DEGREE OF MONOTONE APPROXIMATION

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1. Introduction

We are interested in approximating monotone functions by monotone polynomials. Denote by Π_n, the space of algebraic polynomials of degree $\leq n$, and by Π_n^*, the set of those polynomials in Π_n which are monotone non-decreasing on the interval $[-1,1]$. If f is monotone non-decreasing on $[-1,1]$ ($f\uparrow$), we define the degree of monotone approximation of f as

$$E_n^*(f) = \inf_{P\in\Pi_n^*} ||f-P||, \quad n=1,2,\ldots$$

with $||.||$ the supremum norm on $[-1,1]$. The problem then is to give estimates for $E_n^*(f)$ in terms of the structural properties of f. The hope is to obtain the same order of estimates as for the unconstrained degree

$$E_n(f) = \inf_{P\in\Pi_n} ||f-P||, \quad n=1,2,\ldots .$$

The first estimates of the degree of monotone approximation were given by O. Shisha[8] with much refinement given later by J. Roulier[7]. In 1968, G.G. Lorentz and

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K. Zeller [4] gave the estimate

$$(1.1) \qquad E_n^*(f) \leq C\omega(f,n^{-1}),\ n=1,2,\ldots .$$

This result is interesting because it shows that we can obtain the same estimates for $E_n^*(f)$ as is guaranteed by the Jackson theorem for $E_n(f)$. Later, G.G. Lorentz [3] extended (1.1) to include differentiable functions by showing

$$(1.2) \qquad E_n^*(f) \leq C\, n^{-1}\omega(f',n^{-1}),\ n=1,2,\ldots$$

which again is the same order as in the Jackson theorem for unconstrained approximation. Here, the matter has now stood for some time with no answer to the question of whether the higher order estimates

$$(1.3) \qquad E_n^*(f) \leq C_r n^{-r}\omega(f^{(r)},n^{-r}),\ n=1,2,\ldots$$

hold for $r \geq 2$?

While (1.3) has not been settled, there have been some negative results obtained, again by Lorentz and Zeller [5]. They have shown that there exist functions f with

$$\sup_n \frac{E_n^*(f)}{E_n(f)} = \infty.$$

This shows that in general there is some loss in the degree of approximation with the monotone constraint.

In Section 2, we will give a new proof of (1.1) and (1.2). Our method is somewhat more direct than that of Lorentz and Zeller since we do not have to go to the trigonometric case. Our techniques can also be used for what is called co-monotone approximation (see the remarks in Section 5).

In Section 3, we give estimates for $E_n^*(f)$ when f has an $r^{\underline{th}}$ derivative of bounded variation. These results are obtained very simply from a theorem of G. Freud [2] on one-sided approximation. These results are interesting because for this class of functions the estimates of $E_n^*(f)$ are of the same order as for $E_n(f)$.

The main contribution of this paper is in Section 4 where we give estimates of $E_{2n}^*(x^{2n+1})$. This problem is the monotone analogue of the Chebyshev problem for unconstrained approximation. We show that

$$C_1 n^{\alpha} 2^{-2n} \leq E_{2n}^*(x^{2n+1}) \leq C_2 n^{\beta} 2^{-2n},\ n=1,2,\ldots$$

where $1/4 < \alpha = \log_3 4 - 1$ and $\beta = 1 - \log_a 2 < 1/2$, with $a = 2 + 3^{\frac{1}{2}}$. This tightens the previously known estimates

$$2^{-2n} \leq E_{2n}^*(x^{2n+1}) \leq C\, n 2^{-2n}. \tag{1.4}$$

The lower estimate in (1.4) is obvious from $E_{2n}^*(x^{2n+1}) \geq E_{2n}(x^{2n+1})$, while the upper estimate was shown by Lorentz [3]. Here we see that there is a loss in the efficiency of the monotone approximation of the functions x^{2n+1} as compared with the unconstrained approximation.

2. Jackson-Type Estimates for Monotone Approximation

We want to give a new proof of the following theorem of Lorentz and Zeller [4] and Lorentz [3].

THEOREM 1. *There exists a constant* $C>0$ *such that for each* $f\uparrow$ *and continuous*

$$E_n^*(f) \leq C\omega(f, n^{-1}),\ n=1,2,\ldots \tag{2.1}$$

<u>and for each $f\uparrow$ and continuously differentiable</u>

$$E_n^*(f) \le C\,n^{-1}\omega(f,n^{-1}),\ n=1,2,\dots \tag{2.2}$$

PROOF. We will construct a sequence of linear polynomial operators which preserve monotonicity and provide the estimates of the theorem. It will be notationally more convenient to work on $[-\frac{1}{4},\frac{1}{4}]$ rather than on $[-1,1]$. Let (λ_n) be a sequence of positive algebraic polynomials of degree n with the properties:

$$\int_{-1}^{1}\lambda_n(t)\,dt = 1,\ n=1,2,\dots \tag{2.3}$$

$$\int_{-1}^{1}t^2\lambda_n(t)\,dt \le C_1 n^{-2},\ n=1,2,\dots \tag{2.4}$$

$$\sup_{|t|\ge\frac{1}{4}}\lambda_n(t) \le C_2 n^{-2},\ n=1,2,\dots \tag{2.5}$$

General methods for constructing such polynomials are given in [1], Chapter 6. One example is to take

$$\lambda_{2n-4}(t) = c_n\left(\frac{P_n(t)}{t^2-x_{1,n}^2}\right)^2,$$

where P_n is the Legendre polynomial of degree n and $x_{1,n}$ is its smallest positive zero. c_n is the normalizing constant for (2.3). For such a sequence (λ_n), the operators

$$L_n(f,x) = \int_{-\frac{1}{2}}^{\frac{1}{2}} f(t)\lambda_n(x-t)\,dt \tag{2.6}$$

provide the estimate (see [1])

$$\|f-L_n(f)\|[-\tfrac{1}{4},\tfrac{1}{4}] \le \begin{cases} C\,\omega(f,n^{-1}) \\ C\,n^{-1}\,\omega(f',n^{-1}) \end{cases},\ n=1,2,\dots \tag{2.7}$$

The notation $||.||[-\frac{1}{4},\frac{1}{4}]$ indicates that the norm is the supremum norm over the interval $[-\frac{1}{4},\frac{1}{4}]$. Since f needs to be defined on $[-\frac{1}{2},\frac{1}{2}]$ in the definition of $L_n(f)$, let us first observe that any $f\uparrow$ on $[-\frac{1}{4},\frac{1}{4}]$ can be extended to an increasing function on $[-\frac{1}{2},\frac{1}{2}]$, without changing its modulus of continuity. This is done by defining f to be $f(\frac{1}{4})$ on $[\frac{1}{4},\frac{1}{2}]$ and $f(-\frac{1}{4})$ on $[-\frac{1}{2},-\frac{1}{4}]$. Similarily if f is continously differentiable, we can define f on $[\frac{1}{4},\frac{1}{2}]$ by $f(\frac{1}{4})+f'(\frac{1}{4})(x-\frac{1}{4})$ and on $[-\frac{1}{2},-\frac{1}{4}]$ by $f(-\frac{1}{4})+f'(-\frac{1}{4})(x+\frac{1}{4})$.

Now, take $f\uparrow$ on $[-\frac{1}{4},\frac{1}{4}]$ and extend f to $[-\frac{1}{2},\frac{1}{2}]$ as described above. We can assume that f is non-negative on $[-\frac{1}{2},\frac{1}{2}]$, since adding and subtracting constants does not effect the monotonicity of f or the approximating polynomials. The polynomial $L_n(f)$ provides the necessary estimates because of (2.7), so we need only check for monotonicity. An integration by parts, followed by a differentiation shows that for $x\in[-\frac{1}{4},\frac{1}{4}]$

$$\begin{aligned} L_n(f)'(x) &= -f(\tfrac{1}{2})\lambda_n(x-\tfrac{1}{2})+f(-\tfrac{1}{2})\lambda_n(x+\tfrac{1}{2})+\int_{-\frac{1}{2}}^{\frac{1}{2}}\lambda_n(x-t)\,df(t) \\ &\geq -f(\tfrac{1}{2})\lambda_n(x-\tfrac{1}{2}) \geq -f(\tfrac{1}{2})C_2n^{-2},\ n=1,2,\ldots, \end{aligned}$$

where in the first inequality we used the fact that λ_n, f, and df are non-negative, and in the second, we used (2.5). Thus, while $L_n(f)$ is not necessarily monotone, the polynomial $\bar{L}_n(f,x) = L_n(f,x)+C_2n^{-2}f(\frac{1}{2})x$ is monotone and also provides the estimate

$$||f-\bar{L}_n(f)||[-\tfrac{1}{4},\tfrac{1}{4}] \leq C\omega(f,n^{-1})+C_2f(\tfrac{1}{2})n^{-2} \leq C'\omega(f,n^{-1}),$$

for n=1,2,... . Similarily, when f is differentiable

$$||f-\bar{L}_n(f)||[-\tfrac{1}{4},\tfrac{1}{4}] \leq C'n^{-1}\omega(f',n^{-1})$$

for n=1,2,... . This proves the theorem.

3. Monotone Approximation when $f^{(r)}$ is of Bounded Variation

There is a simple connection between monotone approximation and what is called one-sided L^1 approximation. If f is in $L^1[-1,1]$, let $\Pi_n(f)$ denote the set of those polynomials P in Π_n with $P \geq f$ on $[-1,1]$. The error of one-sided approximation to f from Π_n is defined by

$$E_n(f,\text{one-sided}) = \inf_{P \in \Pi_n(f)} ||f-P||_{L^1}.$$

If $f\uparrow$ and P is a polynomial with $P \geq f'$ on $[-1,1]$, then $P \geq 0$ on $[-1,1]$ and so

$$Q(x) = f(-1) + \int_{-1}^{x} P(t)\, dt$$

is a monotone polynomial on $[-1,1]$ and

$$||f-Q|| \leq ||f'-P||_{L^1}.$$

Thus, whenever $f\uparrow$ with f' in $L^1[-1,1]$, we find

$$E_n^*(f) \leq E_n(f',\text{one-sided}),\ n=1,2,\ldots . \tag{3.1}$$

This easily derived inequality actually gives sharp estimates for certain classes of functions. Let Λ_r denote the class of functions with $f^{(r-1)}$ absolutely continuous and $f^{(r)}$ of bounded variation on $[-1,1]$. There is a theorem of G. Freud[2] which shows that if $f \in \Lambda_r$, then

$$E_n(f,\text{one-sided}) \leq c_r n^{-r-1} \operatorname{Var}(f^{(r)}) \tag{3.2}$$

Combining (3.1) with (3.2) proves the following theorem.

THEOREM 2. *There exist constants* $C_r > 0$, *such that whenever*

$f\uparrow$ with $f\in\Lambda_r$, then

$$E_n^*(f) \le C_r n^{-r}\mathrm{Var}(f^{(r)}),\ n=1,2,\ldots . \tag{3.3}$$

The estimate (3.3) is the same as for unconstrained approximation and it can not be improved for the class Λ_r. For example, the function

$$f_o(x) = x_+^r = \begin{cases} 0, & x \le 0 \\ x^r, & x > 0 \end{cases}$$

is in Λ_r and it is well known that there is a constant $C>0$, for which $E_n(f_o) \ge Cn^{-r}$, $n=1,2,\ldots$. This shows that (3.3) can not be improved for the class Λ_r.

4. Monotone Approximation of x^{2n+1}

The function x^{2n+1} is monotone increasing on $[-1,1]$ and so we can ask for estimates of $E_{2n}^*(x^{2n+1})$.

THEOREM 3. Let $\alpha = \log_3 4 - 1$, and $\beta = 1-\log_a 2$, with $a=2+3^{\frac{1}{2}}$. There exist constants $C_1, C_2>0$, such that

$$C_1 n^{\alpha} 2^{-2n} \le E_{2n}^*(x^{2n+1}) \le C_2 n^{\beta} 2^{-2n},\ n=1,2,\ldots . \tag{4.1}$$

PROOF. We will first prove the lower estimate. Suppose there are arbitrarily large values of n for which

$$E_{2n}^*(x^{2n+1}) < n^{\alpha} 2^{-2n-5}. \tag{4.2}$$

We will derive a contradiction, which in turn proves the lower estimate in (4.1). Throughout, n denotes a value for which (4.2) holds. Let $P_n\uparrow$ be a polynomial of degree 2n such that

$$||x^{2n+1}-P_n(x)|| \le n^{\alpha} 2^{-2n-5}. \tag{4.3}$$

Consider the polynomials $Q_n(x)=x^{2n+1}-P_n(x)$ and $R_n(x)=(2n+1)^{-1}Q_n'(x)$. We begin with the following estimate for R_n.

$$(4.4)\quad \int_{-\frac{1}{2}}^{\frac{1}{2}}(1-x^2)^{\frac{1}{2}}|R_n(x)|dx \le \int_{-\frac{1}{2}}^{\frac{1}{2}}x^{2n}dx + (2n+1)^{-1}\int_{-\frac{1}{2}}^{\frac{1}{2}}P_n'(x)dx$$

$$\le (2n+1)^{-1}(2^{-2n}+P_n(\tfrac{1}{2})-P_n(-\tfrac{1}{2})) \le (2n+1)^{-1}2^{-2n}(2+n^{\alpha}2^{-4})$$

$$\le n^{\alpha-1}\,2^{-2n-5}$$

provided n is sufficiently large. In the second to last inequality, we used (4.3) to estimate $P_n(\frac{1}{2})$ and $P_n(-\frac{1}{2})$ each by $2^{-2n-1}+n^{\alpha}2^{-2n-5}$.

We can also estimate the integral over $[-1,1]$ by using Markov's inequality. Namely,

$$(4.5)\quad \int_{-1}^{1}(1-x^2)^{\frac{1}{2}}|R_n(x)|dx \le 2||(1-x^2)^{\frac{1}{2}}R_n(x)||$$

$$\le 2||Q_n|| \le n^{\alpha}2^{-2n-4}$$

again using (4.3). Recall that $||\cdot||$ always denotes the supremum norm over $[-1,1]$.

If S is any polynomial of degree m with leading coefficient 1, then

$$(4.6)\quad \int_{-1}^{1}(1-x^2)^{\frac{1}{2}}|S(x)|dx \ge \int_{-1}^{1}(1-x^2)|S(x)|dx$$

$$\ge \int_{-1}^{1}|U_{m+2}(x)|dx = 2^{-m-1},$$

where U_k denotes the Chebyshev polynomial of the second kind of degree k, normalized with leading coefficient 1. This is the well known extremal property that U_k has the smallest $L^1[-1,1]$ norm among all polynomials of degree k with leading coefficient 1. We now construct a V_λ which multiplied by R_n will give a polynomial S that will contradict (4.6). Let

$$V_\lambda(x)=(3/8)^\lambda T_\lambda((8x^2-5)/3)$$

where $T_\lambda(x)=2^{-\lambda+1}\cos\lambda\arccos x$ is the Chebyshev polynomial, normalized to have leading coefficient one. The value of λ will be prescribed shortly. We can estimate

$$(4.7)\qquad \|V_\lambda\|[-1,-\tfrac{1}{2}]=(3/8)^\lambda 2^{-\lambda+1},\quad \|V_\lambda\|[\tfrac{1}{2},1]=(3/8)^\lambda 2^{-\lambda+1}$$

and

$$(4.8)\qquad \|V_\lambda\|[-\tfrac{1}{2},\tfrac{1}{2}]=(3/8)^\lambda T_\lambda(5/3)=(3/8)^\lambda 2^{-\lambda}(3^\lambda+3^{-\lambda}) \le (9/8)^\lambda 2^{-\lambda+1}.$$

We take $S=V_\lambda R_n$, which is a polynomial of degree $2n+2\lambda$ with leading coefficient 1 and

$$(4.9)\qquad \int_A (1-x^2)^{\frac{1}{2}}|S(x)|dx \le 2\cdot(3/16)^\lambda \int_{-1}^{1}(1-x^2)^{\frac{1}{2}}|R_n(x)|dx \le (3/4)^\lambda n^\alpha 2^{-2n-2\lambda-3},\ A=[-1,-\tfrac{1}{2}]\cup[\tfrac{1}{2},1],$$

because of (4.5) and (4.7). Also,

$$(4.10)\qquad \int_{-\frac{1}{2}}^{\frac{1}{2}}(1-x^2)^{\frac{1}{2}}|S(x)|dx \le 2\cdot(9/16)^\lambda \int_{-\frac{1}{2}}^{\frac{1}{2}}(1-x^2)^{\frac{1}{2}}|R_n(x)|dx \le (9/4)^\lambda n^{\alpha-1}2^{-2n-2\lambda-4},$$

for n sufficiently large, because of (4.4) and (4.8).

Now, let $\lambda=[\log_3 n]+1$, so that

$$(3/4)^\lambda n^\alpha < (3/4)^{\log_3 n}\, n^\alpha = 1.$$

This shows that the integral in (4.9) is smaller than $2^{-2n-2\lambda-3}$. Similarily,

$$(9/4)^{\lambda} n^{\alpha-1} \leq 3^{\lambda} n^{-1} < 3,$$

and so the integral in (4.10) is smaller than $2^{-2n-2\lambda-2}$. Combining these two estimates gives that

$$\int_{-1}^{1} (1-x^2)^{\frac{1}{2}} |S(x)| dx < 2^{-2n-2\lambda-1},$$

which contradicts (4.6) and therefore proves the lower estimate in (4.1).

To prove the upper estimate in (4.1), we use the same sort of ideas as were used in the proof of the lower estimate. Given $n \geq 2$, let $\lambda=[\log_a n]+1$, with $a=2+3^{\frac{1}{2}}$, and define m by the equation $2n=\lambda+m+1$. Consider the polynomial

$$P_n(x)=2^{-\lambda}(x^2-1)T_m(x)T_\lambda(2x)-24n^{\beta}2^{-2n}x = x^{2n+1}-Q_n(x).$$

We will show that Q_n is monotone increasing and has the desired degree of approximation to x^{2n+1}.

We first show that Q_n is monotone. Recall that if S is a polynomial of degree k, then Markov's inequality gives

$$(4.11) \qquad ||(x^2-1)S'(x)|| \leq ||(x^2-1)^{\frac{1}{2}}S'(x)|| \leq k||S||.$$

Also, for $T_\lambda(2x)$, we have

$$(4.12) \qquad |T_\lambda(2x)| \leq 2^{-\lambda+1}, \ |x| \leq 1/2,$$

and if $\delta \geq 1/2$, then

$$(4.13) \quad |T_\lambda(2x)| \leq T_\lambda(2\delta)=2^{-\lambda}[(2\delta+(4\delta^2-1)^{\frac{1}{2}})^{\lambda}+(2\delta-(4\delta^2-1)^{\frac{1}{2}})^{\lambda}]$$

$$\le 2(\delta+(\delta^2-\tfrac{1}{4})^{\frac{1}{2}})^\lambda, \quad |x| \le \delta.$$

Now, we have

$$(4.14) \qquad Q_n'(x)=(2n+1)x^{2n}-P_n'(x)$$
$$=(2n+1)x^{2n}+24n^\beta 2^{-2n}-2^{-\lambda}[(x^2-1)T_m(x)T_\lambda(2x)]'.$$

We need to estimate $R(x)=2^{-\lambda}[(x^2-1)T_m(x)T_\lambda(2x)]'$. We do this by

$$(4.15) \qquad |R(x)| \le 2^{-\lambda}[|(x^2-1)T_m'(x)T_\lambda(2x)|+$$
$$+|(x^2-1)T_m(x)(T_\lambda(2x))'| + |2xT_m(x)T_\lambda(2x)|].$$

On the interval $[-\frac{1}{2},\frac{1}{2}]$, we use (4.11) and (4.12) in (4.15) to find

$$|R(x)| \le 2^{-\lambda}[m2^{-m-\lambda+2}+\lambda 2^{-m-\lambda+2}+2\cdot 2^{-m-\lambda+2}]$$
$$\le (2n+1)2^{-2n-\lambda+3} \le 24n2^{-2n-\lambda}$$
$$\le 24n\cdot n^{\beta-1}2^{-2n} = 24n^\beta 2^{-2n}.$$

Here, recall that $\beta=1-\log_a 2$. The last estimate when used in (4.14) shows that

$$Q_n'(x) \ge (2n+1)x^{2n} \ge 0, \quad |x| \le 1/2.$$

Next, we consider the case when $1/2 \le |x| \le 9/16$. We want to use (4.13) with $\delta=9/16$, in which case $(\delta+(\delta^2-\frac{1}{4})^{\frac{1}{2}})^\lambda$ is smaller than 1/8, for n sufficiently large. Using this and (4.11) in (4.15) shows that for n sufficiently large

$$|R(x)| \le 2^{-\lambda-1}[m2^{-m}+\lambda 2^{-m}+2\cdot 2^{-m}] \le (2n+1)2^{-2n}.$$

We use this last estimate in (4.14) to find

$$Q_n'(x) \geq (2n+1)2^{-2n} - |R(x)| > 0, \; 1/2 \leq |x| \leq 9/16.$$

For $9/16 \leq |x| \leq 1$, we again use (4.11) and (4.13) with $\delta=1$ (and so $\delta+(\delta^2-\frac{1}{4})^{\frac{1}{2}} = a/2$) in (4.15) to find that

$$|R(x)| \leq 2^{-\lambda}[m\cdot a^{\lambda}2^{-m-\lambda+2}+\lambda\cdot a^{\lambda}2^{-m-\lambda+2}+2\cdot a^{\lambda}2^{-m-\lambda+2}]$$

$$\leq 8(a/2)^{\lambda}(2n+1)2^{-2n}.$$

Using this last estimate in (4.14) gives that

$$Q_n'(x) \geq (2n+1)x^{2n}-|R(x)|$$

$$\geq (2n+1)2^{-2n}[(9/8)^n-8(a/2)^{\lambda}]$$

$$\geq (2n+1)2^{-2n}[(9/8)^n-8an^{\beta}] \geq 0, \; 9/16 \leq |x| \leq 1$$

for n sufficiently large. Thus, we have shown that Q_n is monotone non-decreasing provided n is sufficiently large.

Finally, we need to check the degree of approximation. We have

$$||x^{2n+1}-Q_n(x)|| = ||P_n|| \leq ||2^{-\lambda}(x^2-1)T_m(x)T_{\lambda}(2x)||+$$

$$+24n^{\beta}2^{-2n} \leq 2^{-2\lambda-m+2}\; a^{\lambda}+24n^{\beta}2^{-2n}$$

$$\leq 8(a/2)^{\lambda}2^{-n} +24n^{\beta}2^{-2n} \leq 56\, n^{\beta}2^{-2n},$$

where we have used (4.13) with $\delta=1$ and the fact that $(a/2)^{\lambda} \leq 4n^{\beta}$. This completes the proof of the theorem.

5. Remarks

The method of proof of Theorem 1 can not be extended any further, for example, to answer the question (1.3) for

$r \geq 2$. The problem is that the operators $\bar{L}_n$ are positive polynomial operators and hence can't give a better order of approximation than (n^{-2}), because of the saturation phenomena.

It is possible to delay the saturation phenomena by working with polynomials λ_n which have the following properties

$$\int_{-1}^{1} \lambda_n(t)dt=1, \quad \int_{-1}^{1} t^k\lambda_n(t)dt=0, \; k=1,2,\ldots,r$$

$$\int_{-1}^{1} |t|^r|\lambda_n(t)|dt \leq C_r n^{-r}$$

with C_r independent of n. The operators A_n defined by

$$A_n(f,x)= \int_{-1}^{1} f(t)\lambda_n(x-t)dt$$

will provide the Jackson estimate

$$||f-A_n(f)|| \leq C_r n^{-r}\omega(f^{(r)},n^{-1}), \; n=1,2,\ldots$$

but they do not necessarily preserve monotonicity. One can check that

$$||f'-A_n(f)'|| \leq C_r n^{-r+1}\omega(f^{(r)},n^{-1}), \; n=1,2,\ldots$$

so that $A_n(f)'(x) \geq 0$ unless $f'(x) \leq C_r n^{-r+1}$. Thus, we see that the difficulty in guaranteeing monotonicity occurs when $f'(x)=0$. If f' has only a finite number of zeros, it is always possible to correct for them without a loss in the order of approximation. The problem is when f' has an infinite number of zeros. On the other hand, Theorem 2 gives estimates for $E_n^*(f)$ of the same order as for $E_n(f)$ and the functions in the class Λ_r can have an infinite number of zeros. This indicates that the difficulty in handling functions with many zeros is due to deficiency in our technique and it will probably require a non-linear method of approximation to prove (1.3).

The operators L_n introduced in Section 2 also preserve local monotonicity, in the sense that if f is strictly increasing(or decreasing) on an interval [a,b], then $L_n(f)$ will be strictly increasing(or decreasing) on [a,b], provided n is sufficiently large. Also, in the case that f is merely non-decreasing(or non-increasing) on [a,b], then given any sub-interval $[a',b'] \subset (a,b)$, it is possible to make a correction as is done in the definition of $\overline{L}_n$, so as to get a polynomial whose monotonicity agrees with f on [a',b'], provided n is sufficiently large, and still approximates f with the Jackson order of approximation on $[-\frac{1}{4},\frac{1}{4}]$. There is no problem in making a finite number of such corrections.

Now, suppose f is alternately increasing and decreasing on the intervals $[a_k,a_{k+1}]$, $k=0,1,\ldots,m-1$, with $a_o=-1/4$ and $a_m=1/4$. Given $\varepsilon>0$, we can use L_n, as described above, to construct polynomials $P_n \varepsilon \Pi_n$, which have the same monotonocity as f on each sub-interval $[a_k+\varepsilon,a_{k+1}-\varepsilon]$, $k=0,1,\ldots,m-1$, and these polynomials will approximate f with the Jackson orders $\omega(f,n^{-1})$ and $n^{-1}\omega(f',n^{-1})$. This kind of approximation is called co-monotone approximation and was first studied by D.J. Newman, E. Passow, and L. Raymon[6].

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