}|T(x_k)| &\leq \|T\|_{L^\infty} \leq c_1 \max_{0\leq k\leq 3n-1}|T(x_k)|. \end{aligned}$$

The idea of the proof of (1.3) is to construct a function g in L^p_ω, with $\|g\|_{L^p_\omega} \leq 1$, such that

$$\int_{-\pi}^{\pi} g(x)T(x)dx \geq c\omega(\tfrac{1}{n})\|T\|_{L^q} .$$

We now proceed to the construction of g and some of its properties. If $0<\delta<1/4$, let h_δ denote the 2π-periodic "roof" function defined on $[-\pi,\pi]$ by

$$(2.4) \qquad h_\delta(x) = \begin{cases} 1 - \frac{3n}{2\delta\pi}|x|, & |x| \leq \frac{2\delta\pi}{3n}, \\ 0 \quad , & \frac{2\delta\pi}{3n} \leq |x| \leq \pi. \end{cases}$$

Lemma 1. Let $0<\delta<1/4$ and $T\in\Pi_n$. If $1<p\leq\infty$, define

$$g_\delta(x) = A\omega(n^{-1}) \sum_{k=0}^{3n-1} |T(x_k)|^{q-1} \operatorname{sgn}T(x_k)h_\delta(x-x_k),$$

where

$$A^{-1} = \left(n^{-1} \sum_{k=0}^{3n-1} |T(x_k)|^q\right)^{\frac{1}{p}} .$$

For $p=1$, let k_o be such that $T(x_{k_o}) = \max_k |T(x_k)|$ and define

$$g_\delta(x) = n\omega(n^{-1})\operatorname{sgn}T(x_{k_o})h_\delta(x-x_{k_o}).$$

Then, g_δ is in L^p_ω and there is a constant $c_2(\delta)$, depending only on δ for which

$$(2.5) \qquad \|g_\delta\|_{L^p_\omega} \leq c_2(\delta).$$

Proof: We consider only the case $1<p<\infty$, the other cases are handled similarly. If $3nt \geq 2\delta\pi$, then it follows from (2.4) that

$$\|h_\delta(x+t)-h_\delta(x)\|^p_{L^p} = \int_{-\pi}^{\pi} |h_\delta(x+t)-h_\delta(x)|^p\, dx \leq$$

$$\int_{-t-\frac{2\delta\pi}{3n}}^{-t+\frac{2\delta\pi}{3n}} |h_\delta(x+t)-h_\delta(x)|^p dx + \int_{\frac{-2\delta\pi}{3r}}^{\frac{2\delta\pi}{3n}} |h_\delta(x+t)-h_\delta(x)|^p dx \leq \frac{2^p\cdot 8\delta\pi}{3n} .$$

Here, we have used the fact that $|h_\delta(x)| \leq 1$, for all x.

Now, the functions $h_\delta(x-x_k)$, $k=0,\ldots,3n-1$, have disjoint supports and so

$$\|g_\delta(x+t)-g_\delta(x)\|_{L^p}=A\omega(n^{-1})\left(\sum_{k=0}^{3n-1}|T(x_k)|^q\|h_\delta(x+t-x_k)-h_\delta(x-x_k)\|_{L^p}^p\right)^{\frac{1}{p}}$$

$$\leq 2A\left(\frac{8\delta\pi}{3}\right)^{\frac{1}{p}}\omega(n^{-1})\left(n^{-1}\sum_{k=0}^{3n-1}|T(x_k)|^q\right)^{\frac{1}{p}} = 2\left(\frac{8\delta\pi}{3}\right)^{\frac{1}{p}}\omega(n^{-1})$$

$$\leq 2\left(1+\frac{8\delta\pi}{3}\right)\omega(n^{-1}),$$

where we have used the definition of A.

Since ω is a modulus of continuity, for $3nt \geq 2\delta\pi$

$$\omega(n^{-1}) \leq \left(1+\frac{3}{2\delta\pi}\right)\omega(t).$$

Therefore,

$$\text{(2.6)}\qquad \|g_\delta(x+t)-g_\delta(x)\|_{L^p} \leq c(\delta)\omega(t), \qquad t \geq \frac{2\delta\pi}{3n}\ .$$

When $3nt \leq 2\delta\pi$, we have

$$\|h_\delta(x+t)-h_\delta(x)\|_{L^p}^p \leq \int_{\frac{-4\delta\pi}{3n}}^{\frac{4\delta\pi}{3n}} |h_\delta(x+t)-h_\delta(x)|^p dx \leq \left(\frac{3n}{2\delta\pi}t\right)^p\cdot\frac{8\delta\pi}{3n}.$$

Hence,

$$\text{(2.7)}\qquad \|g_\delta(x+t)-g_\delta(x)\|_{L^p} \leq A\left(\frac{8\delta\pi}{3}\right)^{\frac{1}{p}}\frac{3nt}{2\delta\pi}\,\omega(n^{-1})\left(n^{-1}\sum_{k=0}^{3n-1}|T(x_k)|^q\right)^{\frac{1}{p}}$$

$$\leq c(\delta)nt\omega(n^{-1}) \leq 2c(\delta)\omega(t), \qquad t \leq \frac{2\delta\pi}{3n}\ .$$

In the last inequality, we have used the fact that $t_1^{-1}\omega(t_1) \leq 2t_2^{-1}\omega(t_2)$ when $t_2 < t_1$, which holds for moduli of continuity. The inequalities (2.6), (2.7), and the similar estimate $\|g_\delta\| \leq (n^{-1})$ prove (2.5).

<u>Lemma 2</u>. There are absolute constants c_3, $c_4 > 0$, such that

$$\text{(2.8)}\qquad \int_{-\pi}^{\pi} h_\delta(t)V_n(t) \geq c_3\delta \qquad \text{for each } 0 < \delta < 1/4,$$

$$(2.9)\qquad \int_{-\pi}^{\pi} h_\delta(t)|V_n(t-x_j)|dt \le c_4\delta^2(j-1/2)^{-2}, \quad 0<|x_j|<\pi.$$

Proof: From (2.4), it follows that $h_\delta(t) \ge \frac{1}{2}$ for $3n|t| \le \delta\pi$. From (2.2), we find that on this same interval

$$V_n(t) \ge \frac{\frac{2}{\pi}\cdot\frac{3nt}{2}\cdot\frac{2}{\pi}\cdot\frac{nt}{2}}{3n\left(\frac{t}{2}\right)^2} = 4\pi^{-2}n$$

Thus,

$$\int_{-\pi}^{\pi} h_\delta(t)V_n(t)dt = \int_{\frac{-2\delta\pi}{3n}}^{\frac{2\delta\pi}{3n}} h_\delta(t)V_n(t)dt$$

$$\ge \int_{\frac{-\delta\pi}{3n}}^{\frac{\delta\pi}{3n}} h_\delta(t)V_n(t) \ge \frac{1}{2}\cdot 4\pi^{-2}\, n\cdot\frac{2\delta\pi}{3n} = \frac{4\delta\pi^{-1}}{3} .$$

Note that $V_n(t) \ge 0$ for $|t| \le \frac{2\delta\pi}{3n}$, since $\delta < 1/4$.

To prove (2.9), we return to (2.2) to find, when $|x_j| < \pi$,

$$|V_n(t-x_j)| \le (3n)^{-1}\sin^{-2}\left(\frac{t-x_j}{2}\right)\sin(\delta\pi), \quad |t| \le \frac{2\delta\pi}{3n}.$$

Therefore,

$$\frac{1}{\pi\delta}\int_{-\pi}^{\pi} h_\delta(t)|V_n(t-x_j)|dt \le \int_{\frac{-2\delta\pi}{3n}}^{\frac{2\delta\pi}{3n}} (3n)^{-1}\sin^{-2}\left(\frac{t-x_j}{2}\right)dt$$

$$\le \frac{4\delta\pi}{9n^2}\sin^{-2}\frac{1}{2}\left(|x_j|-\frac{2\delta\pi}{3n}\right) \le \frac{4\delta\pi^3}{9n^2}\left(|x_j|-\frac{2\delta\pi}{3n}\right)^{-2} \le c_4\left(j-\frac{1}{2}\right)^{-2}\delta,$$

where in the second to last inequality we used the fact that $\sin^2 t/2 \ge \pi^{-2}t^2$ for $|t| \le \pi$ and in the last inequality we used our assumption that $\delta < 1/4$.

Theorem 1. There is an absolute constant $c>0$, such that for each $1\le p\le\infty$, $n>0$ and $T\in\Pi_n$, we have

$$(2.10)\qquad \|\ell_T\|_{(L^p_\omega)^*} \ge c\omega(n^{-1})\|T\|_{L^q},$$

where $q = p/p-1$.

Proof: We will prove the theorem for the case $1<p<\infty$, the other cases are handled similarly. Also, we will suppose that n is odd. It is clear that if (2.10) holds for $n = 2m+1$ then it also holds for $n = 2m$ with c replaced by $c/3$. Before we begin the proof it is useful to observe that because of periodicity, the sums in (2.1), (2.3) and the definition of g_δ can be taken over any set of 3n consecutive integers.

Now, let g_δ be the function introduced in Lemma 1 with $0<\delta<1/4$, to be prescribed later. We have from (2.1) that

$$\int_{-\pi}^{\pi} g_\delta(x)\ T(x)dx = n^{-1} \sum_{k=0}^{3n-1} T(x_k) \int_{-\pi}^{\pi} g_\delta(x)\ V_n(x-x_k)dx$$

For each k, let I_k' denote the set of the 3n consecutive integers with middle term k and $I_k = I_k' - \{k\}$. Then, from Lemma 2, we find

$$(A\omega(n^{-1}))^{-1}\ T(x_k) \int_{-\pi}^{\pi} g_\delta(x)\ V_n(x-x_k)dx \geq |T(x_k)|^q \cdot$$

$$\int_{-\pi}^{\pi} h_\delta(x-x_k)V_n(x-x_k)dx - \sum_{j\in I_k} |T(x_k)|\ |T(x_j)|^{q-1}\int_{S_j} h_\delta(x-x_j)\cdot$$

$$V_n(x-x_k)dx \geq c_3\delta|T(x_k)|^q - c_4\delta^2 \sum_{j\in I_k} (j-k-\tfrac{1}{2})^{-2}|T(x_k)|\cdot$$

$$|T(x_j)|^{q-1}\ ,$$

where $S_j = \{x\colon |x-x_j| \leq \frac{2\delta\pi}{3n}\}$.

This last estimate shows that

$$\text{(2.12)}\quad \int_{-\pi}^{\pi} g_\delta(x)T(x)dx \geq c_3\delta A\omega(n^{-1})n^{-1} \sum_{k=0}^{3n-1} |T(x_k)|^q$$

$$- c_4\delta^2 A\omega(n^{-1})n^{-1} \sum_{k=0}^{3n-1} \sum_{j\in I_k} (j-k-\tfrac{1}{2})^{-2}|T(x_k)|\,|T(x_j)|^{q-1}\ .$$

The second sum on the right hand side of (2.12) can be estimated by letting $k-j = \nu$, to find

$$(2.13)\quad \sum_{k=0}^{3n-1} \sum_{j\in I_k} (j-k-\tfrac{1}{2})^{-2} |T(x_k)|\ |T(x_j)|^{q-1}$$

$$\leq \sum_{0<|\nu|<\frac{3n-1}{2}} (\nu-\tfrac{1}{2})^{-2} \sum_{k=0}^{3n-1} |T(x_k)|\ |T(x_{k-\nu})|^{q-1}$$

$$\leq \sum_{0<|\nu|\leq\frac{3n-1}{2}} (\nu-\tfrac{1}{2})^{-2} \left(\sum_{k=0}^{3n-1} |T(x_k)|^q\right)^{\frac{1}{q}} \left(\sum_{k=0}^{3n-1} |T(x_k)|^q\right)^{\frac{1}{p}}$$

$$\leq \sum_{\nu=1}^{\infty} (\nu-\tfrac{1}{2})^{-2} \sum_{k=0}^{3n-1} |T(x_k)|^q .$$

Now, choose $0<\delta<1/4$ sufficiently small that

$$c_3\delta - c_4\delta^2 \sum_{\nu=1}^{\infty} (\nu-\tfrac{1}{2})^{-2} = c_5 > 0$$

and fix δ. Then we use (2.13) in (2.12) to obtain

$$\int_{-\pi}^{\pi} g_\delta(x)T(x)dx \geq c_5 A\omega(n^{-1})n^{-1} \sum_{k=0}^{3n-1} |T(x_k)|^q$$

$$= c_5\omega(n^{-1})\left(n^{-1}\sum_{k=0}^{3n-1} |T(x_k)|^q\right)^{\frac{1}{q}} \geq c_5 c_1^{-1}\omega(n^{-1})\|T\|_{L^q}.$$

We have used (2.3) and the definition of A. Since $\|g_\delta\| \leq c(\delta)$, with $c(\delta)$ now an absolute constant, the theorem is proved.

3. Applications

As we have indicated in the introduction, we wish to use the Banach-Steinhaus theorem together with Theorem 1 to examine convergence theorems for convolution operators. To accomplish this, we must work with spaces in which trigonometric polynomials are dense. Accordingly, we introduce the space $L^p_{\omega,o}$ which is the closure of trigonometric polynomials in L^p_ω. If $t^{-1}\omega(t) \to \infty$ as $t \to 0$, then $L^p_{\omega,o}$ is the collection of all functions f in L^p_ω for which

$$\|f(x+t)-f(x)\|_{L^p} = o(\omega(t)) \qquad (t \to 0).$$

If $\omega(t) = O(t)$, then $L^p_{\omega,o}$ is the collection of all absolutely continuous functions f for which $f' \in L^p$.

For our applications, we will consider $S_n(f)$ the n^{th} partial sum of the Fourier series of f,

$$S_n(f) = \sum_{k=-n}^{n} \hat{f}(k)e^{ikx} = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)D_n(t-x)dt = f*D_n$$

where D_n is the Dirichlet kernel

$$D_n(t) = \frac{\sin(n+1)t/2}{2\ \sin t/2}$$

We first ask what are necessary and sufficient conditions on p and ω to guarantee that $f \in L^p_{\omega,o}$ implies $S_n(f)$ converges uniformly. From the Banach-Steinhaus Theorem, we see that a necessary and sufficient condition is that the operators S_n as mappings from $L^p_{\omega,o}$ to L^∞ be uniformly bounded. The norm of the operator S_n is the same as the norm of the functional

$$S_n(f,0) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)D_n(t)dt$$

in $(L^p_{\omega,o})^*$. Since any function f in L^p_ω with $\|f\|_{L^p_\omega} \le 1$ is the limit in L^p of functions from the unit ball of $L^p_{\omega,o}$, we see that

$$(3.1) \qquad \|S_n\|_{L^p_{\omega,o}} = \|S_n\|_{L^p_\omega} = \frac{1}{\pi}\|\ell_{D_n}\|_{(L^p_\omega)^*}.$$

Therefore, what we are seeking is a necessary and sufficient condition for

$$(3.2) \qquad \|\ell_{D_n}\|_{(L^p_\omega)^*} = O(1)$$

Now, if we apply Theorem 1, we see there is a constant $c_1>0$, with

$$(3.3) \qquad c_1\omega(n^{-1})\|D_n\|_{L^q} \le \|\ell_{D_n}\|_{(L^p_\omega)^*}.$$

It is also true that there is a constant $c_2>0$, such that

$$(3.4) \qquad \|\ell_{D_n}\|_{(L^p_\omega)^*} \leq c_2\omega(n^{-1})\|D_n\|_{L^q} + 3$$

This can be shown by using the de la Vallee Poussin operators V_m, $m=[n/2]$ (see G. Lorentz [2, p. 93]). The operator V_m as an operator from L^p to L^∞ has norm ≤ 3 and there is a constant $c_2 > 0$ such that

$$\|f-V_m(f)\|_{L^p} \leq c_2\omega(n^{-1})\|f\|_{L^p_\omega}$$

Also, $V_m(f) = S_n(V_m(f))$. Hence, if $\|f\|_{L^p_\omega} \leq 1$, then

$$\|S_n(f)\|_{L^\infty} \leq \|S_n(f-V_m(f))\|_{L^\infty} + \|V_m(f)\|_{L^\infty} \leq c_2\omega(n^{-1})\|D_n\|_{L^q} + 3$$

which is (3.4). The inequalities (3.3) and (3.4) show that (3.2) holds if and only if

$$\omega(n^{-1})\|D_n\|_{L^q} = O(1) .$$

Standard calculations give that

$$(3.5) \qquad \|D_n\|_{L^q} \sim \begin{cases} \ln n , & p = \infty, \\ n^{\frac{1}{p}} , & 1 \leq p < \infty . \end{cases}$$

Hence, a necessary and sufficient condition for $f \in L^p_{\omega,o}$ to imply $S_n(f)$ converges uniformly is that

$$(3.6) \qquad \omega(n^{-1})\ln n = O(1), \qquad p = \infty ,$$

$$(3.7) \qquad \omega(n^{-1})n^{\frac{1}{p}} = O(1) , \qquad 1 \leq p < \infty .$$

As a second example, we seek necessary conditions on p and ω, so that $f \in L^p_{\omega,o}$ implies that the Fourier series of f converges absolutely.

Let (T_n) be any sequence of trigonometric polynomials, $T_n \in \Pi_n$ with

$$|\hat{T}_n(k)| \leq 1, \qquad k = 0,\pm 1,\ldots,\pm n \text{ ,}$$

If each $f \epsilon L^p_{\omega,o}$ has an absolutely convergent Fourier series then

$$\|f*T_n\|_{L^\infty} \leq \sum_{-n}^{n} |\hat{f}(k)||\hat{T}_n(k)| \leq \sum_{-n}^{n} |\hat{f}(k)| \leq \sum_{-\infty}^{\infty} |\hat{f}(k)| < +\infty,$$

$$n = 1,2,\ldots$$

Thus, the convolution operators $L_n(f) = f*T_n$ considered as mappings from $L^p_{\omega,o}$ to L^∞ are pointwise and hence uniformly bounded. It then follows from Theorem 1 that there is an absolute constant $M>0$ such that

$$c_1\omega(n^{-1})\|T_n\|_{L^q} \leq \|\ell_{T_n}\|_{(L^p_\omega)^*} = \frac{1}{\pi}\|L_{T_n}\| \leq M.$$

Since M does not depend on T_n, we have

(3.8) $\quad \Lambda_{n,p}\,\omega(n^{-1}) = O(1),$

where

$$\Lambda_{n,p} = \sup_{\substack{T\epsilon\Pi_n \\ |\hat{T}(k)|\leq 1}} \|T\|_{L^q} .$$

That is, (3.8) is a necessary condition on p and ω for each $f\epsilon L^p_{\omega,o}$ to have an absolutely convergent Fourier series. The asymptotic behavior of $\Lambda_{n,p}$ is given by

$$(3.9) \qquad \Lambda_{n,p} \sim \begin{cases} n^{\frac{1}{2}} \;, & 2 \leq p \leq \infty \text{ ,} \\ n^{\frac{1}{p}} \;, & 1 \leq p \leq 2 \text{ ,} \end{cases}$$

The case $2\leq p\leq\infty$ in (3.9) follows from a result of D. J. Newman [3, Theorem 1]. The case $1\leq p\leq 2$ follows from the well known inequality $\|f\|_{L^q} \leq \|\hat{f}\|_{\ell^p}$ (see Zygmund [4, p. 101, Vol. II]) and the previous estimate (3.5). Hence, (3.9) shows that a necessary condition for each $f \in L^p_{\omega,o}$ to have an absolutely convergent Fourier series is that

$$(3.10)\quad \begin{aligned} n^{\frac{1}{p}}\,\omega(n^{-1}) &= O(1)\ , & 1 \le p \le 2. \\ n^{\frac{1}{2}}\,\omega(n^{-1}) &= O(1)\ , & 2 \le p \le \infty. \end{aligned}$$

Most classical theorems on absolute convergence of Fourier series are stated for the classes L^p_ω. Although (3.10) does give necessary conditions for each $f \in L^p_\omega$ to have an absolutely convergent Fourier series, slightly stronger necessary conditions are known. We consider one example, $p=\infty$, which is typical, and refer the reader to Zygmund [4, p. 240, Vol. I] for a more detailed discussion of absolute convergence theorems. When $p=\infty$, there is a function $f \in L^\infty_{t^{1/2}}$, whose Fourier series does not converge absolutely. Our result would only show that for each ω for which $\lim_{t\to 0} t^{-1/2}\,\omega(t) = \infty$, there is a function $f \in L^\infty_{\omega,o}$ whose Fourier series does not converge absolutely.

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References

[1] Butzer, P. L. and R. J. Nessel, Fourier Analysis and Approximation, I. Academic Press, New York, 1970.

[2] Lorentz, G. G., Approximation of Functions. Holt, Rinehart and Winston, New York, 1966.

[3] Newman, D. J., An L^1 extremal problem for polynomials. Proc. A.M.S. 16 (1965), 1287-1290.

[4] Zygmund, A., Trigonometric Series, I, II. Cambridge Univ. Press, Cambridge, 1959, 383 pp. and 364 pp.

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