POINTWISE APPROXIMATION BY POLYNOMIALS AND SPLINES

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1. INTRODUCTION

We want to make some observations about pointwise approximation by polynomials and splines. The importance of pointwise estimates by polynomials is well known, since such estimates are needed to characterize the smoothness of a function in terms of its approximation by algebraic polynomials.

J. Brudnyi [2] has shown that for each $r \geqslant 1$, there is a constant $C_r > 0$ so that for any function $f \in C\,[-1, 1]$ we can find algebraic polynomials P_n, $n=1, 2, \ldots$, with P_n of degree $\leqslant n$ and

$$|f(x) - P_n(x)| \leqslant C_r \omega_r(f, \Delta_n(x)), \quad -1 \leqslant x \leqslant 1, \tag{1.1}$$

where $\Delta_n(x) = n^{-1}(1-x^2)^{1/2} + n^{-2}$. This shows of course that it is possible to improve the degree of approximation near the endpoints in comparison to the usual Jackson theorems. These estimates are then of the right form for converse theorems.

Actually Brudnyi showed more than (1.1) since he constructed linear operators L_n so that $L_n(f)=P_n$ do the job in (1.1). We will show in Section 3 that basically any of the standard linear methods used in proving Jackson's theorem already give (1.1).

There is the interesting question that was posed both by G. G. Lorentz and S. B. Stečkin as to whether it is possible to drop the n^{-2} term that appears in the difinition of $\Delta_n(x)$. This was shown to be possible in the cases $r=1$ and 2 by S. Teljakovskii [7] and I. Gopengauz [5]. We show in Section 4 the validity of (1.1), with the n^{-2} term deleted in the case $r=2$. Here again we are able to use linear methods of approximation. The question whether the n^{-2} term can be dropped in the general case $r > 2$ is not yet settled but there is some hope that the techniques presented here can be refined to give the general case.

One of the main ideas used in our estimates of the approximation is to replace the arbitrary function f by a smooth function h. This of course is no new idea since it has been used succesfully many times before. The actual smoothing can be done in a variety of ways. The most common methods are to use some sort of variation of Stecklov averages. While the actual method of smoothing is not important what is important is the estimates involved, especially for the derivatives of the smoothed functions. In this sense the idea of smoothing interfaces with some concepts in the theory of interpolation of linear operators, especially the Peetre K functional method for generating interpolation spaces.

If $r \geqslant 1$, let $L_\infty^{(r)} = L_\infty^{(r)}[-1, 1] = \{f : f^{(r-1)}$ is absolutely continuous and $f^{(r)} \in L_\infty[-1, 1]\}$. The K functional for interpolation between $C[-1, 1]$ and $L_\infty^{(r)}$ is defined by

$$K_r(f, t) = K(t, f, C, L_\infty^{(r)}) = \inf_{h \in L_\infty^{(r)}} (\|f-h\| + t\|h^{(r)}\|_\infty), \tag{1.2}$$

where $\|\cdot\|$ is as it will be throughout this paper the supremum norm on $[-1, 1]$. Thus K_r measures how well the function f can be approximated by functions h from $L_\infty^{(r)}$ with a control on the size of $\|h^{(r)}\|_\infty$.

G. Freud and V. Popov [4] have shown how to use modifications of the Stecklov averages to obtain the estimates

$$A_1 \omega_r(f, t) \leqslant K_r(f, t^r) \leqslant A_2 \omega_r(f, t), \quad t > 0, \tag{1.3}$$

with A_1 and A_2 constants depending only on r. This last estimate shows then, that for all practical purposes the K_r functional and the modulus of continuity are equally acceptable in obtaining estimates for the degree of approximation. The idea then is to obtain our estimates for smooth functions and than use (1.3) and the definition of K_r to obtain our estimates for the arbitrary function f.

In Section 2, we will show that it is possible to use spline functions in smoothing. Among other things, we will show how spline functions can be used to prove (1.3). Spline functions have the additional advantage of giving local estimates in their approximation of f. Then, when the knots are allowed to coalesce we can get better estimates at the point of coalescence.

Our approach to obtaining spline estimates is that of C. deBoor and G. Fix [1]. This approach besides being very simple and elegant shows that the spline constructions can be made via linear bounded projections onto the appropriate spline space.

2. APPROXIMATION BY SPLINES

In this section, we will follow the ideas of C. deBoor and G. Fix [1] to derive linear methods of spline approximation. We assume for simplicity that the knots are symmetric. That is, if the knots are the set $\Pi = \{x_i\}$ then we suppose that whenever x_i is a knot $-x_i$ is a knot denoted by $x_{-i} = -x_i$. We suppose also that 0 is always a knot and is denoted by $x_0 = 0$. The knot set may be finite or infinite. When it is infinite we assume that $\lim_{i \to \infty} x_i = 1$. With other kinds of labeling we could assume that the knots had an interior limit point but for our purposes this is not interesting. We suppose also that the knots are ordered $-1 \leqslant x_i < x_{i+1} \leqslant 1$. When the knot set is finite, say $\pi = \{x_i\}_{-n}^{n}$ then we define $x_i = -1$, $i < -n$, and $x_i = 1$, $i > n$.

Given the knot set π, we denote by $S_k(\pi)$ the spline space consisting of all functions S, with $S \in C^{(k-2)}[-1, 1]$, and S a polynomial of degree $\leqslant k-1$ on each interval $[x_i, x_{i+1})$. So $S_k(\pi)$ is the space of splines of order k (degree $k-1$) with knots Π.

A canonical basis for the space $S_k(\pi)$ is given by the B-splines

$$N_{i,k}(x) = (x_{i+k} - x_i)\, g_k(x_i, \ldots, x_{i+k}; x), \tag{2.1}$$

where $g_k(s; x) = (s-x)_+^{k-1}$, and in (2.1) we have used standard divided difference notation. (see [3] for properties of B-splines). Each spline $S \in S_k(\pi)$ can be written uniquely as

$$S(x) = \sum_i \lambda_i(S)\, N_{i,k}(x), \qquad x \in (-1, 1), \tag{2.2}$$

where the λ_i are continuous linear functionals on $S_k(\pi)$. Note that for each x at most k of the functions $N_{i,k}$ are non-zero at x.

The deBoor—Fix formula [1] shows that the functionals λ_j can be represented as

$$\lambda_j(S) = \sum_{r<k} \omega_{j,r} S^{(r)}(\tau_j), \tag{2.3}$$

with

$$(k-1)!\,\omega_{j,r} = (-1)^{k-1-r} \psi_j^{(k-1-r)}(\tau_j), \quad r < k,$$

$$\psi_j(x) = (x_{j+1} - x) \ldots (x_{j+k-1} - x),$$

with τ_j any point in (x_j, x_{j+k}). For our purpose, we choose τ_j as follows. Let (a_j, b_j) be the largest interval of the form (x_i, x_{i+1}) with $j \leqslant i < j+k$. Then, $|b_j - a_j| \geqslant A_3 |x_{j+k} - x_j|$. We take τ_j as the mid point of (a_j, b_j), $\tau_j = {}^1/_2 (a_j + b_j)$. From the formula for $\omega_{j,r}$, it follows that

$$|\omega_{j,r}| \leqslant (x_{j+k} - x_j)^r, \text{ for all } j \text{ and } r. \tag{2.4}$$

Thus using Markov's inequality with (2.3) and (2.4), we find

$$|\lambda_j(S)| \leqslant \sum_{r=0}^{k-1} (x_{j+k} - x_j)^r |S^{(r)}(\tau_j)| \leqslant$$

$$\leqslant \sum_{r=0}^{k-1} \left(\frac{x_{j+k} - x_j}{b_j - a_j}\right)^r \|S\|[x_j, x_{j+k}] \leqslant A_4 \|S\|[x_j, x_{j+k}], \tag{2.5}$$

with A_4 a constant depending only on k. The notation $\|\cdot\|[a, b]$ indicates that the norm is taken only over $[a, b]$.

The inequality (2.5) shows that the linear functional λ_j is supported on the interval $[x_j, x_{j+k}]$ and hence is a linear functional on the space $S_k(\pi)$ restricted to $[x_j, x_{j+k}]$. By the Hahn—Banach theorem, λ_j can be extended to a functional $\bar{\lambda}_j$ defined on $C[x_j, x_{j+k}]$ with preservation of the norm of the functional. From the Riesz representation theorem, $\bar{\lambda}_j$ can be represented as

$$\bar{\lambda}_j(f) = \int_{x_j}^{x_{j+k}} f(t)\, d\mu_j(t),$$

with $d\mu_j$ a Borel measure. This last equation serves to define $\bar{\lambda}_j$ for functions in $C[-1, 1]$ as well. The inequality (2.5) now takes the form

$$|\bar{\lambda}_j(f)| \leqslant A_4 \|f\| [x_j, x_{j+k}], \quad f \in C[-1, 1]. \tag{2.6}$$

Consider now the operators

$$L_\pi(f, x) = \sum_i \bar{\lambda}_i(f) N_{i,k}(x),$$

which are projections from $C[-1, 1]$ onto the space $S_k(\pi)$. The spline $L_\pi(f)$ is a good local approximation to the function f as will be shown with the help of the following lemma.

Lemma 1. *There is a constant A_5 depending only on k such that whenever $f \in C[a, b]$, then there is a polynomial P of degree $\leqslant k-1$, such that*

$$\|f - P\| [a, b] \leqslant A_5 \omega_k(f, k^{-1}|b - a|). \tag{2.7}$$

Proof. It will be enough to show this result for the interval $[0, k]$. The general result then holds by transforming the interval $[a, b]$ into the interval $[0, k]$ in the usual linear way.

We suppose that (2.7) does not hold and work for a contradiction. If (2.7) does not hold, then for each $n \geqslant 1$, we can find a function f_n with $\omega_k(f_n, 1) = n^{-1}$ and dist $(f_n, \mathbf{P}_{k-1}) = 1$, with $\mathbf{P}_{k-1}$ the space of algebraic polynomials of degree $\leqslant k-1$ and of course distance is measured with respect to the supremum norm on $[0, k]$. We can also suppose that the zero polynomial is the best approximation to f_n from the space $\mathbf{P}_{k-1}$, since we are at liberty to subtract polynomials of degree $\leqslant k-1$ without changing the k-th order moduli of smoothness of f_n. Thus $\|f_n\| = 1$. It is easy to check that $\{f_n\}_1^\infty$ is an equicontinuous set of functions. Since the functions are also uniformly bounded, we can extract a subsequence (f_{n_j}) which converges to some function f in $C[0, k]$. Then, f will have $\omega_k(f, 1) = 0$, which means that f is a polynomial of degree $\leqslant k-1$. But also, we have dist $(f, \mathbf{P}_{k-1}) = 1$, which means f is not a polynomial of degree $\leqslant k-1$. This is the desired contradiction.

The proof we have given for Lemma 1 is indirect and so we have no estimate for the constant A_5. However, H. Whitney [8] has given a direct but more complicated proof of this lemma that includes an estimate of the constant A_5. It would be interesting to find a simple direct proof of this lemma. Such a proof is easy to construct when $k = 1$ or 2.

The following theorem gives local estimates for the degree of approximation of f by $L_\pi(f)$.

Theorem 1. *Let π be a set of knots. If f is in $C[-1, 1]$, then*

$$|f(x) - L_\pi(f, x)| \leqslant A_6 \omega_k(f, |x_{j+k} - x_{j-k+1}|), \quad x_j \leqslant x \leqslant x_{j+1}, \tag{2.8}$$

with A_6 a constant that depends only on k and A_3.

Proof. Let P be a polynomial which satisfies (2.7) for the interval $[x_{j-k+1}, x_{j+k}]$. Since $x_j \leqslant x \leqslant x_{j+1}$, we have $N_{i,k}(x) = 0$, when $i \notin [j - k + 1, j]$, and $\sum_i N_{i,k}(x) = 1$. Hence,

$$|L_\pi(f - P, x)| \leqslant \sum_{j-k+1}^{j} |\bar{\lambda}_i(f - P)| N_{i,k}(x) \leqslant$$
$$\leqslant \sup_{j-k+1 \leqslant i \leqslant j} |\bar{\lambda}_i(f - P)| \sum_i N_{i,k}(x) \leqslant A_4 \|f - P\| [x_{j-k+1}, x_{j+k}] \tag{2.9}$$

because of (2.6).

The operator L_π preserves polynomials of degree $\leqslant k-1$, and so

$$|L_\pi(f, x) - f(x)| \leqslant |L_\pi(f - P, x)| + |f(x) - P(x)| \leqslant$$
$$\leqslant (A_4 + 1)\|f - P\|[x_{j-k+1}, x_{j+k}] \leqslant$$
$$\leqslant A_5(A_4 + 1)\,\omega_k(f, |x_{j+1} - x_{j-k+1}|) \leqslant A_6\omega_k(f, |x_{j+1} - x_{j-k+1}|),$$

where in the second inequality we used our estimate (2.9), and in the third inequality we used Lemma 1. This completes the proof of the theorem.

It is also possible to estimate the derivatives of $L_\pi(f)$. Such estimates are important when splines are to be used as a tool in other approximation problems. Let $S=L_\pi(f)$ and $r=k-1$. Then S has $r-1$ continuous derivatives and its r-th derivative exists almost everywhere and is a step function. We suppose $r \geqslant 1$ ($k \geqslant 2$), here. We also assume now that the knot set π has the property that there is a constant A_7 such that for all j,

$$|x_{j+k} - x_{j-k}| \leqslant A_7 |x_{j+1} - x_j|. \tag{2.10}$$

Let $M = |S^{(r)}(\tfrac{1}{2}(x_j + x_{j+1}))|$ which is then the absolute value of $S^{(r)}$ on the interval (x_j, x_{j+1}). Take $t = r^{-1}\delta$, with $\delta = |x_{j+1} - x_j|$. Then,

$$|\Delta_t^r(S, x_j)| = \left| \int_{x_j}^{x_j+t} \int_{u_{r-1}}^{u_{r-1}+t} \cdots \int_{u_1}^{u_1+t} S^{(r)}(u)\, du\, du_1 \ldots du_{r-1} \right| \geqslant Mt^r. \tag{2.11}$$

From (2.8) and the fact $\omega_k(f, t) \leqslant 2\omega_r(f, t)$, we find that

$$|\Delta_t^r(S, x_j)| \leqslant |\Delta_t^r(f, x)| + 2^{r+1}A_6\omega_r(f, |x_{j+k} - x_{j-k}|) \leqslant C\omega_r(f, t),$$

where in the last inequality we used (2.8). This together with (2.11) shows that

$$\|S^{(r)}\|[x_j, x_{j+1}] \leqslant Ct^{-r}\omega_r(f, t) \leqslant A_7' |x_{j+1} - x_j|^{-r}\omega_r(f, |x_{j+1} - x_j|). \tag{2.12}$$

Let us now consider the case of equally spaced knots, so that $\pi_n = \{-1 + kn^{-1}\}_0^{2n}$. The estimates (2.8) and (2.12) show the connection between the K functional and the moduli of smoothness mentioned in the introduction. To see this, let r be any positive integer and $k=r+1$. When $t > 0$, choose $n \geqslant 1$, so that $(n+1)^{-1} \leqslant t \leqslant n^{-1}$. Then for $S=L_{\pi_n}(f)$, we have

$$K_r(f, t^r) \leqslant K_r(f, n^{-r}) \leqslant \|f - S\| + n^{-r}\|S^{(r)}\|_\infty \leqslant$$
$$\leqslant A_6\omega_r(f, n^{-1}) + A_7'\omega_r(f, n^{-1}) \leqslant A_2\omega_r(f, (n+1)^{-1}) \leqslant A_2\omega_r(f, t),$$

with A_2 a constant depending only on r. We have used (2.8) to estimate $\|f-S\|$ and (2.12) to estimate $\|S^{(r)}\|_\infty$. This is the right hand side of (1.3).

The proof of the left hand side of (1.3) is quite simple. If h is any function with $h^{(r)}$ in L_∞, then for any $\delta > 0$,

$$|\Delta_\delta^r(f, x)| \leqslant 2^{r+1}\|f - h\| + \delta^r\|h^{(r)}\|_\infty.$$

Taking now on the left side a supremum over all $\delta \leqslant t$ and the permissible x's and on the right hand side a supremum over $\delta \leqslant t$ and an infimum over all $h \in L_\infty^{(r)}$ gives

$$\omega_r(f, t) \leqslant 2^{r+1}K_r(f, t^r).$$

This is the left hand side of (1.3).

3. POINTWISE ESTIMATES FOR APPROXIMATION BY POLYNOMIALS

We want now to show how standard linear methods of approximation will yield the Brudnyi estimates (1.1). To start with let (K_n) be any sequence of non-negative even trigonometric polynomials of degree $\leqslant n$ which satisfy

$$\int_{-\pi}^{\pi} K_n(t)\,dt = 1, \tag{3.1}$$

$$\int_{-\pi}^{\pi} |t|^j K_n(t)\,dt \leqslant A_8 n^{-j}, \quad j=1, 2, \ldots, 2r \text{ and } n=1, 2, \ldots, \tag{3.2}$$

with A_8 a constant depending only on r. The linear operators L_n defined by

$$L_n(g, \theta) = \int_{-\pi}^{\pi} (-\Delta_t^{2r}(g, \theta) + g(\theta)) K_n(t)\,dt \tag{3.3}$$

are standard examples of linear methods of approximating 2π periodic functions g by trigonometric polynomials of degree $\leqslant n$. The best known examples are the Jackson kernels, where for example we can take $K_n(t)=c_n(\sin(mt/2))^{4r}\cdot(\sin(t/2))^{-4r}$, with $m=[n/4r]$ (see e. g. G. G. Lorentz [6, p. 57]).

To get algebraic approximations we can make the usual substitution $x=\cos\theta$, but first we want to preserve polynomials of degree $r-1$. So for $f\in C[-1, 1]$, let $Q_r(f)$ be the algebraic polynomial of degree $\leqslant r-1$ which interpolates f at the equally spaced points $-1+2i/(r-1)$, $i=0, \ldots, r-1$. Then, Q_r is a bounded linear projection from $C[-1, 1]$ onto P_{r-1}. Let us define

$$\bar{L}_n(f, x) = L_n(f(\cos\theta) - Q_r(f, \cos\theta), \arccos x) + Q_r(f, x). \tag{3.4}$$

Then, $(\bar{L}_n)$ is a uniformly bounded sequence of polynomial operators, which as we shall see give the Brudnyi estimate. We begin with

L e m m a 2. *If g is a 2π-periodic and continuous function with $|g^{(\nu)}(\theta)| \leqslant M|\sin\theta|^{\mu}$, a. e., with μ and ν non-negative integers, $\mu+\nu\leqslant 2r$, then*

$$|L_n(g, \theta) - g(\theta)| \leqslant A_9 M n^{-\nu}(|\sin\theta|^{\mu} + n^{-\mu}),$$

with A_9 a constant depending only on r.

P r o o f. In the proof, C always denotes a constant depending at most on r. Since $\nu\leqslant 2r$, we have

$$|\Delta_t^{2r}(g, \theta)| \leqslant C|\Delta_t^{\nu}(g, \theta)| \leqslant CM \sup_{0\leqslant|u|\leqslant\nu t} |\sin(\theta+u)|^{\mu}|t|^{\nu}.$$

When $2\nu|t|\leqslant\pi$, we have that for any $|u|\leqslant t$,

$$|\sin(\theta+u)| \leqslant |\sin\theta| + |\sin u| \leqslant |\sin\theta| + |\sin\nu t| \leqslant C(|\sin\theta| + |t|).$$

But this inequality automatically holds for $2\nu|t|\geqslant\pi$. Hence,

$$|\Delta_t^{2r}(g, \theta)| \leqslant CM(|\sin\theta| + |t|)^{\mu}|t|^{\nu} \leqslant CM(|\sin\theta|^{\mu} + |t|^{\mu})|t|^{\nu}. \tag{3.5}$$

Going now to the definition of L_n, we find that from (3.3) and (3.5)

$$|L_n(g, \theta) - g(\theta)| \leqslant CM\int_{-\pi}^{\pi} |\sin\theta|^{\mu}|t|^{\nu}K_n(t)\,dt +$$

$$+ CM\int_{-\pi}^{\pi} |t|^{\mu+\nu}K_n(t)\,dt \leqslant CMn^{-\nu}(|\sin\theta|^{\mu} + n^{-\mu}),$$

where in the last inequality, we used the moment estimate (3.2). This is the desired result and the lemma is proved.

Our next lemma establishes Brudnyi's theorem for functions $f\in L_\infty^{(r)}$.

L e m m a 3. *Let $r\geqslant 1$. If $f\in L_\infty^{(r)}$, with $\|f^{(r)}\|_\infty = 1$, then*

$$|\bar{L}_n(f, x) - f(x)| \leqslant A_{10}(\Delta_n(x))^r, \quad -1\leqslant x\leqslant 1, \tag{3.6}$$

with A_{10} a constant depending only on r.

P r o o f. Since L_n preserves polynomials of degree $r-1$, and $\omega_r(f)$ does not change with the addition of a polynomial of degree $r-1$, we can suppose that $f(0)=f'(0)=\ldots=f^{(r-1)}(0)$ and hence $\|f\|\leqslant 1$. Let $h=f-Q_r(f)$. It follows from Rolle's theorem that $h^{(k)}$ has a zero in $(-1, 1)$, $k=0, \ldots, r-1$. This gives the estimate $\|h^{(k)}\|<2\|h^{(k+1)}\|$. Hence,

$$\|h^{(k)}\|<2\|h^{(r)}\|=2^r\|f^{(r)}\|=2^r, \quad k=0, 1, \ldots, r-1. \tag{3.7}$$

Now, let $g(0)=h(\cos 0)$. We want to show that g can be decomposed into a finite sum of functions which satisfy Lemma 2.

Differentiate g r times. Then, $g^{(r)}(\theta)=(-\sin\theta)^r f^{(r)}(\cos\theta)+g^{(r)}_{2,0}(\theta)$. Let $g_{1,0}$ be the 2π-periodic function with mean value zero whose r-th derivative is $(-\sin\theta)^r f^{(r)}(\cos\theta)$. Then $g=g_{1,0}+g_{2,0}$ which serves to define $g_{2,0}$ as well. The function $g^{(r)}_{2,0}$ has only terms involving $h^{(k)}$, with $k\leqslant r-1$ and hence has another derivative. Write $g^{(r+1)}_{2,0}=g^{(r+1)}_{1,1}+g^{(r+1)}_{2,1}$, with $g^{(r+1)}_{1,1}$ being the collection of terms involving $f^{(r)}$ and $g^{(r+1)}_{2,1}$ the remaining terms. Continuing, at the k-th stage, we have $g^{(r+k)}_{2,k-1}=g^{(r+k)}_{1,k}+g^{(r+k)}_{2,k}$, with $g^{(r+k)}_{1,k}$ being the collection of terms involving $f^{(r)}$ and $g^{(r+k)}_{2,k}$ the remaining terms. In this way, we end up with the decomposition $g=g_{1,0}+g_{1,1}+\ldots+g_{1,r-1}+g_{2,r-1}$.

One verifies simply that $g_{1,k}$, satisfies for $0\leqslant k\leqslant r-1$,

$$|g^{(r+k)}_{1,k}(\theta)|\leqslant M|\sin\theta|^{r-k}|f^{(r)}(\cos\theta)|\leqslant M|\sin\theta|^{r-k}, \quad \text{a. e.}, \tag{3.8}$$

with M depending only on r. For example $M=(4r)^{2r}$ surely works. Moreover, we have

$$|g^{(2r)}_{2,r-1}(\theta)|\leqslant M, \text{ a. e.} \tag{3.9}$$

Now, we use Lemma 2 on each term $g_{1,k}$ and the term $g_{2,r}$ to find

$$|L_n(g, \theta)-g(\theta)|\leqslant A_9 M\sum_{k=0}^{r}(|\sin\theta|^k n^{-2r+k}+n^{-2r})\leqslant A_{10}(\Delta_n(\cos\theta))^r, \tag{3.10}$$

with A_{10} depending only on r. Returning to f, with $x=\cos\theta$, we have

$$|L_n((f-Q_r(f))(\cos\theta), \arccos x)-(f-Q_r(f))(x)|\leqslant\frac{1}{2}A_{10}(\Delta_n(x))^r.$$

The trigonometric polynomial $T_r(\theta)=Q_r(f, \cos\theta)$ satisfies $\|T^{(2r)}\|\leqslant \leqslant C\|Q_r(f)\|\leqslant C'\|f\|=C'$, with C' depending only on r. Hence from Lemma 2 again

$$|L_n(T_r, \theta)-T_r(\theta)|\leqslant A_9 C' n^{-2r}\leqslant {}^1/_2 A_{10}(\Delta_n(x))^r.$$

Using these last two inequalities with the definition (3.4) gives (3.6).

Our next theorem shows that L_n gives Brudnyi's estimate.

T h e o r e m 2. *Let (K_n) be a sequence of trigonometric polynomials satisfying (3.1)—(3.3) and define L_n by (3.4). Then there is a constant C_r depending only on r so that for each $f\in C[-1, 1]$, we have*

$$|L_n(f, x)-f(x)|\leqslant C_r\omega_r(f, \Delta_n(x)), \quad x\in[-1, 1], \quad n=1, 2, \ldots \tag{3.11}$$

P r o o f. We know that the operators L_n are uniformly bounded. Let $\|L_n\|\leqslant M$, $n=1, 2, \ldots$, with M depending only on r. Take $-1\leqslant x\leqslant 1$ and fix x. If h is any function in $L^{(r)}_\infty$, then

$$\begin{aligned}|L_n(f, x)-f(x)|&\leqslant\|f-h\|_\infty(\|L_n\|+1)+|L_n(h, x)-h(x)|\leqslant\\ &\leqslant(M+1)\|f-h\|+A_{10}\|h^{(r)}\|_\infty(\Delta_n(x))^r\leqslant\\ &\leqslant A_{11}(\|f-h\|+\|h^{(r)}\|_\infty(\Delta_n(x))^r)\end{aligned} \tag{3.12}$$

with A_{11} depending only on r. Here, we have used Lemma 3 to estimate the second term on the right hand side of the inequality. Taking an infimum over all such h and recalling the definition of the K functional, we have

$$|L_n(f, x) - f(x)| \leqslant A_{11} K_r(f, (\Delta_n(x))^r) \leqslant A_{11} A_2 \omega_r(f, \Delta_n(x)), \tag{3.13}$$

where we have used (1.3). This is the desired inequality and the theorem is proved.

4. FINER ESTIMATES

In the case $r=2$, we can give better estimates than those given in Theorem 2. First we need an estimate for the approximation of f'.

L e m m a 4. *Under the assumptions of Theorem 2 and $r=2$, we have that whenever f' is absolutely continuous and $|f''| \leqslant M$, a. e., then*

$$|(L_n(f, x))' - f'(x)| \leqslant C_1 M \Delta_n(x), \quad x \in [-1, 1], \quad n = 1, 2, \ldots, \tag{4.1}$$

with C_1 an absolute constant.

P r o o f. The proof is similar to Lemma 3. We can assume $M=1$. Let $g(\theta) = f(\cos\theta)$, where just as in Lemma 3, we can suppose that $f(0) = f'(0) = 0$. The decomposition of Lemma 3 now takes the form $g = g_1 + g_2 + g_3$, with $g_1''(\theta) = (\sin\theta)^2 f''(\cos\theta)$, $g_2^{(3)}(\theta) = \sin\theta\cos\theta f''(\cos\theta)$ and $g_3^{(3)} = \sin\theta f'(\cos\theta)$. So from Lemma 2, we find

$$|L_n(g', \theta) - g'(\theta)| \leqslant C(n^{-1}\sin^2\theta + n^{-2}|\sin\theta| + n^{-3}) \leqslant$$
$$\leqslant C(n^{-1}|\sin\theta| + n^{-2})^2 n, \tag{4.2}$$

with C now an absolute constant. In the same way, we have

$$|L_n(g'', \theta) - g''(\theta)| \leqslant C(n^{-1}|\sin\theta| + n^{-2})^2 n^2. \tag{4.3}$$

If we let $T_n(\theta) = L_n(g, \theta)$, then because of the convolution structure of the operators L_n, we have $T_n' = L_n(g')$ and $T_n'' = L_n(g'')$. Thus (4.2) is

$$|T_n'(\theta) - g'(\theta)| \leqslant C(n^{-1}|\sin\theta| + n^{-2})^2 n. \tag{4.4}$$

This last estimate can be improved some near 0. The function $T_n - g$ is even and hence $T_n'(0) - g'(0) = 0$. Using this with (4.3), we have that when $|\sin\theta| \leqslant n^{-1}$, then $|T_n''(\theta) - g''(0)| \leqslant Cn^{-3}$ and so for $\theta \in [-\pi/2, \pi/2]$,

$$|T_n'(\theta) - g'(\theta)| \leqslant |T_n''(\xi) - g''(\xi)||\theta| \leqslant Cn^{-3}|\sin\theta|, \quad |\sin\theta| \leqslant n^{-1}, \tag{4.5}$$

where of course we used the mean value theorem with $\xi \in [-0, \theta]$. Similarly this inequality holds when $\theta \notin [-\pi/2, \pi/2]$ provided $|\sin\theta| \leqslant n^{-1}$, now because $T_n'(\pi) - g'(\pi) = 0$.

If we superimpose the inequalities (4.4) and (4.5), we have

$$|T_n'(\theta) - g'(\theta)| \leqslant C|\sin\theta|(n^{-1}|\sin\theta| + n^{-2}). \tag{4.6}$$

This last inequality when rewritten in terms of f' and $(L_n(f))'$ gives the estimate (4.1).

Now we need only make a linear adjustment to get operators that give the Teljakovskii type estimate for ω_2. When f is in $C[-1, 1]$ let $l_n(f, x) = -{}^1/_2(x-1)(f(-1) - L_n(f, -1)) + {}^1/_2(x+1)(f(1) - L_n(f, 1))$, so that $l_n(f)$ is the linear function which interpolate $f - L_n(f)$ at the endpoints. The operator l_n clearly has norm $\leqslant 2(\|L_n\| + 1)$ on $C[-1, 1]$. Now define the operator M_n by

$$M_n(f) = L_n(f) + l_n(f). \tag{4.7}$$

So $\|M_n\| \leqslant C$ with C independent of n. We can now show that the $M_n(f)$ give the Teljakovskii type estimate for ω_2.

Theorem 3. *Let* (L_n) *satisfy the hypothesis of Theorem* 2 *with* $r=2$ *and define* M_n *as in* (4.7). *Then, for each* $f \in C\,[-1,\,1]$, *we have*

$$|M_n(f,\,x)-f(x)| \leqslant C_2\omega_2(f,\,n^{-1}(1-x^2)^{1/2}), \qquad -1 \leqslant x \leqslant 1, \qquad n=1,\,2,\,\ldots, \tag{4.8}$$

with C_2 *an absolute constant.*

Proof. The proof is like that of Theorem 2. We check first functions f with $|f''| \leqslant 1$ a. e. on $[-1,\,1]$. For such f, $|l_n(f,\,x)| \leqslant Cn^{-4}$, $-1 \leqslant x \leqslant 1$, because $|L_n(f,\pm 1)-f(\pm 1)| \leqslant Cn^{-4}$ according to Theorem 2. Hence, from Theorem 2, we have

$$|M_n(f,\,x)-f(x)| \leqslant |L_n(f,\,x)-f(x)|+|l_n(f,\,x)| \leqslant$$
$$\leqslant C\,((\Delta_n(x))^2+n^{-4}) \leqslant C\,(\Delta_n(x))^2. \tag{4.9}$$

We can improve this estimate near the end points using the fact that $M_n(f,\,\pm 1)=f(\pm 1)$. For example, if $0 \leqslant x \leqslant 1$, then

$$|M_n(f,\,x)-f(x)| \leqslant |x-1|\,|(M_n(f))'(\xi)-f'(\xi)| \leqslant C\,|x-1|\,|(L_n(f))'(\xi)-$$
$$-f'(\xi)|+n^{-4}) \leqslant C\,|x-1|\,(\Delta_n(\xi)+n^{-4}) \leqslant C\,(1-x^2)\,\Delta_n(x), \tag{4.10}$$

where in the first inequality we used the mean value theorem with $x \leqslant \xi \leqslant 1$, in the second inequality we used the fact that $|(l_n(f))'| \leqslant Cn^{-4}$, in the third inequality we used Lemma 4, and in the last inequality we used the fact that $1-x \leqslant 1-x^2$, $0 \leqslant x \leqslant 1$ and $\Delta_n(\xi) \leqslant \Delta_n(x)$, because $x \leqslant \xi$. The same inequality holds when $-1 \leqslant x \leqslant 0$.

When we superimpose inequalities (4.9) and (4.10), we find

$$|M_n(f,\,x)-f(x)| \leqslant C\,(1-x^2)\,n^{-2}, \qquad -1 \leqslant x \leqslant 1, \tag{4.11}$$

with C an absolute constant. Hence, if f' is absolutely continuous and $|f''| \leqslant M$, a. e., then

$$|M_n(f,\,x)-f(x)| \leqslant CM\,(1-x^2)\,n^{-2}, \qquad -1 \leqslant x \leqslant 1. \tag{4.12}$$

Now let f be an arbitrary function from $C\,[-1,\,1]$ and h an arbitrary function with $h'' \in L_\infty\,[-1,\,1]$. Take $x \in [-1,\,1]$ and fix x. We have

$$|M_n(f,\,x)-f(x)| \leqslant \|f-h\|(\|M_n\|+1)+|M_n(h,\,x)-h(x)| \leqslant$$
$$\leqslant C\,(\|f-h\|+\|h''\|_\infty\,(1-x^2)\,n^{-2}).$$

Taking an infimum over all such h on the right hand side and using the definition of the K functional, we have

$$|M_n(f,\,x)-f(x)| \leqslant CK_2(f,\,(1-x^2)\,n^{-2}) \leqslant C_2\omega_2(f,\,n^{-1}(1-x^2)^{1/2}),$$

where in the last inequality we used (1.3) with $r=2$. Since x was an arbitrary point in $[-1,\,1]$ and C_2 does not depend on x or n, we have proved the theorem.

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О НЕКОТОРЫХ СВОЙСТВАХ ДИФФЕРЕНЦИРУЕМЫХ ФУНКЦИЙ НЕСКОЛЬКИХ ПЕРЕМЕННЫХ

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Пусть $\|f\|_{\mathbf{p}}$ — норма смешанного лебеговского класса $L_{\mathbf{p}}(R_n)$, где R_n — n-мерное евклидово пространство точек $\mathbf{x}=(x_1, \ldots, x_n)$; $\mathbf{p}=(p_1, \ldots, p_n)$, $1 \leqslant p_k \leqslant \infty$, $k=1, \ldots, n$. Далее, пусть $\varphi(\mathbf{x})>0$ — фиксированная непрерывная функция, для которой $\int\limits_0^\infty \frac{\ln \alpha(t)}{1+t^2}\,dt$ конечен, где $\alpha(t)=\sup\limits_{\mathbf{x},\mathbf{y}\in R_n;\ |\mathbf{y}|\leqslant t} \frac{\varphi(\mathbf{x}+\mathbf{y})}{\varphi(\mathbf{x})}$, а $\mathbf{y}=(y_1, \ldots, y_n)$, $|\mathbf{y}|=\sqrt{y_1^2+\ldots+y_n^2}$. Обозначим через $L_{\mathbf{p}\varphi}(R_n)$ класс функций $f(\mathbf{x})$, для которых конечна величина $\|f\|_{\mathbf{p}\varphi}=\left\|\frac{f}{\varphi}\right\|_{\mathbf{p}}$. Для $f\in L_{\mathbf{p}\varphi}(R_n)$ положим

$$\omega_{kx_i}(f;\ \delta e_i)_{\mathbf{p}\varphi}=\sup_{|h|\leqslant\delta}\|\Delta^k_{hx_i} f\|_{\mathbf{p}\varphi},$$

$$E_\nu(f)_{\mathbf{p}\varphi}=\inf_{g_\nu}\|f-g_\nu\|_{\mathbf{p}\varphi},$$

где $\Delta^k_{hx_i} f$ — конечная разность k-го порядка функции $f(\mathbf{x})$ относительно переменного x_i, а в определении $E_\nu(f)_{\mathbf{p}\varphi}$, $\nu=(\nu_1, \ldots, \nu_n)$, $\nu_j>0$, $j=1, \ldots, n$, нижняя грань распространяется на всевозможные $g_\nu(\mathbf{x})\in L_{\mathbf{p}\varphi}(R_n)$, являющиеся целыми функциями степени $<\nu_1, \ldots, \nu_n$ по совокупности переменных $x_1, \ldots, x_n$.

Пусть функции $\psi_j(\delta)$, $j=1, \ldots, n$, удовлетворяют одновременно условиям (S) и $(S\lambda_j)$ Н. К. Бари и С. Б. Стечкина и, кроме того, они и неотрицательные целые числа r_j, $j=1, \ldots, n$, таковы, что существуют положительные числа $\rho_1, \ldots, \rho_n$, а также положительные числа c_1 и c_2, такие, что при любом $a>1$ и любом неотрицательном k выполняются соотношения

$$c_1 T_k(a)\leqslant a^{-kr_j/\rho_j}\psi_j(a^{-k/\rho_j})\leqslant c_2 T_k(a),$$

где c_1, c_2 и $T_k(a)$ от j $(j=1, \ldots, n)$ не зависят. При выполнении указанных выше условий мы скажем, что $\mathbf{r}=(r_1, \ldots, r_n)$ и $\psi(t)=(\psi_1(t), \ldots, \psi_n(t))$ удовлетворяют $\Gamma(T_\nu, \rho, \lambda)$ условию, где $\rho=(\rho_1, \ldots, \rho_n)$, $\lambda=(\lambda_1, \ldots, \lambda_n)$.

Далее, пусть $0<\theta<\infty$, $\varkappa(t)$ — непрерывная на $[0, 1]$ функция, удовлетворяющая условиям: $\varkappa(0)=0$, $\varkappa(2t)\leqslant c_3\varkappa(t)$, $t^\beta\leqslant c_4\varkappa(t)$, где c_3 и c_4 — постоянные, $t\in[0, 1]$, а также функции $\psi_j^{-1}(t)\,t^{\lambda_j-\rho_j\beta}$ удовлетворяют условию (S), $j=1, \ldots, n$.

Обозначим через $B^{(\mathbf{r},\psi,\varkappa)}_{\mathbf{p}\varphi\theta\beta}(R_n)$ класс всех функций из $L_{\mathbf{p}\varphi}(R_n)$, для которых существуют соболевские производные $\frac{\partial^{r_i} f}{\partial x_i^{r_i}}\in L_{\mathbf{p}\varphi}(R_n)$, $i=1, \ldots, n$, и для