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Saturation theorems for discretized linear operators

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There is a well developed theory of saturation for positive convolution operators on C^*, the space of 2π-periodic and continuous functions. Up until recently, this was suitable to handle almost all the important approximation processes on C^*. Now, however, some new and interesting sequences of operators have been obtained for C^* by discretizing convolution operators [1], [5]. Such a discretized operator L has a simple form because the value of $L(f)$ depends only on a finite number of values of f. Since these operators are not given by convolution, the existing saturation theorems do not apply to determine their saturation properties.

Our interest in this note is to prove a saturation theorem for positive operators that map C^* to C^*, with no additional structure assumptions on the operators (such as convolution). Thus, this saturation theorem will determine the saturation properties of the discretized convolution operators, provided the other hypotheses of the saturation theorem are satisfied. Our theorem may be considered either as an extension of TURECKIĬ's saturation theorem for convolution operators [2, p. 69] or as the trigonometric analogue of MÜHLBACH's theorem [4] for $C[-1, 1]$. In fact, Tureckiĭ's theorem is a special case of our theorem when one assumes that the operators are given by convolution. Our proof will use a modification of the parabola technique (see [2], Ch. 5), which is used in the proof of Mühlbach's theorem.

If $\{L_n\}$ is a sequence of positive linear operators, we define

$$\mu_n(x) = 4L_n\left(\sin^2\left(\frac{t-x}{2}\right), x\right). \tag{1}$$

We will make the following restriction throughout

(2) *for each* $x\in[-\pi, \pi]$ *there is an infinite number of* n *for which* $\mu_n(x)\neq 0$.

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Theorem. *Let $\{L_n\}$ be a sequence of positive linear operators from C^* to C^*. Define μ_n by* (1) *and assume that* (2) *holds. In addition, suppose that*

$$1-L_n(1,x) = o_x(\mu_n(x)), \quad x\in[-\pi,\pi]; \tag{3}$$

$$L_n(\sin(t-x),x) = o_x(\mu_n(x)), \quad x\in[-\pi,\pi]; \tag{4}$$

$$L_n\left(\sin^4\left(\frac{t-x}{2}\right),x\right) = o_x(\mu_n(x)), \quad x\in[-\pi,\pi]. \tag{5}$$

Then the following are equivalent for $M\geqq 0$,

$$f\in \mathrm{Lip}_M^* 2 = \{f\in C^*\colon |f(x+t)+f(x-t)-2f(x)| \leqq 2Mt^2;\quad x,t\in[-\pi,\pi]\}, \tag{6}$$

$$|L_n(f,x)-f(x)| \leqq M\mu_n(x)+o_x(\mu_n(x)), \quad x\in[-\pi,\pi]. \tag{7}$$

Remarks. In particular, if

$$L_n(f,x)-f(x) = o_x(\mu_n(x)),$$

then (6) shows that f is linear and hence constant because of periodicity. This is the "o" part of the saturation theorem. The equivalence of (6) and (7) is stronger than the typical saturation theorem since we have an exact matchup for the constant M in (6) and (7).

Proof. We first want to show that (6) implies (7). If $f\in\mathrm{Lip}_M^* 2$, then

$$|f'(x+t)-f'(x)| \leqq 2M|t|;\quad x,t\in[-\pi,\pi]$$

(see, e.g., [2], (1.3.6)) and so for each $x\in[-\pi,\pi]$

$$|f(t)-f(x)-f'(x)(t-x)| = \left|\int_x^t [f'(u)-f'(x)]\,du\right| \leqq$$

$$\leqq \left|\int_x^t |f'(u)-f'(x)|\,du\right| \leqq 2M\left|\int_x^t |u-x|\,du\right| = M(t-x)^2.$$

Hence,

$$|f(t)-f(x)-f'(x)\sin(t-x)| \leqq M(t-x)^2+|f'(x)|\,|\sin(t-x)-(t-x)|. \tag{8}$$

We can estimate

$$|(t-x)-\sin(t-x)| \leqq C_1|t-x|^3 \leqq C_2\left|\sin\left(\frac{t-x}{2}\right)\right|^3, \quad |t-x|\leqq\pi, \tag{9}$$

for a fixed constant C_2. Also,

$$\begin{aligned}(t-x)^2 &= 4\sin^2\left(\frac{t-x}{2}\right)+(t-x)^2-4\sin^2\left(\frac{t-x}{2}\right) \leqq \\ &\leqq 4\sin^2\left(\frac{t-x}{2}\right)+C_3\sin^4\left(\frac{t-x}{2}\right), \quad |t-x|\leqq\pi.\end{aligned} \tag{10}$$

Now, let us use the last two estimates in (8) to find

$$|f(t)-f(x)-f'(x)\sin(t-x)| \leqq$$
$$\leqq 4M\sin^2\left(\frac{t-x}{2}\right)+C_2|f'(x)|\left|\sin\left(\frac{t-x}{2}\right)\right|^3+C_3M\sin^4\left(\frac{t-x}{2}\right), \quad |t-x|\leqq\pi.$$

Because the functions on both sides of this last inequality are periodic with period 2π, the inequality must hold for all x and t. Applying L_n and using the last inequality shows that

$$|L_n(f,x)-f(x)| \leqq M\mu_n(x)+|f(x)|\,|1-L_n(1,x)|+$$
$$+|f'(x)|\,|L_n(\sin(t-x),x)|+C_3ML_n\left(\sin^4\left(\frac{t-x}{2}\right),x\right)+ \tag{11}$$
$$+C_2|f'(x)|L_n\left(\left|\sin\left(\frac{t-x}{2}\right)\right|^3,x\right).$$

The second, third, and fourth terms on the right hand side of (11) are obviously $o_x(\mu_n(x))$ because of (3), (4), and (5). The last term is also $o_x(\mu_n(x))$ because of the Cauchy—Schwarz inequality for positive linear operators. Namely,

$$L_n\left(\left|\sin\left(\frac{t-x}{2}\right)\right|^3,x\right) \leqq$$
$$\leqq\left(L_n\left(\sin^2\left(\frac{t-x}{2}\right),x\right)\right)^{1/2}\left(L_n\left(\sin^4\left(\frac{t-x}{2}\right),x\right)\right)^{1/2}=o_x(\mu_n(x)) \tag{12}$$

because of (1) and (5). This shows that

$$|L_n(f,x)-f(x)| \leqq M\mu_n(x)+o_x(\mu_n(x))$$

which is the estimate (7), as desired.

We turn now to proving that (7) implies (6). We will modify the parabola technique described in [2, Ch. 5]. Suppose that f satisfies (7) but not (6). Then, there is an $M'>M$ and x_0, h such that

$$|f(x_0+h)+f(x_0-h)-2f(x_0)| \geqq 2M'h^2.$$

By working with $-f$ in place of f if necessary, we can assume that

$$f(x_0+h)+f(x_0-h)-2f(x_0) \leqq -2M'h^2.$$

Then for $M''=\frac{1}{2}(M+M')$ we have from [2, p. 138] that there is a point $x_1\in(x_0-h, x_0+h)$ and a $\delta>0$ such that the parabola

$$Q(x) = -M''(t-x_1)^2+\alpha(t-x_1)+f(x_1)$$

satisfies

$$Q(t) \geqq f(t), \quad |t-x_1| \leqq \delta.$$

Now let

$$R(t) = -4M'' \sin^2\left(\frac{t-x_1}{2}\right) + \alpha \sin(t-x_1) + f(x_1).$$

Then, by using (9) and (10), we see that

$$Q(t) \leqq R(t) + C_3 M'' \sin^4\left(\frac{t-x_1}{2}\right) + C_2|\alpha| \left|\sin\left(\frac{t-x_1}{2}\right)\right|^3 = S(t), \quad |t-x_1| \leqq \delta.$$

Choose $C_4 > 0$ so that

$$f(t) \leqq S(t) + C_4 \sin^4\left(\frac{t-x_1}{2}\right), \quad |t-x_1| \leqq \pi. \tag{13}$$

This is possible because both f and S are bounded on $[x_1-\pi, x_1+\pi]$ and $\sin^4\left(\frac{t-x_1}{2}\right)$ is strictly positive for $\delta \leqq |t-x_1| \leqq \pi$.

We use (13) to estimate

$$L_n(f, x_1) - f(x_1) \leqq L_n(S, x_1) - S(x_1) + C_4 L_n\left(\sin^4\left(\frac{t-x_1}{2}\right), x_1\right) =$$

$$= L_n(S, x_1) - S(x_1) + o(\mu_n(x_1))$$

because of (5). For S we have

$$L_n(S, x_1) - S(x_1) = L_n(R, x_1) - R(x_1) + o(\mu_n(x_1))$$

because of (5) and (12). Going one step further,

$$L_n(R, x_1) - R(x_1) = -M'' \mu_n(x) + \alpha L_n(\sin(t-x_1), x_1) + f(x_1)(L_n(1, x_1) - 1) =$$

$$= -M'' \mu_n(x_1) + o(\mu_n(x_1)),$$

because of (3) and (4). Putting this all together shows that

$$L_n(f, x_1) - f(x_1) \leqq -M'' \mu_n(x_1) + o(\mu_n(x_1))$$

and hence

$$|L_n(f, x_1) - f(x_1)| \geqq M'' \mu_n(x_1) + o(\mu_n(x_1)) > M\mu_n(x_1) + o(\mu_n(x_1))$$

for all n such that $\mu_n(x_1) \neq 0$. Since this is true for infinitely many n because of (2), we have a contradiction to our assumption (7). This shows that (6) must hold and proves the theorem.

Now we want to apply our theorem to determine the saturation properties of discretized positive convolution operators. Let $\Phi_n(t)\geqq 0$ be given by

$$\Phi_n(t)=\sum_{k=0}^{l_n}\varrho_{k,n}\cos kt,\quad \varrho_{0,n}=\frac{1}{2},$$

and for $t_j=\frac{2\pi j}{m_n}$ $(j=1,2,\dots,m_n)$ define

(14) $$L_n(f,x)=\frac{2}{m_n}\sum_{j=1}^{m_n}f(t_j)\Phi_n(t_j-x).$$

The operators L_n are called discretized convolution operators since (14) can be considered as an approximation to the convolution

$$\frac{1}{\pi}\int_{\pi}^{\pi}f(t)\Phi_n(t-x)\,dt$$

by using a quadrature formula with m_n equally spaced points. This quadrature formula is exact for all polynomials of degree $\leqq m_n-1$. That is if T is a trigonometric polynomial of degree $\leqq m_n-1$, then

(15) $$\frac{1}{\pi}\int_{-\pi}^{\pi}T(t)\,dt=\frac{2}{m_n}\sum_{j=1}^{m_n}T(t_j).$$

The following characterization of the saturation properties of the operators $\{L_n\}$ will follow from our theorem.

Corollary. *If $\{L_n\}$ is a sequence of positive linear operators given by* (14) *and satisfying*

(16) $$\sum_{k=a_n}^{l_n}|\varrho_{k,n}|=o(1-\varrho_{1,n})\quad (a_n=\min(m_n-2,l_n))$$

and

(17) $$3-4\varrho_{1,n}+\varrho_{2,n}=o(1-\varrho_{1,n}),$$

then $\{L_n\}$ is saturated with order $1-\varrho_{1,n}$ and saturation class Lip* 2. *More precisely,*

$$\|f-L_n(f)\|\leqq M(1-\varrho_{1,n})+o(1-\varrho_{1,n})$$

if and only if $f\in \mathrm{Lip}^*_M\,2$.

We note that when $m_n\geqq l_n+3$, then (16) is superfluous.

Proof. We only have to check (2)—(5). We have

(18) $$\mu_n(x)=2-2L_n(\cos(t-x),x)=$$
$$=2-\frac{4}{m_n}\sum_{k=0}^{l_n}\sum_{j=1}^{m_n}\varrho_{k,n}\cos(t_j-x)\cos k(t_j-x).$$

When $k \leqq m_n - 2$ and x is fixed $\cos(t-x)\cos k(t-x)$ is a trigonometric polynomial of degree $\leqq m_n - 1$ and so

$$\frac{4}{m_n}\sum_{j=1}^{m_n}\cos(t_j - x)\cos k(t_j - x) = \begin{cases}0, & k \neq 1;\\ 2, & k = 1\end{cases}$$

because of (15). When $k \geqq m_n - 1$, then

$$\frac{4}{m_n}\sum_{j=1}^{m_n}|\cos(t_j - x)\cos k(t_j - x)| \leqq 4.$$

These last two estimates when put into (18) show that

$$\mu_n(x) = 2(1-\varrho_{1,n}) + O\left(\sum_{k=m_n-1}^{l_n}|\varrho_{k,n}|\right) = 2(1-\varrho_{1,n}) + o(1-\varrho_{1,n})$$

because of (16).

Similar estimates show that

$$|1 - L_n(1,x)| \leqq 2\sum_{k=m_n}^{l_n}|\varrho_{k,n}| = o(1-\varrho_{1,n}),$$

$$L_n(\sin(t-x), x) = O\left(\sum_{k=m_n-1}^{l_n}|\varrho_{k,n}|\right) = o(1-\varrho_{1,n}),$$

$$L_n\left(\sin^4\left(\frac{t-x}{2}\right), x\right) = \frac{1}{8}(3 - 4\varrho_{1,n} + \varrho_{2,n}) + O\left(\sum_{k=m_n-2}^{l_n}|\varrho_{k,n}|\right) = o(1-\varrho_{1,n}),$$

where in the last equation we used both (16) and (17). This completes the proof.

Let us apply this corollary to some interesting special cases.

1. *Jackson operators.* Here

$$\Phi_n(t) = \frac{3}{2n(2n^2+1)}\left(\frac{\sin\frac{nt}{2}}{\sin\frac{t}{2}}\right)^4$$

and

$$2n(2n^2+1)\varrho_{k,n} = \begin{cases}4n^3 - 6nk^2 + 3k^3 + 2n - 3k & \text{if } 1 \leqq k \leqq n+1,\\ (2n-k+1)(2n-k)(2n-k-1) & \text{if } n-1 \leqq k \leqq 2n-2 = l_n.\end{cases}$$

Our corollary will handle the saturation of the discretized Jackson operators when

$$\limsup_{n\to\infty}(2n - m_n)n^{-1/4} \leqq 0. \tag{19}$$

Since then

$$1-\varrho_{1,n}=\frac{3}{2n^2+1},\quad 3-4\varrho_{1,n}+\varrho_{2,n}=\frac{9}{n(2n^2+1)},$$

and when $m_n\leqq 2n-2$

$$\sum_{k=m_n}^{l_n}|\varrho_{k,n}|=(2n-m_n)^4O(n^{-3})=o(1-\varrho_{1,n}).$$

Thus, the saturation order is n^{-2}. Among other cases, when $m_n=2n-2$, i.e., for the Jackson—Bojanic—Shisha operator [1], [3], the saturation problem is solved.

2. *de la Vallée Poussin operators.* Now

$$\Phi_n(t)=\frac{(2n)!!}{2(2n-1)!!}\cos^{2n}\frac{t}{2}$$

and

$$\varrho_{k,n}=\frac{(n!)^2}{(n-k)!\,(n+k)!}\quad (k=1,2,\dots,n).$$

In particular,

$$1-\varrho_{1,n}=\frac{1}{n+1},\quad 3-4\varrho_{1,n}+\varrho_{2,n}=\frac{6}{(n+1)(n+2)}.$$

The conditions of our corollary will be satisfied if

$$\liminf_{n\to\infty} m_n(n\log n)^{-1/2}\geqq 2^{1/2}. \tag{20}$$

Indeed, for $m_n\leqq n$

$$\sum_{k=m_n}^{n}|\varrho_{k,n}|=\sum_{k=m_n}^{n}\frac{n}{n+k}\sum_{j=1}^{k-1}\frac{n-j}{n+j}\leqq\sum_{k=m_n}^{n}\sum_{j=1}^{k-1}\left(1-\frac{2j}{n+j}\right)\leqq$$

$$\leqq\sum_{k=m_n}^{n}\exp\left(-2\sum_{j=1}^{k-1}\frac{j}{n+j}\right)\sim\sum_{k=m_n}^{n}\exp\left(-2(k-1)+2n\log\frac{n+k-1}{n-1}\right)\sim$$

$$\sim\sum_{k=m_n}^{n}\exp\left(-2k+2n\left(\frac{k}{n}-\frac{k^2}{n}\right)\right)=\sum_{k=m_n}^{n}\exp\left(-\frac{k^2}{n}\right)\leqq n\exp\left(-\frac{m_n^2}{n}\right)=o(n^{-1}).$$

Thus the de la Vallée Poussin operators are saturated with order n^{-1} provided that (20) holds.

3. *Fejér—Korovkin operators.* In this case

$$\Phi_n(t)=\frac{\sin^2\frac{\pi}{n}}{n}\left(\frac{\cos\frac{nt}{2}}{\cos t-\cos\frac{\pi}{n}}\right)^2$$

and

$$\varrho_{k,n} = \frac{2}{n}\sum_{j=1}^{n-k} \sin\frac{k+j}{n}\pi \sin\frac{j}{n}\pi \quad (1 \leqq k \leqq n-2 = l_n).$$

In particular,

$$\varrho_{1,n} = \cos\frac{\pi}{n}, \quad 3-4\varrho_{1,n}+\varrho_{2,n} = \frac{2\pi^2}{n^3}+O(n^{-4}) = o(1-\varrho_{1,n}).$$

We can use our corollary if

$$\text{(21)} \qquad \liminf_{n\to\infty}(n-m_n)n^{-1/4} \leqq 0.$$

For then we have for $m_n \leqq n$

$$\sum_{k=m_n}^{n} |\varrho_{k,n}| \leqq \frac{2\pi^2}{n^3}\sum_{k=m_n}^{n}\sum_{j=1}^{n-k}(n-k-j)j \leqq \frac{\pi^2}{2n^3}\sum_{k=m_n}^{n}(n-k)^3 =$$
$$= (n-m_n)^4 O(n^{-3}) = o(1-\varrho_{1,n}).$$

Thus the Fejér—Korovkin operators are saturated with order n^{-2} provided that (21) holds.

It would be interesting to know whether (19)—(21) are necessary for the corresponding discretized operators to have saturation class Lip*2.

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Теоремы о насыщении для дискретных линейных операторов

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Пусть $\{L_n\}$—последовательность линейных положительных операторов, отображающих пространство 2π-периодических непрерывных функций в себя. Предположим, что для каждого $x\in[-\pi, \pi]$ существует бесконечно много номеров n, для которых $\mu_n(x)= =4L_n\left(\sin^2\frac{t-x}{2}, x\right)\neq 0$. Предположим также, что для всех $x\in[-\pi, \pi]$ каждая из величин $1-L_n(1, x)$, $L_n(\sin(t-x), x)$ и $L_n\left(\sin^4\frac{t-x}{2}, x\right)$ есть $o_x(\mu_n(x))$. Тогда при $M\geqq 0$ следующие ут-

верждения эквивалентны:

$$f \in \mathrm{Lip}_M^* 2 = \{f \in C^* : |f(x+t)+f(x-t)-2f(x)| \leqq 2Mt^2; \quad x, t \in [-\pi, \pi]\}$$

и

$$|L_n(f, x)-f(x)| \leqq M\mu_n(x)+o_x(\mu_n(x)), \quad x \in [-\pi, \pi].$$

Эта общая теорема о насыщении применяется к так называемым дискретным операторам свертки, в частности, с различным дискретным вариантом интеграла Джексона, операторов Фейера—Коровкина и интеграла Валле Пуссена.

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