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# Minimum Moduli of Differential Operators from the Viewpoint of Approximation Theory

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## 1. INTRODUCTION

One of the prettiest results in approximation theory is an old theorem of S. Bernstein which states that if $f^{(n)}$ is absolutely continuous on $[-1, 1]$ and $f^{(n+1)}$ is in $L^\infty[-1, 1]$, then

$$\operatorname{dist}_\infty (f, \mathbf{P}_n) \leqslant \frac{2^{-n}}{(n+1)!} \| f^{(n+1)} \|_\infty , \tag{1}$$

where the distance is measured in the $L^\infty[-1, 1]$ norm, $\| \cdot \|_\infty$, and $\mathbf{P}_n$ is the space of algebraic polynomials of degree $\leqslant n$. The inequality (1) is sharp, since when $f(x) = 2^{-n}/(n+1)! \cos(n+1) \operatorname{arc\,cos} x$, it becomes an equality. In approximation theoretic terms, (1) provides an estimate for the error in approximating $f$ in $\| \cdot \|_\infty$ by elements of $\mathbf{P}_n$. Of course, it is of interest to obtain results for other norms (e.g., $L^p$) and other spaces. This paper makes a modest contribution in this direction.

In order to provide the proper setting for our generalizations, we introduce the idea of the minimum modulus of a differential operator. Suppose $T$ is an ordinary linear differential operator of order $n$ with domain $\mathscr{D}(T) := \{f \in L^p[-1, 1]: f^{(n-1)} \text{ a.c.}, Tf \in L^q[-1, 1]\}$. Denote by $\mathscr{N}(T)$, the null space of $T$. When considered as a mapping on $\mathscr{D}(T)/\mathscr{N}(T)$ into $L^q[-1, 1]$, $T$ has an inverse. The reciprocal of the norm of this inverse operator is called the minimum modulus of $T$, $\gamma(T, p, q)$. It is easy to verify the formula

$$(\gamma(T, p, q))^{-1} = \sup_{\|Tf\|_q \leqslant 1} \operatorname{dist}_p(f, \mathscr{N}(T)). \tag{2}$$

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If there is a function $f$ in $\mathscr{D}(T)$ for which the supremum on the right-hand side of (2) is attained, we say $f$ is extremal.

We can now restate (1) in terms of the minimum modulus. We use $n+1$ instead of $n$ and consider the operator $T = D^{n+1}$. The null space of $T$ is $\mathbf{P}_n$. Hence, the inequality (1) and the fact that it is sharp is exactly the same as saying

$$\gamma(D^{n+1}, \infty, \infty) = 2^n(n+1)!\,.$$

Also, the function $2^{-n}/(n+1)! \cos(n+1) \operatorname{arc\,cos} x$ is extremal.

In this sense, we see that (1) is really just the determination of $\gamma(D^{n+1}, \infty, \infty)$. It is in this spirit that we seek generalizations of (1). Thus, we will replace $D^{n+1}$ by more general operators $T$, and replace $\mathbf{P}_n$ by $\mathscr{N}(T)$. We would also like to replace $L^\infty$ by other $L^p$ spaces. The problem then is to determine $\gamma(T, p, q)$. When this is accomplished, we have the inequality

$$\operatorname{dist}_p(f, \mathscr{N}(T)) \leqslant (\gamma(T, p, q))^{-1} \| Tf \|_q\,, \tag{3}$$

as our generalization to (1), and of course (3) is sharp.

Our techniques will be applicable to operators $T$ of the form

$$T = (D + \lambda_n(x)) \cdots (D + \lambda_1(x)), \qquad \lambda_k \in C^{(n-k)}[-1, 1]. \tag{4}$$

The functions $\lambda_k$ are assumed to be real valued. These requirements on $T$ guarantee, among other things, that $\mathscr{N}(T)$ is a Chebyshev space of dimension $n$.

In Section 2, we will determine $\gamma(T, p, \infty)$, for each $1 \leqslant p \leqslant \infty$. The case $\gamma(T, \infty, \infty)$ is a result of M. Zedek [7]. The value of $\gamma(T, \infty, \infty)$ was also obtained by T. Rivlin [6] with a different point of view. Our approach differs from Rivlin's and Zedek's. It is more in line with the traditional proofs of Bernstein's inequality. In Section 3, we will determine $\gamma(T, 1, 1)$. Here, our approach is approximation theoretic, relying heavily on duality and characterizations of best approximations in $L^1$.

The reader will find that conspicuously absent is the determination of $\gamma(T, 2, 2)$, which on the surface would appear to be the most manageable because of all the structure in $L^2$. This case can be handled in a theoretical sense using a calculus of variations approach as is done in the paper of S. Goldberg and A. Meir [3]. However, the determination of the numerical value of $\gamma(T, 2, 2)$, even for $T = D^n$, appears to be a formidable problem.

## 2. THE DETERMINATION OF $\gamma(T, p, \infty)$

When $T$ is a differential operator of the form (4), then $\mathscr{N}(T)$ is a Chebyshev space of dimension $n$ (Zedek [7]). This means that we can interpolate any

$n$ values by functions in $\mathcal{N}(T)$. We will use a remainder formula for this interpolation which is the analogue of Cauchy's formula for Lagrange interpolation.

LEMMA 1. *If $f \in C^{(n)}[-1, 1]$ has $(n+1)$ distinct zeros in $[-1, 1]$, then there is a point $\xi$ in $(-1, 1)$ for which $Tf(\xi) = 0$.*

*Proof.* This is a result of Zedek [7].

LEMMA 2. *Suppose $\psi$ is a solution to the equation $T\psi = 1$ on $[-1, 1]$ which has exactly $n$ distinct zeros $x_1, x_2, \ldots, x_n$ in $[-1, 1]$. If $f \in C^{(n)}[-1, 1]$ and $P \in \mathcal{N}(T)$ interpolates $f$ at each point $x_1, x_2, \ldots, x_n$, i.e., $P(x_i) = f(x_i)$, $i = 1, 2, \ldots, n$, then for each $x \in [-1, 1]$ there is a $\xi_x \in (-1, 1)$ such that*

$$f(x) - P(x) = Tf(\xi_x)\, \psi(x). \tag{5}$$

*Proof.* The proof is an exact mimic of the proof of Cauchy's formula for $T = D^n$. The formula (5) is clear when $x$ is one of the points $x_1, x_2, \ldots, x_n$. When $x \neq x_i$, $i = 1, 2, \ldots, n$, let $\alpha = (\psi(x))^{-1} \cdot (f(x) - P(x))$. The function $f(t) - P(t) - \alpha\psi(t)$ vanishes at the $n+1$ distinct points $x, x_1, x_2, \ldots, x_n$, and hence by Lemma 1 there is a point $\xi_x$ for which $T(f - P - \alpha\psi)(\xi_x) = 0$. Since, $TP = 0$ and $T\psi = 1$ on $[-1, 1]$, the last equation can be rewritten as $\alpha = Tf(\xi_x)$ which gives (5) by the very definition of $\alpha$.

Now, let $\psi$ be any function in $C^{(n)}[-1, 1]$ for which $T\psi = 1$ on $[-1, 1]$. We want to approximate $\psi$ by elements of $\mathcal{N}(T)$ in the $L^p[-1, 1]$ norm. The existence and uniqueness of such approximants are classical results [4]. Denote by $P_p{}^*$, the best $L^p[-1, 1]$ approximation to $\psi$ from $\mathcal{N}(T)$, so that

$$\| \psi - P_p{}^* \|_p = \inf_{P \in \mathcal{N}(T)} \| \psi - P \|_p .$$

LEMMA 3. *For each $1 \leq p \leq \infty$, the function $\psi_p = \psi - P_p{}^*$ has exactly $n$ distinct zeros in $[-1, 1]$ and changes sign at each of these zeros.*

*Proof.* We first show that $\psi_p$ has at least $n$ changes of sign in $[-1, 1]$. For $p = 1, \infty$, this follows from the classical alternation theorems. For $1 < p < \infty$, the proof is simple enough. By the duality theorem for $L^p$ approximation [4, p. 84], the function $h_p = |\psi_p|^{p-1} \operatorname{sgn} \psi_p$ is orthogonal to $\mathcal{N}(T)$, i.e.,

$$\int_{-1}^{1} P(x)\, h_p(x)\, dx = 0, \qquad P \in \mathcal{N}(T). \tag{6}$$

Notice that $h_p$ changes sign precisely at the points where $\psi_p$ changes sign. If $\psi_p$ and hence $h_p$ has less than $n$ changes of sign in $(-1, 1)$, then we can construct a function $P \in \mathcal{N}(T)$ which changes sign precisely at these points (this is a well-known property of Chebyshev systems [5, p. 30]). This makes it clear that

$$\int_{-1}^{1} P(x)\, h_p(x)\, dx \neq 0,$$

which is the desired contradiction. Thus, $\psi_p$ has at least $n$ changes of sign in $(-1, 1)$.

That $\psi_p$ can have no more than $n$ zeros in $[-1, 1]$ follows from Lemma 1. For if not there would be a point $\xi \in (-1, 1)$, with $T\psi_p(\xi) = 0$ which contradicts the fact that $T\psi_p = 1$ on $[-1, 1]$.

We can now easily prove the main result of this section.

THEOREM 1. *If $T$ is of the form* (4), *then for* $1 \leqslant p \leqslant \infty$, $\gamma(T, p, \infty) = \|\psi_p\|_p^{-1}$ *and* $\psi_p$ *is an extremal function.*

*Proof.* Let $x_1, x_2, \ldots, x_n$ be the $n$ distinct zeros of $\psi_p$ in $[-1, 1]$. Suppose first that $f \in C^{(n)}[-1, 1]$ and $P_f \in \mathcal{N}(T)$ is the function which interpolates $f$ at $x_1, x_2, \ldots, x_n$. From Lemma 2, we see that if $x \in [-1, 1]$, there is a point $\xi_x \in (-1, 1)$ such that

$$f(x) - P_f(x) = Tf(\xi_x)\, \psi_p(x).$$

So,

$$|f(x) - P_f(x)| \leqslant \|Tf\|_\infty\, |\psi_p(x)|, \qquad -1 \leqslant x \leqslant 1.$$

Taking the norm in $L^p$, we find

$$\|f - P_f\|_p \leqslant \|Tf\|_\infty\, \|\psi_p\|_p,$$

or

$$\operatorname{dist}_p(f, \mathcal{N}(T)) \leqslant \|\psi_p\|_p\, \|Tf\|_\infty. \tag{7}$$

This result also holds for all $f$ with $f^{(n-1)}$ absolutely continuous and $Tf \in L^\infty[-1, 1]$, because of the denseness of $C^{(n)}[-1, 1]$ in this space. The estimate (7) shows that $\gamma(T, p, \infty) \geqslant \|\psi_p\|^{-1}$. The opposite inequality is immediate when we take $f = \psi_p$.

When $T = D^n$, the functions $\psi_p$ and the values of $\gamma(D^n, p, \infty)$ are easily obtained for $p = 1, 2, \infty$. For $p = 1$, $\psi_1$ is the normalized Chebyshev polynomial of the second kind and $\gamma(D^n, 1, \infty) = 2^{n-1}n!$. For $p = 2$, $\psi_2$ is the normalized Legendre polynomial of degree $n$ and $\gamma(D^n, 2, \infty) = (n + 1/2)^{1/2}(2n)!/2^n n!$. For $p = \infty$, $\psi_\infty$ is the Chebyshev polynomial of degree $n$ and $\gamma(D^n, \infty, \infty) = 2^{n-1}n!$, which is again Bernstein's result (1).

## 3. The Determination of $\gamma(T, 1, 1)$

In this case the situation is a little more subtle, mainly because there is no extremal function in $\mathscr{D}(T)$. We will use a well-known duality theorem for $L^1$ approximation which we now state.

Lemma 4. *Let* $f \in L^1[-1, 1]$ *and* $\mathscr{M}$ *be a finite dimensional subspace of* $L^1[-1, 1]$. *Then,*

$$\operatorname{dist}_1(f, \mathscr{M}) = \sup_{h \in \mathscr{M}^\perp} \int_{-1}^{1} f(x)\, h(x)\, dx,$$

*where* $\mathscr{M}^\perp = \{h \in L^\infty : \| h \|_\infty = 1$ *and* $\int_{-1}^{1} P(x)\, h(x)\, dx = 0,\ P \in \mathscr{M}\}$.

Recall, the function $\psi_1$ introduced in the last section. Since the span of $\mathscr{N}(T) \cup \{\psi_1\}$ is a Chebyshev space (use Lemma 1), we also have available Markov's theorem for $L^1$ approximation [4, p. 67].

Lemma 5. *Let* $f \in C[-1, 1]$ *and* $P_f \in \mathscr{N}(T)$ *be that function which interpolates* $f$ *at* $x_1, x_2, \ldots, x_n$ *the zeros of* $\psi_1$. *If* $f - P_f$ *changes sign precisely at* $x_1, x_2, \ldots, x_n$, *then* $P_f$ *is the best* $L^1[-1, 1]$ *approximation to* $f$ *from* $\mathscr{N}(T)$ *and*

$$\operatorname{dist}_1(f, \mathscr{N}(T)) = \left| \int_{-1}^{1} f(x) \operatorname{sgn} \psi_1(x)\, dx \right|.$$

Lemma 6. *If* $f \in C^{(n)}[-1, 1]$ *with* $Tf > 0$ $(Tf < 0)$ *on* $[-1, 1]$, *then* $P_f$ *is the best* $L^1[-1, 1]$ *approximation to* $f$ *from* $\mathscr{N}(T)$ *and*

$$\operatorname{dist}_1(f, \mathscr{N}(T)) = \left| \int_{-1}^{1} f(x) \operatorname{sgn} \psi_1(x)\, dx \right|.$$

*Proof.* From Lemma 2, we see that

$$f(x) - P_f(x) = Tf(\xi_x)\, \psi_1(x).$$

Since $Tf > 0$, $f - P_f$ will change sign precisely when $\psi_1$ does and thus the result follows from Lemma 5.

We can now determine $\gamma(T, 1, 1)$. Let $T^*$ be the adjoint operator to $T$ (see [2, p. 1285]). Denote by $\Psi_1$, the solution to the differential equation $T^*\Psi = \operatorname{sgn} \psi_1$, with initial conditions $\Psi_1^{(k)}(-1) = 0$, $k = 1, 2, \ldots, n$. When $f \in C^{(n)}[-1, 1]$ and $P \in \mathscr{N}(T)$ with $P^{(k)}(1) = f^{(k)}(1)$, $k = 1, 2, \ldots, n$, then

$$\begin{aligned}\int_{-1}^{1} f(x)\,\operatorname{sgn}\psi_1(x)\,dx &= \int_{-1}^{1} (f(x)-P(x))\operatorname{sgn}\psi_1(x)\,dx\\ &= \int_{-1}^{1} (f(x)-P(x))\,T^*\Psi_1(x)\,dx\\ &= \int_{-1}^{1} T(f-P)(x)\,\Psi_1(x)\,dx\\ &= \int_{-1}^{1} Tf(x)\,\Psi_1(x)\,dx. \qquad (8)\end{aligned}$$

Similarly, if $h$ is any function in $\mathcal{N}(T)^\perp$ and $H$ satisfies $T^*H = h$, with $H^{(k)}(-1) = 0$, $k = 1, 2,\ldots, n$, then

$$\int_{-1}^{1} f(x)\,h(x)\,dx = \int_{-1}^{1} Tf(x)\,H(x)\,dx \qquad (9)$$

THEOREM 2. *If $T$ is of the form* (4), *then* $\gamma(T, 1, 1) = \|\Psi_1\|_\infty^{-1}$.

*Proof.* First, let $f_0 \in C^{(n)}[-1, 1]$ with $\|Tf_0\|_1 = 1$ and $h \in \mathcal{N}(T)^\perp$. Then, by (9)

$$\int_{-1}^{1} f_0(x)\,h(x)\,dx = \int_{-1}^{1} Tf_0(x)\,H(x)\,dx \leqslant \sup_{\|Tf\|_1=1} \int_{-1}^{1} Tf(x)\,H(x)\,dx. \qquad (10)$$

It is to be understood in (10) and the sequel that the suprema are taken only over functions in $C^{(n)}[-1, 1]$, unless explicitly stated otherwise. Now, the supremum in (10) is the $L^\infty[-1, 1]$ norm of $H$. If $\|H\|_\infty = H(x_0)$ for some $x_0 \in [-1, 1]$, then the supremum can be attained by considering only functions with $Tf > 0$. Hence, in this case,

$$\begin{aligned}\int_{-1}^{1} f_0(x)\,h(x)\,dx &\leqslant \sup_{\substack{Tf>0\\ \|Tf\|_1=1}} \int_{-1}^{1} Tf(x)\,H(x)\,dx\\ &= \sup_{\substack{Tf>0\\ \|Tf\|_1=1}} \int_{-1}^{1} f(x)\,h(x)\,dx \leqslant \sup_{\substack{Tf>0\\ \|Tf\|_1=1}} \left|\int_{-1}^{1} f(x)\operatorname{sgn}\psi_1(x)\,dx\right|\\ &= \sup_{\substack{Tf>0\\ \|Tf\|_1=1}} \left|\int_{-1}^{1} Tf(x)\,\Psi_1(x)\,dx\right| \leqslant \|\Psi_1\|_\infty .\end{aligned}$$

Here, for the second inequality, we used Lemmas 6 and 4.

Similarly, when $\|H\|_\infty = -H(x_0)$, we need only consider $f$'s with $Tf < 0$. Arguing as we have above, we find that

$$\int_{-1}^{1} f_0(x)\,h(x)\,dx \leqslant \|\Psi_1\|_\infty ,$$

for all $h \in \mathcal{N}(T)$. Hence, from Lemma 4, it follows that

$$\operatorname{dist}_1(f_0, \mathcal{N}(T)) \leqslant \| \Psi_1 \|_\infty, \tag{11}$$

whenever $f_0 \in C^{(n)}[-1, 1]$, with $\| Tf_0 \|_1 = 1$. The restriction $f_0 \in C^{(n)}$ is removed by a denseness argument. Thus, (11) shows that

$$\gamma(T, 1, 1) \geqslant \| \Psi_1 \|_\infty^{-1}. \tag{12}$$

To see the reverse inequality, consider any $f \in C^{(n)}[-1, 1]$, with $Tf > 0$ $(Tf < 0)$ and $\| Tf \|_1 = 1$. Then, from Lemma 6

$$\operatorname{dist}_1(f, \mathcal{N}(T)) = \left| \int_{-1}^{1} f(x) \operatorname{sgn} \psi_1(x)\, dx \right| = \left| \int_{-1}^{1} Tf(x)\, \Psi_1(x)\, dx \right|.$$

Taking a supremum over all such $f$ we see that the right-hand side becomes the $L^\infty[-1, 1]$ norm of $\Psi_1$, so that

$$\sup_{\| Tf \|_1 = 1} \operatorname{dist}_1(f, \mathcal{N}(T)) \geqslant \| \Psi_1 \|_\infty .$$

In other words,

$$\gamma(T, 1, 1) \leqslant \| \Psi_1 \|_\infty^{-1}.$$

This is the reverse inequality to (12) and proves the theorem.

When we take $T = D^n$, the function $\psi_1$ is the Chebyshev polynomial of the second kind of degree $n$. Hence $\operatorname{sgn} \psi_1 = \operatorname{sgn} \sin(n + 1) \arccos x$. This means that $\operatorname{sgn} \psi_1$ changes sign at the points $\cos(k\pi/(n + 1))$, $k = 1, 2, \ldots, n$. The points $\cos(k\pi/(n + 1))$ are spaced so that the distance between consecutive points increases as we move from $-1$ to $0$ and decreases as we move from $0$ to $1$. Because of this, an induction argument shows that $\| \Psi_1 \|_\infty$ is $| \Psi_1(0) |$ when $n$ is odd and $| \Psi_1(\cos(n + 2)\pi/(2n + 2)) |$ when $n$ is even.

Rather than try to determine $\| \Psi_1 \|_\infty$ directly, it is easier to return to the ideas used in the proof of Theorem 2. Consider the case when $\| \Psi_1 \|_\infty = \Psi_1(0)$. Then,

$$\Psi_1(0) = \int_{-1}^{1} \Psi_1(x)\, d\mu(x),$$

where $d\mu$ is the Dirac measure with unit mass at 0. The measure $d\mu$ is not the $n$th derivative of a function from $\mathscr{D}(D^n)$ which is why we dont have an extremal function. However,

$$\begin{aligned} \Psi_1(0) &= \int_{-1}^{1} \Psi_1(x)\, d\mu(x) \\ &= \frac{1}{(n-1)!} \int_{-1}^{1} x_+^{n-1} \operatorname{sgn} \psi_1(x)\, dx = \frac{1}{(n-1)!} \operatorname{dist}_1(x_+^{n-1}, \mathbf{P}_{n-1}), \end{aligned}$$

where $x_+^{n-1}$ is defined to be 0 if $x < 0$ and $x^{n-1}$ if $x > 0$.

Even though there is no extremal function in the strict sense ($x_+^{n-1}$ is not in $\mathscr{D}(D^n)$), the function $x_+^{n-1}/(n-1)!$ still serves the purpose of determining $\gamma(D^n, 1, 1)$ when $n$ is odd. Similarly,

$$(\gamma(D^n, 1, 1))^{-1} = \frac{1}{(n-1)!} \operatorname{dist}_1((x\text{-}\cos(n+2)\,\pi/(2n+2))_+^{n-1}, \mathbf{P}_{n-1}),$$

when $n$ is even. The problem of determining $\operatorname{dist}_1(x_+^{n-1}, \mathbf{P}_{n-1})$ is solved explicitly in [1] by means of a finite but complicated sum which we do not reproduce here. When $n$ is even, the results of [1] do not determine $\gamma(D^n, 1, 1)$ explicitly but do provide asymptotic estimates.

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