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MONOTONE APPROXIMATION BY POLYNOMIALS*

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Abstract. We prove Jackson type estimates for the approximation of monotone functions by monotone polynomials. The results are given in terms of the modulus of continuity of $f^{(k)}$, for any $k \geqq 0$. The estimates are of the same order as for the unconstrained approximation by polynomials.

1. Introduction. In the preceding paper [2], we have developed Jackson type theorems for monotone approximation by splines. Here, we want to give similar results for monotone approximation by algebraic polynomials.

Let Π_n denote the space of algebraic polynomials of degree $\leqq n$ and Π_n^*, the set of those polynomials in Π_n which are monotone nondecreasing on $[0, 1]$. If $f \in C[0, 1]$ is monotone nondecreasing on $[0, 1]$ ($f \uparrow$), then we define the error of monotone approximation of f by polynomials of degree $\leqq n$ by

$$E_n^*(f) = \inf_{P \in \Pi_n^*} \|f - P\|,$$

with $\|\cdot\|$ the supremeum norm on $[0, 1]$.

Our main result is the following theorem which gives an estimate for $E_n^*(f)$ in terms of the smoothness of f.

THEOREM 1. *If $k \geqq 0$ and $f^{(k)} \in C[0, 1]$, $f \uparrow$, then for $n \geqq k+1$*

$$E_n^*(f) \leqq C n^{-k} \omega(f^{(k)}, n^{-1}), \tag{1.1}$$

where C is a constant that depends only on k.

Thus (1.1) is the same as the classical Jackson theorem for unconstrained approximation by polynomials, and shows that at least in this sense there is no loss in the degree of approximation caused by the monotone constraint. We should remark that there are known examples with a loss in the degree of monotone approximation given by G. G. Lorentz and K. Zeller [6], and the author [1].

The cases $k = 0$, and $k = 1$ of Theorem 1 have been obtained previously by G. G. Lorentz and K. Zeller [5] and G. G. Lorentz [4], respectively. They have used linear methods of approximation in their cases. This is not possible in the general case since such a sequence of linear operators would have to preserve the positivity of f' and hence be restricted in their effectiveness of approximation by the saturation phenomena for positive operators.

The proof of Theorem 1 relies heavily on the results and techniques of [2]. In fact, the proof is developed in much the same way with the major exception being the fact that there is no direct analogue of B-splines. Instead, we have to construct polynomials which are large on a given interval and fall off fast outside of this interval. These polynomials are constructed in § 2.

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Using the results of [2] on the degree of monotone approximation by splines, we can simplify the kind of functions for which we need to prove Theorem 1. In fact, it will be enough to show the following simpler case of Theorem 1.

THEOREM 2. *If $k \geqq 2$ and $f^{(k)}$ is absolutely continuous with $\|f^{(k+1)}\|_{L^\infty[0,1]} \leqq 1$, then*

$$E_n^*(f) \leqq Cn^{-k-1}, \qquad n \geqq N, \tag{1.2}$$

where C and N are constants that depend only on k.

Let us observe why it is enough to prove Theorem 2. Assuming that we have proved Theorem 2 and f is an arbitrary function in $C^{(k)}[0, 1]$, then by Theorem 1 of [2], there is a spline $S \in \mathcal{S}(k+2, n)$ (following the notation of [2]), with $S\uparrow$ and

$$\|f-S\| \leqq C'n^{-k}\omega(f^{(k)}, n^{-1}), \tag{1.3}$$

where C' depends only on k. From (1.3), it follows that when $|t| \leqq n^{-1}$

$$|\Delta_t^{k+1}(S, x)| \leqq |\Delta_t^{k+1}(f, x)| + 2^{k+1}C'n^{-k}\omega(f^{(k)}, n^{-1}) \leqq C''n^{-k}\omega(f^{(k)}, n^{-1}). \tag{1.4}$$

Now, S is a spline of degree $k+1$, and so $S^{(k+1)}$ is a step function. If $\|S^{(k+1)}\| = \gamma$, then there is an interval $[a, b]$ of length n^{-1} on which $S^{(k+1)}$ either equals γ or $-\gamma$. If it equals γ and $t = (k+1)^{-1}n^{-1}$, then

$$\Delta_t^{k+1}(S, a) = \int_a^{a+t}\int_{x_k}^{x_k+t}\cdots\int_{x_1}^{x_1+t} S^{(k+1)}(x)\,dx\,dx_1\cdots dx_k \geqq \gamma t^{k+1}. \tag{1.5}$$

This together with (1.4) shows that

$$\|S^{(k+1)}\| = |\gamma| \leqq Dn\omega(f^{(k)}, n^{-1}),$$

with D a constant depending only on k. This same result holds for $-\gamma$.

Now, by Theorem 2, there is a polynomial $P \in \Pi_n^*$, such that

$$\|S-P\| \leqq CDn^{-k}\omega(f^{(k)}, n^{-1})$$

and thus

$$\|f-P\| \leqq \|f-S\| + \|S-P\| \leqq (C'+CD)n^{-k}\omega(f^{(k)}, n^{-1}),$$

which gives Theorem 1 for those values of $n \geqq N$.

We need also to check the values of n with $k+1 \leqq n < N$. For this, let Q be a polynomial of degree $\leqq n$ with

$$\|f' - Q\| \leqq C(n-1)^{-k+1}\omega(f^{(k)}, (n-1)^{-1}) \leqq C'n^{-k}\omega(f^{(k)}, n^{-1}).$$

The existence of such polynomials Q follows from the usual unconstrained Jackson theorems. The polynomial

$$P(x) = f(0) + \int_0^x (Q(t) + C'n^{-k}\omega(f^{(k)}, n^{-1}))\,dt$$

monotone nondecreasing and we have

$$\|f-Q\| \leqq \|f-Q\| + C'n^{-k}\omega(f^{(k)}, n^{-1}).$$

Thus, we have (1.2) also for the values of n with $k+1 \leqq n < N$. This then shows that Theorem 2 implies Theorem 1. Thus in the sequel, it will be enough to prove Theorem 2 and so we can restrict ourselves to functions f with $\|f^{(k+1)}\|_\infty \leqq 1$.

There were certain constants that played an important role in the statement and results of [2]. In this paper, we will sometimes have to redefine these constants to fit the needs of polynomial approximation. We will use the same symbolization for these constants. This is allowable because the constants that are redefined all had only the restriction on them that certain inequalities hold. In the case that the constants are $\geqq 1$, these inequalities will still hold if we redefine the constants to be larger, which we do. When the constants are $\leqq 1$, these inequalities will still hold if we redefine the constants to be smaller, which we do.

2. Some special polynomials. We need to construct some polynomials which mimic the B-splines. These polynomials will be used for corrections in the same way that the B-splines were used in [2]. Let $T_m(x) = \cos m(\arccos x)$ be the Chebyshev polynomials of degree m. If m is odd, then $T_m(0)=0$. Near the end of this section, we will prescribe an even positive integer r, which will depend only on k and will be larger than $4k+2$. Thus any constants that appear and depend only on r will in turn depend only on k. Let m be the largest odd integer such that $mr \leqq n$, and define

$$Q_{m,r}(x) = \int_{-1}^{x} c_{m,r}(m^{-1}t^{-1}T_m(t))^r\,dt,$$

with $c_{m,r}$ a normalizing constant chosen so that $Q_{m,r}(1)=1$.

We want first to estimate $c_{m,r}$. If $|t| \leqq m^{-1}$, then $|m^{-1}t^{-1}T_m(t)| \geqq 2\pi^{-1} \geqq 2^{-1}$, and so

$$\int_{-1}^{1} (m^{-1}t^{-1}T_m(t))^r\,dt \geqq 2^{-r+1}m^{-1}.$$

Also, since $|T_m(t)| \leqq 1$, $-1 \leqq t \leqq 1$, we have

$$\int_{|t| \geqq m^{-1}} (m^{-1}t^{-1}T_m(t))^r\,dt \leqq 2m^{-1}\int_{1}^{m} t^{-r}\,dt \leqq 2m^{-1},$$

since $r \geqq 4$. Now, $|m^{-1}t^{-1}T_m(t)| \leqq 1$, $-1 \leqq t \leqq 1$, and so

$$\int_{|t| \leqq m^{-1}} (m^{-1}t^{-1}T_m(t))^r\,dt \leqq 2m^{-1}.$$

These last three inequalities show that

$$\frac{m}{4} \leqq c_{m,r} \leqq 2^{r-1}m. \tag{2.1}$$

This last inequality for $c_{m,r}$, together with the fact that $|T_m(t)| \leqq 1$, $-1 \leqq t \leqq 1$, shows that

$$|Q_{m,r}(x)| \leqq \left|\frac{mx}{2}\right|^{-r+1}, \qquad -1 \leqq x \leqq 0, \tag{2.2}$$

(2.3)
$$|1-Q_{m,r}(x)| \le \left|\frac{mx}{2}\right|^{-r+1}, \qquad 0 \le x \le 1.$$

If $I=[a, b]$ is an interval contained in $[0, 1]$ with length $\ge rn^{-1}$, then denote by $|I|$, the length of I and define

$$\lambda_I(x) = c_I(Q_{m,r}(x-a') - Q_{m,r}(x-b')) + n^{-r}|I|^{-1},$$

with $a' = a - 8rn^{-1}$, $b' = b + 8rn^{-1}$, and c_I a normalizing constant chosen so that

$$\int_0^1 \lambda_I(t)\, dt = 1.$$

λ_I is then a polynomial of degree $\le n-1$, which as we shall see is large on I and falls off fast outside of I. First, we want to show that $c_I \sim |I|^{-1}$.

When $n \ge 3r$, then $m \ge 2$ and $8rn^{-1} \ge 4m^{-1}$. We will only consider values of n larger than $3r$ throughout but this is permissible since r depends only on k. If we use (2.2) and (2.3), we find

$$Q_{m,r}(x-a') - Q_{m,r}(x-b') \ge 1 - 2^{-r+1} - 2^{-r+1} \ge \tfrac{1}{2}, \qquad x \in I.$$

Also, since $|Q_{m,r}(x)| \le 1$, $-1 \le x \le 1$, we have

(2.4)
$$\tfrac{1}{2} \le Q_{m,r}(x-a') - Q_{m,r}(x-b') \le 2, \qquad x \in I.$$

We can also estimate outside of I. If $x \in [0, 1]$ and $\delta' = \operatorname{dist}(x, [a', b']) \ge n^{-1}$, then $\operatorname{dist}(x, I) \le 9r\delta'$. Thus,

(2.5)
$$\begin{aligned}&|Q_{m,r}(x-a') - Q_{m,r}(x-b')|\\ &\quad \le \left(\frac{m\delta'}{2}\right)^{-r+1} \le \left(\frac{m \operatorname{dist}(x, I)}{18r}\right)^{-r+1}, \qquad \text{when } \operatorname{dist}(x, I) \ge 9m^{-1},\end{aligned}$$

because of (2.2) and (2.3). Since $|I| \ge m^{-1}$, we have

(2.6)
$$\frac{|I|}{2} \le \int_0^1 (Q_{m,r}(x-a') - Q_{m,r}(x-b'))\, dx \le (38 + 9(4r)^r)|I|$$

where the left side was estimated using (2.4) and the right side was estimated by considering the integral over two sets. The first set is where $\operatorname{dist}(x, I) \le 9m^{-1}$, on which we used the facts that $|Q_{m,r}(x)| \le 1$, $x \in [-1, 1]$, and that this set has measure $\le 19|I|$. The integral over the second set S was estimated by using (2.5), to see that

$$\begin{aligned}\int_S (Q_{m,r}(x-a') - Q_{m,r}(x-b'))\, dx &\le 2\left(\frac{18r}{m}\right)^{r-1} \int_{9m^{-1}}^1 t^{-r+1}\, dt\\ &\le 9(4r)^{r-1}m^{-1} \le 9(4r)^r|I|,\end{aligned}$$

where we have used the fact that $|I| \ge m^{-1}$ and $n \le 2rm$.

The estimate (2.6) shows that there are constants $a_1, a_2 > 0$, such that

$$a_1|I|^{-1} \le c_I \le a_2|I|^{-1};$$

so $c_I \sim |I|^{-1}$ as we have previously stated.

If J is any interval we define

$$d_n(x, J) = \max(1, n \operatorname{dist}(x, J)).$$

Now, because of (2.5) and (2.7), it follows that if we choose α_2 of equation (2.2) of [2] large enough, then

$$\lambda_I(x) \leqq \alpha_2 |I|^{-1}(d_n(x, I))^{-r+1}, \quad \text{for } x \in [0, 1]. \tag{2.8}$$

Note that this inequality automatically holds when $\operatorname{dist}(x, I) \leqq 9m^{-1}$, because $|Q_{m,r}(x)| \leqq 1$, $x \in [-1, 1]$.

We will also need an estimate for $\lambda_I(x)$ from below. Namely the constant α_1 of equation (2.1) of [2] can be chosen small enough that

$$\alpha_1 |I|^{-1}(d_n(x, I))^{-r} \leqq \lambda_I(x), \qquad x \in [0, 1]. \tag{2.9}$$

This estimate holds for $x \in I$, because of (2.4). In the same way that we have proved (2.4), we can show that (2.9) holds when $\operatorname{dist}(x, I) \leqq m^{-1}$. Also, (2.9) automatically holds when $\operatorname{dist}(x, I) \geqq \frac{1}{4}$, because of the term $|I|^{-1}n^{-r}$ that appears in the definition of λ_I.

To see that (2.9) holds when $m^{-1} \leqq \operatorname{dist}(x, I) \leqq \frac{1}{4}$, let $\xi'_\nu = \cos((3\nu+1)/3)\pi m^{-1}$ and $\xi''_\nu = \cos((3\nu-1)/3)\pi m^{-1}$. Then, $|T_m(x)| \geqq \frac{1}{2}$, for $x \in [\xi'_\nu, \xi''_\nu]$, and $|\xi'_\nu - \xi''_\nu| \geqq m^{-1}$, whenever $[\xi'_\nu, \xi''_\nu] \subseteq [-\frac{1}{2}, \frac{1}{2}]$. Suppose that $x < a$ and $\delta = \operatorname{dist}(x, I) = |x - a|$. Then the interval $[\delta, \delta + 8m^{-1}]$ contains a set S, which consists of the parts of the intervals $[\xi'_\nu, \xi''_\nu]$ which intersect $[\delta, \delta + 8m^{-1}]$ and the set S has measure $|S| \geqq m^{-1}$. Therefore,

$$\begin{aligned}\int_{x-b'}^{x-a'} c_{m,r}(m^{-1}t^{-1}T_m(t))^r\,dt &\geqq \int_\delta^{\delta+8m^{-1}} c_{m,r}(m^{-1}t^{-1}T_m(t))^r\,dt \\ &\geqq \int_S c_{m,r}(2mt)^{-r}\,dt \geqq (2m)^{-r}(\delta + 8m^{-1})^{-r}|S|c_{m,r} \\ &\geqq C(n\delta)^{-r}\end{aligned}$$

where C is a constant that depends only on r. Here, we have used our estimate for $c_{m,r}$ in (2.1), the definition of m, and the fact that $\delta \geqq m^{-1}$.

This last estimate and our estimate for c_I in (2.7) show that when $x < a$ and $\operatorname{dist}(x, I) \geqq m^{-1}$

$$\lambda_I(x) \geqq c_I \int_{x-b'}^{x-a'} c_{m,r}(m^{-1}t^{-1}T_m(t))^r\,dt \geqq \alpha_1 |I|^{-1}(d_n(x, I))^{-r}.$$

The same estimate holds when $x > b$ and $\operatorname{dist}(x, I) \geqq m^{-1}$, and so we have proved (2.9).

The polynomials λ_I will be used to correct the derivatives of our approximating polynomials. The primitive

$$\Lambda_I(x) = \int_0^x \lambda_I(t)\,dt$$

will therefore be the correction to the approximating polynomials themselves.

Because of our normalization, $\Lambda_I(1)=1$. It follows from (2.8) that

$$(2.10)\quad |\Lambda_I(x)| \leqq \alpha_2 |I|^{-1} \int_0^x (d_n(t,I))^{-r+1}\,dt \leqq 2\alpha_2 |I|^{-1} n^{-1} (d_n(x,I))^{-r+2}, \qquad x \leqq a.$$

Similarly,

$$(2.11)\quad |1-\Lambda_I(x)| \leqq 2\alpha_2 |I|^{-1} n^{-1} (d_n(x,I))^{-r+2}, \qquad x \geqq b.$$

On I, we have $|\lambda_I(t)| \leqq \alpha_2 |I|^{-1}$, and so

$$(2.12)\quad |\Lambda_I(x)-\Lambda_I(y)| \leqq \left|\int_x^y \lambda_I(t)\,dt\right| \leqq \alpha_2 |I|^{-1} |x-y| \quad \text{when } x, y \in I.$$

We will need one other correcting polynomial. Let k be the integer in Theorem 1, $k \geqq 2$, and now let m be chosen so that it is the largest odd integer with $m(2k+2) \leqq n-2$. Consider,

$$\phi_{m,r}(x) = (r^2(2n)^{-2} - x^2)(m^{-1}x^{-1}T_m(x))^{2k+4}$$

which is a polynomial of degree $\leqq n-1$ that is positive on $[-r(2n)^{-1}, r(2n)^{-1}]$, and negative outside of this interval. We first want an estimate for the integral of $\phi_{m,r}$ over $[-1, 1]$.

When $|x| \leqq kn^{-1} \leqq m^{-1}$, then $|m^{-1}x^{-1}T_m(x)| \geqq \frac{1}{2}$. Also, since $r \geqq 4k$, we have $(r^2(2n)^{-2} - x^2) \geqq 8^{-1}r^2n^{-2}$, when $|x| \leqq kn^{-1}$. Hence,

$$(2.13)\quad \int_{-r(2n)^{-1}}^{r(2n)^{-1}} \phi_{m,r}(x)\,dx \geqq 8^{-1}r^2n^{-2}2^{-2k-4}(2kn^{-1}) = 2^{-2k-6}r^2kn^{-3}.$$

We can also estimate the integral over the set $S = [-1, 1] - [-r(2n)^{-1}, r(2n)^{-1}]$. Now, $|T_m(x)| \leqq 1$, $x \in [-1, 1]$, and $\phi_{m,r}$ is negative on S, and so

$$\left|\int_S \phi_{m,r}(x)\,dx\right| \leqq 2\int_{r(2n)^{-1}}^{1} x^2(mx)^{-2k-4}\,dx$$
$$\leqq 2m^{-2k-4}(2nr^{-1})^{2k+1} \leqq 2\left(\frac{2n}{mr}\right)^{2k+1}\left(\frac{n}{m}\right)^3 n^{-3}.$$

Since k is fixed and $n \leqq 4mk$, we can choose r sufficiently large, depending only on k, so that $r \geqq 4k+2$, and

$$(2.14)\quad \left|\int_S \phi_{m,r}(x)\,dx\right| \leqq \frac{1}{2}\int_{-r(2n)^{-1}}^{r(2n)^{-1}} \phi_{m,r}(x)\,dx.$$

We fix this value of r for the rest of the paper.

If I is an interval contained in $[0, 1]$ of length m^{-1} and midpoint a, then define

$$\phi_I(x) = d_I\phi_{m,r}(x-a), \qquad \int_0^1 \phi_I(x)\,dx = 1.$$

Because of (2.13) and (2.14), it follows that there are constants $a_1', a_2' > 0$, such that

$$a_1'n^3 \leqq d_I \leqq a_2'n^3.$$

We can also require that α_2 is chosen large enough that when dist $(x, I) \geqq 2rn^{-1}$, then

(2.15) $|\phi_I(x)| \leqq C|r^r(2n)^{-2}-(x-a)^2|(m \text{ dist }(x, I))^{-2k-4}n^3 \leqq \alpha_2 n(d_n(x, I))^{-2k-2}$,

where we have used the facts that $d_n(x, I) \geqq 1$, for all x and $|r^2(2n)^{-2}-(x-a)^2| \leqq 2(\text{dist}(x, I))^2$, when dist $(x, I) \geqq 2rn^{-1}$. If α_2 is large enough then this inequality will also hold when dist $(x, I) \leqq 2rn^{-1}$, because in this case $|\phi_I(x)| \leqq Cn$, with C depending only on k. Therefore,

$$|\phi_I(x)| \leqq \alpha_2 n(d_n(x, I))^{-2k-2}, \qquad x \in [0, 1]. \tag{2.16}$$

Define the primitive

$$\Phi_I(x) = \int_0^x \phi_I(t)\, dt,$$

which is a polynomial of degree $\leqq n$. From (2.16), it follows that α_2 can be chosen so large that

$$|\Phi_I(x)| \leqq \alpha_2(d_n(x, I))^{-2k-1}, \qquad x \leqq a, \tag{2.17}$$

$$|1-\Phi_I(x)| \leqq \alpha_2(d_n(x, I))^{-2k-1}, \qquad x \geqq a, \tag{2.18}$$

where as before, a is the midpoint of I.

3. A decomposition of f. In [2], we have given a decomposition for monotone functions which we will also use here. We decompose $[0, 1]$ into a union of certain pairwise disjoint intervals I_j^*, $j = 1, 2, \cdots, m$ and J_j^*, $j = 0, 1, \cdots, m$, with I_j^* to the right of J_{j-1}^* and to the left of J_j^*. For any of these intervals I, we define

$$f_I(x) = \int_0^x f'(t)\chi_I(t)\, dt,$$

where χ_I is the characteristic function of the interval I. Then our decomposition for f is

$$f(x) = f(0) + \sum_{j=0}^{m} f_{J_j^*} + \sum_{j=1}^{m} f_{I_j^*}. \tag{3.1}$$

Each of the functions f_I is monotone nondecreasing and our proof of Theorem 2 will be to approximate each f_I by monotone polynomials to get our approximation to f.

The intervals J_j^* and I_j^* have special properties that we summarize. Recall that to prove Theorem 2, we need only consider functions with $\|f^{(k+1)}\|_{L^\infty[0,1]} = 1$. Each interval J_j^* has length $\geqq r^2 n^{-1}$ and

(3.2) *if I is any of the intervals J_j^*, then $f_I'(x) = f'(x) \leqq An^{-k}$, $x \in I$, with A a constant depending only on k.*

On the other hand, each interval I_j^* has length $\geqq 2r^2n^{-1}$ and

(3.3) *if I is any of the intervals I_j^*, then there is an interval $[l_0 n^{-1}, (l_0+r)n^{-1}]$, contained in I on which $f_I'(x) = f'(x) \geqq B_2 n^{-k}$.*

Also, if we denote $I=[i_1n^{-1}, i_2n^{-1}]$, $i_2-i_1=\lambda r^2+\mu$, with $\lambda>1$ and $0\leqq\mu<r^2$, and $x_\nu=(i_1+\nu r^2)n^{-1}$, then

(3.4) *if I is any of the intervals J_j^*, then for each $1\leqq\nu\leqq\lambda$, there is an interval $[l_\nu n^{-1}, (l_\nu+r)n^{-1}]$ contained in $[x_{\nu-1}, x_\nu]$ on which $f_I'(x)=f'(x)\geqq B_1n^{-k}$.*

We should remark that the actual statements of the results (3.3) and (3.4) in [2] (these are Lemmas 3 and 4 in [2]) are stated with $n^{-k+1}\omega(f^{(k)}, n^{-1})$ in place of n^{-k}. However, the proof goes over exactly the same with n^{-k}.

The constant B_1 of (3.14) is equal to $2^{-r^4}A_1=100r^2C_2\alpha_1^{-1}\alpha_2^2$, where $A_1=100r^22^{r^4}C_2\alpha_1^{-1}\alpha_2^2$. As we have remarked earlier, the constants α_1, α_2, C_2 were introduced in [2] so that certain inequalities hold. We have redefined α_1 and α_2 in § 2, preserving the original inequalities, and requiring that some new inequalities hold. In a similar vein, we will redefine C_2 in § 5, so that a new inequality holds while retaining the old inequalities that involved C_2. The constant $B_1=2^{-r^4}\alpha_3A_1^2$ where we will use the same value of α_3 as in [2]. The only importance in the value of α_3 for this paper is that it is bigger than 1.

If I is one of the intervals I_j^*, then we have a control over f' immediately to the right and left of I, because of (3.2). This estimate was actually given in a more precise way in [2] and we will need this more precise version:

(3.5) *if $I=[a,b]$ is one of the intervals I_j^*, then $f'(x)\leqq A_1n^{-k}$, $x\in([a-m^{-1}, a]\cup[b, b+m^{-1}])\cap[0,1]$.*

The intervals $[a-m^{-1}, a]\cap[0,1]$ and $[b, b+m^{-1}]\cap[0,1]$ are contained in what we called intervals of type 1 in [2].

4. Approximation of the functions $f_{J_j^*}$. We can approximate the functions $f_{J_j^*}$ using the technique of G. G. Lorentz and K. Zeller [5]. Let I be one of the intervals J_j^*. The function f_I is in Lip 1, in fact

$$(4.1)\qquad |f'(x)|\leqq An^{-k},\qquad \text{a.e. in } [0,1],$$

because of (3.2). Thus, the Lorentz–Zeller theorem gives that there is a polynomial $P_I\in\Pi_n^*$, such that

$$(4.2)\qquad |f_I(x)-P_I(x)|\leqq Cn^{-k-1},\qquad x\in[0,1],$$

with C an absolute constant. We want to observe more, namely that P_I is a better approximation away from the interval I, due to the fact that $f_I'(x)=0$, outside of I.

LEMMA 1. *If I is one of the intervals J_j^*, $j=0,1,\cdots,m$, then there is a polynomial $P_I\in\Pi_n^*$, such that*

$$(4.3)\qquad |f_I(x)-P_I(x)|\leqq CAn^{-k-1}(d_n(x,I))^{-k},\qquad x\in[0,1],$$

with C depending only on k.

Proof. The basic idea is to go to the trigonometric case via the substitution $x=\cos^2\theta$, and then use the Jackson operators. Let r be the integer defined in § 2, and K_n the Jackson kernel

$$(4.4)\qquad K_n(t)=c_n\left(\frac{\sin(n't/2)}{\sin(t/2)}\right)^{2r+2},\qquad \int_{-\pi}^{\pi}K_n(t)\,dt=1,$$

where n' is chosen as the largest integer such that $(2r+2)n' \leqq n$. So, K_n is a trigonometric polynomial of degree $\leqq n$, and we have the following estimates for the moments of K_n (see G. G. Lorentz [3, p. 57]):

$$\int_{-\pi}^{\pi} |t|^j K_n(t)\, dt \leqq C_1 n^{-j}, \qquad j = 0, 1, \cdots, 2r, \tag{4.5}$$

with C_1 a constant that depends only on k.

If h is a 2π periodic function, we define

$$L_n(h, \theta) = \int_{-\pi}^{\pi} h(\theta + t) K_n(t)\, dt.$$

It will be notationally more convenient to work on $[-\frac{1}{2}, \frac{1}{2}]$, then on $[0, 1]$ and so we introduce $\tilde{f}_I(x) = f_I(x + \frac{1}{2})$. Let $g(\theta) = \tilde{f}_I(\cos\theta)$, and define

$$g_n(\theta) = \begin{cases} g(k\pi(n')^{-1}), & \nu\pi(n')^{-1} \leqq \theta \leqq (\nu+1)\pi(n')^{-1}, \quad \nu = 0, \cdots, n'-1, \\ g_n(-\theta), & \text{for } \theta < 0. \end{cases}$$

Since the function g_n is even, we have that $L_n(g_n, \theta)$ is an even trigonometric polynomial of degree $\leqq n$. Hence, $\tilde{P}_I(x) = L_n(g_n, \arccos x)$ is an algebraic polynomial of degree $\leqq n$. Lorentz and Zeller have only used the operators L_n when $r = 1$, but the proof of the monotonicity of $\tilde{P}_I$ and the verification of (4.2) with $P_I(x) = \tilde{P}_I(x - \frac{1}{2})$ is exactly the same in the general case.

We need to get a better estimate than (4.2) outside of I. If S is any set, let $\tilde{S} = \{x : x + \frac{1}{2} \in S\}$ and $\tilde{S}' = \{\theta : \cos\theta \in S\}$. Note first that $|g_n(\theta) - g(\theta)| \leqq CA\pi n^{-k-1}$ for all θ and $g_n(\theta) = g(\theta)$ if $\operatorname{dist}(\theta, \tilde{I}') \geqq \pi n^{-1}$. Hence, if we take $\theta \notin \tilde{I}'$ and let $\delta = \operatorname{dist}(\theta, \tilde{I}')$ and assume that $\delta \geqq \pi n^{-1}$, then we have

$$|g_n(\theta+t) - g(\theta)| \leqq |g_n(\theta+t) - g(\theta+t)| + |g(\theta+t) - g(\theta)|$$

and so $|g_n(\theta+t) - g(\theta)| = 0$, $|t| \leqq \delta$ and $\leqq CAn^{-k}|t|$ when $|t| > \delta$. This gives

$$\left| \int_{-\pi}^{\pi} (g_n(\theta+t) - g(\theta)) K_n(t)\, dt \right| \leqq CAn^{-k} \int_{|t| \geqq \delta} |t| K_n(t)\, dt$$

$$\leqq CAn^{-k}\delta^{-r+1} \int_{-\pi}^{\pi} |t|^r K_n(t)\, dt \leqq CAn^{-r-k}\delta^{-r+1},$$

because of (4.5). Translating this to $x \notin \tilde{I}$, using $\operatorname{dist}(\theta, \tilde{I}') \geqq \operatorname{dist}(x, \tilde{I})$, we find

$$|\tilde{f}_I(x) - \tilde{P}_I(x)| \leqq CAn^{-k-1}(d_n(x, \tilde{I}))^{-r+1}, \qquad x \in [-1, 1],$$

with C a constant depending only on k, where we have also used (4.2). When we restate this last inequality in terms of f_I and P_I and make the simple observations that $d_n(x, I) \geqq 1$ and $r \geqq k+1$, we get (4.3).

5. Approximation of the functions $f_{I_j}^*$. The approximation of the functions $f_{I_j}^*$ is more complicated. In this section, we will use standard techniques to get a good polynomial approximation to $f_{I_j}^*$ but this polynomial may not be monotone and so we will have to make corrections to this polynomial in the next section. Again, it is more convenient to work on $[-\frac{1}{2}, \frac{1}{2}]$ than on $[0, 1]$. Let $I = [a, b]$ be one of the intervals I_j^*, and denote as before $\tilde{I}=\{x: x+\frac{1}{2}\in I\}$, $\tilde{f}_I(x)=f_I(x+\frac{1}{2})$, and $g(\theta)=\tilde{f}_I(\cos\theta)$.

We will approximate first with the Jackson operators of order r. Let

$$M_n(g,\theta)=-\int_{-\pi}^{\pi}\left(\sum_1^r(-1)^\nu\binom{r}{\nu}g(\theta+\nu t)\right)K_n(t)\,dt,$$

where K_n is the kernel of (4.4). Then $M_n(g,\theta)$ is an even trigonometric polynomial of degree $\leqq n$, and so $\tilde{P}(x)=M_n(g, \arccos x)$ is an algebraic polynomial of degree $\leqq n$. The polynomial $P(x)=\tilde{P}(x-\frac{1}{2})$ is a good approximation to f_I.

Let $E=E_1\cup E_2$, where $E_1=[a-rn^{-1}, a+rn^{-1}]\cap[0,1]$ and $E_2=[b-rn^{-1}, b+rn^{-1}]\cap[0,1]$. The following lemma establishes the approximating properties of P and in the process redefines the constant C_2 of [2].

LEMMA 2. *The constant C_2 can be chosen so large that*

(5.1) $$|f_I(x)-P(x)|\leqq C_2A_1n^{-k-1}(d_n(x,I))^{-r},\qquad x\in[0,1],$$

(5.2) $$|f_I'(x)-P'(x)|\leqq C_2(A_1(d_n(x,E))^{-r}+(d_n(x,I))^{-r})n^{-k},\qquad x\in[0,1].$$

Proof. It will be important to observe that our choice of C_2 does not depend on any of the other constants, particularly A_1 and A_2. Throughout the proof C and C' denote constants that depend on k but are independent of all of the other constants.

If $I=[0,1]$, then (5.1) and (5.2) follow from the usual Jackson theorems on the simultaneous approximation of a function and its derivatives. Hence, we can assume that $I\neq[0,1]$. This will allow us to control the derivatives of f provided n is sufficiently large. Indeed, we know that $|f'(x)|\leqq A_1n^{-k}$ on an interval $\mathcal{J}$ of length $\geqq n^{-1}$ because of (3.5). This and the fact that $|f^{(k+1)}(x)|\leqq 1$ on $\mathcal{J}$ give that $|f^{(i)}(x)|\leqq CA_1n^{-k+i-1}$, $x\in\mathcal{J}$, $i=1,2,\cdots,k+1$, with C a constant depending only on k.

Now that we have $f^{(i)}$ controlled on $\mathcal{J}$, it is easy to get an estimate for all x. For example, if $x_0\in\mathcal{J}$, we have

$$|f^{(k)}(x)|\leqq|f^{(k)}(x_0)|+\left|\int_{x_0}^{x}f^{(k+1)}(t)\,dt\right|\leqq(CA_1n^{-1}+1)$$

where we used the fact that $\|f^{(k+1)}\|_\infty=1$. Continuing in this way, we get

$$\|f^{(i)}\|\leqq(C'A_1n^{-1}+1),\qquad i=1,2,\cdots,k+1,$$

with C' depending only on k. Thus for n sufficiently large $C'A_1n^{-1}\leqq 1$ and we have $\|f^{(i)}\|\leqq 2$, $i=1,2,\cdots,k+1$.

We now proceed to the actual proof of (5.1) and (5.3). We need only verify these inequalities for n sufficiently large since they hold automatically for $n\leqq N$. The proof consists of showing the corresponding result for the approximation of g by $M_n(g)$. To do this, we first need some estimates for $\Delta_t^r(g,\theta)$. If S is any subset of $[0,1]$, then as in § 4, we denote $\tilde{S}=\{x: x+\frac{1}{2}\in S\}$, and $\tilde{S}'=\{\theta: \cos\theta\in\tilde{S}\}$.

Since f_I is constant outside of I, we have

(5.3) $$\Delta_t^r(g,\theta)=0,\quad\text{when }[\theta,\theta+rt]\cap\tilde{I}'=\phi.$$

Also, since $f_I(x)=f(x)-f(a)$ on I, we have

$$(5.4)\quad |\Delta_t^r(g,\theta)|\leqq 2^r|\Delta_t^{k+1}(g,\theta)|\leqq C\|f^{(k+1)}\|_\infty t^{k+1}\leqq Ct^{k+1},\quad \text{when } [\theta,\theta+rt]\subseteq \tilde{I}'$$

with C a constant depending only on k. In the second inequality, we used the fact that the $(k+1)$st derivative of g can be expressed in terms of the $f^{(i)}$, $i=1,\cdots,k+1$ and we know $\|f^{(i)}\|\leqq 2$, $i=1,2,\cdots,k+1$.

We also need an estimate for $\Delta_t^r(g,\theta)$ in the remaining case. Let $F=([a-m^{-1},a]\cup[b,b+m^{-1}])\cap[0,1]$. Because of (3.5), we know that

$$(5.5)\quad |f_I(x)-(f(x)-f(a))|\leqq\int_F f'(t)\,dt\leqq 2rA_1n^{-k-1},\qquad x\in F,$$

where we have used the fact that the two intervals that make up F each have length m^{-1}, and $f_I'(x)=0$, $x\in F$.

Let $\tilde{f}(x)=f(x+\frac{1}{2})$ and $h(\theta)=\tilde{f}(\cos\theta)$. It follows from (5.5) that

$$(5.6)\quad |g(\theta)-(h(\theta)-f(a))|\leqq 2rA_1n^{-k-1},\qquad \theta\in\tilde{F}'.$$

Now, suppose that $[\theta,\theta+rt]$ is not contained in either $\tilde{I}'$, or $\mathscr{C}\tilde{I}'$, and $|t|\leqq n^{-1}$. Then, $\theta\in\tilde{E}'$ and for each value of ν either $\theta+\nu t\in\tilde{F}'$ or $\theta+\nu t\in\tilde{I}'$, and so

$$
\begin{aligned}
|\Delta_t^r(g,\theta)|&\leqq\left|\sum_0^r(-1)^\nu\binom{r}{\nu}h(\theta+\nu t)\right|+\left|\sum{}'(-1)^\nu\binom{r}{\nu}(g(\theta+\nu t)-h(\theta+\nu t)+f(a))\right|\\
(5.7)\qquad &\leqq|\Delta_t^r(h,\theta)|+2^{r+1}rA_1n^{-k-1}\leqq CA_1n^{-k-1},\qquad \theta\in\tilde{E}',\quad |t|\leqq n^{-1},
\end{aligned}
$$

with C a constant depending only on k and the set E introduced at the beginning of this section. The $\sum'$ indicates that this sum is taken only over those ν for which $\theta+\nu t$ is in F'. This sum was estimated using (5.6).

The estimate (5.7) holds for all other values of θ, if $|t|\leqq n^{-1}$, because in the other cases either (5.3) or (5.4) holds. Hence,

$$\omega_r(g,n^{-1})\leqq CA_1n^{-k-1},$$

again with C a constant depending only on k. Hence for any t and θ,

$$(5.8)\quad |\Delta_t^r(g,\theta)|\leqq\omega_r(g,t)\leqq(1+nt)^r\omega_r(g,n^{-1})\leqq CA_1n^{-k-1}(1+nt)^r.$$

Now, the estimates (5.3) and (5.8), can be used to prove (5.1). First for any θ, we have

$$
\begin{aligned}
|g(\theta)-M_n(g,\theta)|&\leqq\int_{-\pi}^{\pi}|\Delta_t^r(g,\theta)|K_n(t)\,dt\\
(5.9)\qquad &\leqq CA_1n^{-k-1}\int_{-\pi}^{\pi}(1+nt)^rK_n(t)\,dt\leqq C'A_1n^{-k-1},
\end{aligned}
$$

with C' a constant depending only on k. Here, we have used (5.8), the fact that K_n has integral 1 and the estimates for the moments of K_n given in (4.5).

We need to improve this estimate when θ is not in $\tilde{I}'$. For such θ, let $\delta = \operatorname{dist}(\theta, \tilde{I}')$. If $\delta > 0$, then

$$
\begin{aligned}
|g(\theta)-M_n(g,\theta)| &\le \int_{|t|>\delta r^{-1}} |\Delta_t^r(g,\theta)|K_n(t)\,dt \le r^{r-\delta-r}\int_{-\pi}^{\pi} t^r|\Delta_t^r(g,\theta)|K_n(t)\,dt \\
&\le r^r C A_1 \delta^{-r} n^{-k-1}\int_{-\pi}^{\pi} t^r(1+nt)^r K_n(t)\,dt \\
&\le C' A_1 (n\delta)^{-r} n^{-k-1},
\end{aligned}
\tag{5.10}
$$

with C' a constant depending only on k. Here, we have used (5.3), (5.8), and (4.5).

The last inequality coupled with (5.9) shows that

$$|g(\theta)-M_n(g,\theta)| \le C' A_1 n^{-k-1}(d_n(\theta,\tilde{I}'))^{-r}. \tag{5.11}$$

This together with the fact that $d_n(x,\tilde{I}) \le d_n(\theta,\tilde{I}')$, $x=\cos\theta$, gives the estimate (5.1), when everything is restated in terms of f_I and P.

The estimate (5.2) is established in much the same way. In exactly the same way that we have proved (5.3), (5.4), and (5.8), we can show that

$$\Delta_t^r(g',\theta)=0, \quad \text{when } [\theta,\theta+rt]\cap\tilde{I}'=\phi, \tag{5.12}$$

$$|\Delta_t^r(g',\theta)| \le Ct^k, \quad \text{when } [\theta,\theta+rt]\subseteq\tilde{I}', \tag{5.13}$$

$$|\Delta_t^r(g',\theta)| \le CA_1 n^{-k}(1+nt)^r, \quad \text{for any } t \text{ and } \theta. \tag{5.14}$$

In exactly the same way that we have established (5.11), we can use (5.12) and (5.14) to see that

$$|g'(\theta)-(M_n(g))'(\theta)| \le CA_1 n^{-k}(d_n(\theta,\tilde{I}'))^{-r}, \quad \text{for any } t \text{ and } \theta. \tag{5.15}$$

Here, we have used the fact that $M_n(g',\theta)=(M_n(g))'(\theta)$. Restating this last inequality in terms of $\tilde{f}_I$, we find

$$
\begin{aligned}
|\tilde{f}_I'(x)-\tilde{P}'(x)| &\le CA_1 n^{-k}(d_n(x,\tilde{I}))^{-r}(1-x^2)^{-1/2} \\
&\le 2CA_1 n^{-k}(d_n(x,\tilde{I}))^{-r}, \qquad -\tfrac{1}{2}\le x\le\tfrac{1}{2}.
\end{aligned}
\tag{5.16}
$$

For x in $\tilde{I}$, we can improve this last estimate. When $\theta\in\tilde{I}'$, let $\delta=\operatorname{dist}(\theta,\tilde{E}')$. Then

$$
\begin{aligned}
|g'(\theta)-(M_n(g))'(\theta)| &\le \int_{|t|\le r^{-1}\delta} |\Delta_t^r(g',\theta)|K_n(t)\,dt \\
&\quad + \int_{|t|\ge r^{-1}\delta} |\Delta_t^r(g',\theta)|K_n(t)\,dt \\
&= \Sigma_1+\Sigma_2.
\end{aligned}
\tag{5.17}
$$

We use (5.14) on Σ_2 and estimate exactly as in (5.10) to find that

$$\Sigma_2 \le C' A_1 n^{-k}(n\delta)^{-r}. \tag{5.18}$$

For Σ_1, we use (5.13) to find

$$\Sigma_1 \le C\int_{-\pi}^{\pi} |t|^k K_n(t)\,dt \le C n^{-k}, \tag{5.19}$$

because of (4.5). These last two estimates when put back into (5.17) give

$$(5.20)\qquad |g'(\theta)-(M_n(g))'(\theta)|\le C'(A_1(d_n(\theta,\tilde{E}'))^{-r}+1)n^{-k},\qquad \theta\in\tilde{I}',$$

where we have used (5.15) to replace $n\delta$ by $d_n(\theta,\tilde{E}')$ in our estimate (5.18) of Σ_2.

Restating (5.20) in terms of f_I and using the fact that $d_n(x,\tilde{I})=1, x\in\tilde{I}$, gives

$$(5.21)\qquad |f_I'(x)-P'(x)|\le 2C'(A_1(d_n(x,\tilde{E}))^{-r}+(d_n(x,\tilde{I}))^{-r})n^{-k},\qquad x\in\tilde{I}.$$

This inequality also holds for any $x\in[-\frac{1}{2},\frac{1}{2}]$ because of (5.16) and the fact that outside of $\tilde{I}$, $d_n(x,\tilde{E})\le d_n(x,\tilde{I})$. Restating (5.21) in terms of f_I and P gives (5.2).

6. Monotone approximation of the functions $f_{I_j^*}$. We can now give a monotone approximation to the functions $f_{I_j^*}$ analogous to that given in Lemma 1 for $f_{I_j^*}$.

LEMMA 3. *If I is one of the intervals I_j^*, $j=1,\cdots,m$, then there is a polynomial $P_I\in\Pi_n^*$, such that*

$$(6.1)\qquad |f_I(x)-P_I(x)|\le Cn^{-k-1}(d_n(x,I))^{-k},\qquad x\in[0,1].$$

Proof. Let P be a polynomial of degree $\le n$ which satisfies Lemma 2. Since P need not be nondecreasing, we must make some corrections. The correcting polynomials will not vanish outside I as was the case for splines. Instead, these polynomials will fall off due to the factor $d_n(x,I)$. This means that all our estimates will contain terms involving $d_n(x,I)$. While this complicates matters some, the basic idea is the same as the approximation of f_I by splines given in [2].

While P' is not necessarily positive, we do have from (5.2) that

$$(6.2)\qquad P'(x)\ge f'(x)-C_2(A_1(d_n(x,E))^{-r}+(d_n(x,I))^{-r})n^{-k},\qquad x\in[0,1].$$

Let $\gamma_1=C_2A_1\alpha_1^{-1}|E_1|n^{-k}=C_2A_1r\alpha_1^{-1}n^{-k-1}$, $\gamma_2=C_2\alpha_1^{-1}|I|n^{-k}$, and define

$$Q_1(x)=\gamma_1(\Lambda_{E_1}(x)+\Lambda_{E_2}(x))+\gamma_2\Lambda_I(x),$$

where E_1 and E_2 are the two intervals that make up E and the Λ polynomials are as defined in § 2. Now, $\Lambda_I'(x)=\lambda_I(x)$ and therefore from (2.9) and (6.2), it follows that

$$(6.3)\qquad P'(x)+Q_1'(x)\ge f'(x)\ge 0,\qquad x\in[0,1].$$

However, we may have added too much error and so we must take it away.

As in § 3, let $I=[i_1n^{-1}, i_2n^{-1}]$, with $i_2-i_1=r^2\lambda+\mu$, $1<\lambda$, $0\le\mu<r^2$, and $x_\nu=(i_1+\nu r^2)n^{-1}$, $\nu=0,1,\cdots,\lambda$. From (3.3), we know that there is an interval $[l_0n^{-1},(l_0+r)n^{-1}]\subseteq I$, on which

$$(6.4)\qquad P'(x)+Q_1'(x)\ge f'(x)\ge B_2n^{-k}.$$

Also, from (3.4), we know that for each $1\le\nu\le\lambda$ there is an interval $[l_\nu n^{-1},(l_\nu+r)n^{-1}]\subseteq[x_{\nu-1},x_\nu]$, on which

$$(6.5)\qquad P'(x)+Q_1'(x)\ge f'(x)\ge B_1n^{-k}.$$

Define

$$a_1=\Lambda_I(x_1)-\Lambda_I(0),\qquad a_\lambda=\Lambda_I(1)-\Lambda_I(x_{\lambda-1}),$$

$$a_\nu=\Lambda_I(x_\nu)-\Lambda_I(x_{\nu-1}),\qquad 2\le\nu\le\lambda-1.$$

Then, because of (2.10)–(2.12), we have

(6.6) $$|a_\nu| \leqq 4r^2\alpha_2|I|^{-1}n^{-1}, \qquad \nu = 1, 2, \cdots, \lambda.$$

where the r^2 appears because $|x_\nu - x_{\nu-1}| = r^2 n^{-1}$, and the 4 appears because the estimate of a_λ uses (2.12) twice and (2.11) once.

Recall the Φ polynomials introduced in § 2. Let us use the notation $\Phi_\nu = \Phi_{[l_\nu n^{-1}, (l_\nu+r)n^{-1}]}$ and $\phi_\nu = \phi_{[l_\nu n^{-1}, (l_\nu+r)n^{-1}]} = \Phi'_\nu$. Define

$$Q_2(x) = 2\gamma_1\Phi_0(x) + \gamma_2 \sum_1^\lambda a_\nu \Phi_\nu(x).$$

The polynomial $P_I = P + Q_1 - Q_2$ will be our approximation to f_I.

First, we want to show that P_I is nondecreasing. The polynomial $\phi_\nu = \Phi'_\nu$ is only positive on the interval $[l_\nu n^{-1}, (l_\nu+r)n^{-1}]$ and so because of (6.3), we need only check that P'_1 is positive on these intervals. Let's first consider $[l_0 n^{-1}, (l_0+r)n^{-1}]$. This interval can intersect at most two of the other intervals $[l_\nu n^{-1}, (l_\nu+r)n^{-1}]$ and so from the definition of Q_2, we find

$$\begin{aligned} Q'_2(x) &\leqq 2\gamma_1\alpha_2 n + \gamma_2(4r^2\alpha_2|I|^{-1}n^{-1})\alpha_2 n \\ &\leqq 2rC_2A_1\alpha_1^{-1}\alpha_2 n^{-k} + 4r^2C_2\alpha_2^2\alpha_1^{-1}n^{-k} \\ &\leqq B_2 n^{-k} \leqq P'(x) + Q'_1(x), \qquad x \in [l_0 n^{-1}, (l_0+r)n^{-1}], \end{aligned}$$

where the first inequality uses (2.16) and (6.6), the second uses the definition of γ_1 and γ_2, the third used the value of $B_2 = \alpha_3 2^{-r^4} A_1^2 \geqq 100r^2\alpha_1^{-1}\alpha_2^2 C_2 A_1$, and the fourth inequality is (6.4). Thus $P'_I(x) \geqq 0$ on $[l_0 n^{-1}, (l_0+r)n^{-1}]$.

For the other intervals $[l_\nu n^{-1}, (l_\nu+r)n^{-1}]$, $1 \leqq \nu \leqq \lambda$, we need only check on the parts of these intervals that don't intersect $[l_0 n^{-1}, (l_0+r)n^{-1}]$. For such an interval, we have

$$\begin{aligned} Q'_2(x) &\leqq \gamma_2|a_\nu|\,|\phi_\nu(x)| \leqq C_2\alpha_1^{-1}|I|n^{-k}(4r^2\alpha_2|I|^{-1}n^{-1})\alpha_2 n \\ &\leqq B_1 n^{-k} \leqq P'(x) + Q'_1(x), \qquad x \in [l_\nu n^{-1}, (l_\nu+r)n^{-1}] \setminus [l_0 n^{-1}, (l_0+r)n^{-1}], \end{aligned}$$

where the second inequality uses the definition of γ_2, (6.6), and (2.16). The third inequality uses the value of $B_1 = 100r^2C_2\alpha_1^{-1}\alpha_2^2$, and the last inequality is (6.5). This shows that P_I is indeed monotone nondecreasing.

To finish the proof of Lemma 3, we need to verify (6.1). To this end, it is enough to show that

(6.7) $$|Q_1(x) - Q_2(x)| \leqq Cn^{-k-1}(d_n(x, I))^{-k}, \qquad x \in [0, 1],$$

with C depending only on k. Consider first the polynomial $2\gamma_1\Phi_0 - \gamma_1(\Lambda_{E_1} + \Lambda_{E_2})$. If dist $(x, I) \geqq 2rn^{-1}$, then dist $(x, I) \leqq 2$ dist (x, E_1), dist $(x, I) \leqq 2$ dist (x, E_2), and dist $(x, I) \leqq$ dist $(x, [l_0 n^{-1}, (l_0+r)n^{-1}])$. Hence, if we use (2.10) and (2.17), we find that for $x \leqq a - 2rn^{-1}$, $I = [a, b]$, we have

(6.8) $$\begin{aligned} |2\gamma_1\Phi_0(x) - \gamma_1(\Lambda_{E_1}(x) + \Lambda_{E_2}(x))| &\leqq 6\gamma_1\alpha_2(d_n(x, I))^{-2k-1} \\ &\leqq Cn^{-k-1}(d_n(x, I))^{-k}, \end{aligned}$$

where in the first inequality, we have used the facts that $r - 2 \geqq 2k - 1$ and $|E_1| = rn^{-1}$. In the second inequality we used the fact that $\gamma_1 \leqq$ const. n^{-k-1}, with the constant depending only on k.

Similarly, when $x \geqq b + 2rn^{-1}$,

$$
\begin{aligned}
|2\gamma_1\Phi_0(x) - \gamma_1(\Lambda_{E_1}(x) + \Lambda_{E_2}(x))| &\leqq |2\gamma_1(1 - \Phi_0(x)) - \gamma_1(1 - \Lambda_{E_1}(x)) \\
&\qquad - \gamma_1(1 - \Lambda_{E_2}(x))| \\
&\leqq Cn^{-k-1}(d_n(x, I))^{-k},
\end{aligned} \tag{6.9}
$$

because of (2.11) and (2.18).

The estimates (6.8) and (6.9) also hold when dist $(x, I) \leqq 2rn^{-1}$, because the polynomials Φ_0, Λ_{E_1}, and Λ_{E_2} all have supremum norm equal to 1 on $[0, 1]$. Therefore,

$$
|2\gamma_1\Phi_0(x) - \gamma_1(\Lambda_{E_1}(x) + \Lambda_{E_2}(x))| \leqq Cn^{-k-1}(d_n(x, I))^{-k}, \qquad x \in [0, 1], \tag{6.10}
$$

with C depending only on k.

We will now prove an estimate like (6.10) for the polynomial $\gamma_2(\Lambda_I - \sum_1^\lambda a_\nu\Phi_\nu)$. Let $x_{-1} = 0$, $x_{\lambda+1} = 1$, and $a_0 = 0$. Then, if $x_{\nu_0} \leqq x \leqq x_{\nu_0+1}$,

$$
\begin{aligned}
\left|\gamma_2\left(\Lambda_I(x) - \sum_0^\lambda a_\nu\Phi_n(x)\right)\right| &\leqq \gamma_2\left|\sum_{-1}^{\nu_0} a_\nu(1 - \Phi_\nu(x))\right| + \gamma_2|\Lambda_I(x) - \Lambda_I(x_{\nu_0})| \\
&\quad + \gamma_2\left|\sum_{\nu_0+1}^{\lambda} a_\nu\Phi_\nu(x)\right| \\
&= \Sigma_1 + \Sigma_2 + \Sigma_3.
\end{aligned} \tag{6.11}
$$

To estimate Σ_1, we need only observe that because of (6.6), we have $\gamma_2 \max|a_\nu| \leqq Cn^{-k-1}$, with C depending only on k. Also, if we let $s_\nu = [l_\nu n^{-1}, (l_\nu + r)n^{-1}] \subseteq [x_{\nu-1}, x_\nu] \subseteq I$ then $d_n(x, I) \leqq d_n(x, s_\nu)$, for any $x \in [0, 1]$ and $\nu = 1, \cdots, \lambda$. Hence, from (2.18)

$$
\begin{aligned}
\Sigma_1 &\leqq \alpha_2\gamma_2 \max|a_\nu| \sum_1^{\nu_0} (d_n(x, s_\nu))^{-2k-1} \\
&\leqq \alpha_2 Cn^{-k-1}(d_n(x, I))^{-k} \sum_1^{\nu_0} (d_n(x, s_\nu))^{-2} \leqq C'n^{-k-1}(d_n(x, I))^{-k},
\end{aligned} \tag{6.12}
$$

with C' depending only on k. Here, we used the fact that $\sum_1^\lambda (d_n(x, s_\nu))^{-2}$ is uniformly bounded on $[0, 1]$, because $|x_{\nu-1} - x_\nu| = r^2n^{-1}$.

The sum Σ_3 can be estimated in exactly the same way as Σ_1 except that now we use (2.17) in place of (2.18) to find

$$
\Sigma_3 \leqq C'n^{-k-1}(d_n(x, I))^{-k}, \qquad x \in [0, 1]. \tag{6.13}
$$

If $x \in I$, then Σ_2 is estimated by using (2.12) to find

$$
\Sigma_2 \leqq \gamma\alpha_2|I|^{-1}|x - x_{\nu_0}| \leqq C'n^{-k-1}(d_n(x, I))^{-k}, \qquad x \in I,
$$

because $d_n(x, I) = 1$, $x \in I$, and $|x - x_{\nu_0}| \leqq 2r^2n^{-1}$. When $x \notin I$, we use either (2.10) or (2.11) as appropriate to find

$$
\begin{aligned}
\Sigma_2 \leqq \gamma_2|\Lambda_I(x) - \Lambda_I(x_{\nu_0})| &\leqq 2\gamma_2\alpha_2|I|^{-1}n^{-1}(d_n(x, I))^{-r+2} \\
&\leqq C'n^{-k-1}(d_n(x, I))^{-k}, \qquad x \in I,
\end{aligned}
$$

because $r - 2 \geqq k$.

Putting our estimates for Σ_1, Σ_2, and Σ_3 back into (6.11) gives

$$\left|\gamma_2\left(\Lambda_I(x)-\sum_1^{\lambda} a_\nu\Phi_\nu(x)\right)\right| \leqq 3C'n^{-k-1}(d_n(x,I))^{-k}, \qquad x\in[0,1]. \tag{6.14}$$

Finally, when we use (6.14) and (6.8), it follows that

$$|Q_1(x)-Q_2(x)| \leqq (3C'+C)n^{-k-1}(d_n(x,I))^{-k}, \qquad x\in[0,1],$$

and so using (5.1), we have

$$|f_I(x)-P_I(x)| \leqq |f_I(x)-P(x)|+|Q_1(x)-Q_2(x)| \leqq Cn^{-k-1}(d_n(x,I))^{-k}$$

for all $x\in[0,1]$, with C depending only on k. This proves Lemma 3.

7. Proofs of theorem. It is easy to prove Theorem 2, using the results in Lemmas 1 and 3. If $\|f^{(k+1)}\|_{L^\infty[0,1]}=1$, then

$$f(x)=f(0)+\sum_0^m f_{J_j^*}(x)+\sum_1^m f_{I_j^*}(x)$$

as in (3.1). Let

$$P(x)=f(0)+\sum_0^m P_{J_j^*}(x)+\sum_1^m P_{I_j^*}(x),$$

where the polynomials $P_{J_j^*}$ are given in Lemma 1 and the polynomials $P_{I_j^*}$ are given in Lemma 3. P is then a monotone polynomial.

Now, $k\geqq 2$, $|J_j^*|\geqq r^2n^{-1}$, and $|I_j^*|\geqq r^2n^{-1}$. Hence,

$$\sum_0^m (d_n(x,J_j^*))^{-k}+\sum_1^m (d_n(x,I_j^*))^{-k} \leqq D, \qquad x\in[0,1],$$

with D depending only on k. Therefore,

$$|f(x)-P(x)| \leqq \sum_0^m |f_{J_j^*}(x)-P_{J_j^*}(x)|+\sum_1^m |f_{I_j^*}(x)-P_{I_j^*}(x)|$$

$$\leqq Cn^{-k-1}\left(\sum_0^m (d_n(x,J_j^*))^{-k}+\sum_1^m (d_n(x,I_j^*))^{-k}\right) \leqq CDn^{-k-1},$$

for any $x\in[0,1]$, which proves Theorem 2. As we have shown in the introduction, Theorem 1 follows from Theorem 2.

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