A CONSTRUCTIVE THEORY FOR APPROXIMATION BY SPLINES WITH AN ARBITRARY SEQUENCE OF KNOT SETS

by

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Introduction. We are interested in the global constructive theory for splines. By global we mean that we consider the global approximation of functions by splines and wish to compare this to the global smoothness of the function as described by properties of the modulus of continuity. Splines are more suited for local approximation and when we study global results, we are forced to accept essentially the worst of the local. Thus, for example, global direct theorems (i.e. estimates of the degree of spline approximation) are given in terms of the maximum distance between adjacent knots. In this paper, we will develope an inverse theory (i.e. a measure of the global smoothness of the function in terms of its degree of approximation) in terms of the minimum distance between adjacent knots. Our main contribution is that we give inverse theorems for completely arbitrary sequences of knot sets whereas prior results require some additional assumptions, usually some sort of mixing condition on the knot sequence.

Let $\Delta: 0 = x_0 < x_1 < \ldots < x_m = 1$ be a set of knots. If $r > 0$ and $-1 \le \rho \le r-2$, we let $\mathcal{S}_{r,\rho}(\Delta)$ denote the space of splines of order r (degree r-1) with knots Δ and smoothness ρ. That is, $S \in \mathcal{S}_{r,\rho}(\Delta)$ if and only if $S \in C^{\rho}[0,1]$, and on $(x_{\nu-1}, x_\nu)$ S is a polynomial of degree $\le r-1$, for $\nu = 1,2,\ldots,m$. When $\rho = -1$, no continuity is assumed but we do make the convention that S is continuous from the right at each knot and from the left at 0.

The degree of approximation by splines from $\mathcal{S}_{r,\rho}(\Delta)$ can be measured in terms of the upper mesh length $\overline{\Delta} = \max_{1\le i\le m} |x_{i-1} - x_i|$. Namely, see e.g. [4] if $f \in L_p[0,1]$, then there is a spline $S \in \mathcal{S}_{r,\rho}(\Delta)$ such that

$$(1.1) \qquad \|f - S\|_p \le C\omega_r(f,\overline{\Delta})_p$$

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with $\omega_r(f,\cdot)_p$ the r-th order modulus of smoothness as measured in L_p and C depending only on r. If $(\Delta_n)_1^\infty$ is a sequence of knot sets then (1.1) gives that for each $f \in L_p[0,1]$ there are splines $S_n \in \mathcal{S}_{r,\rho}(\Delta_n)$ so that

$$\|f - S_n\|_p \le C\omega_r(f,\bar{\Delta}_n)_p, \quad n = 1,2,\ldots . \tag{1.2}$$

Our interest in this paper is the inverse problem to (1.2). In Section 3, we give a general inverse theorem for approximation by splines from $\mathcal{S}_{r,\rho}(\Delta_n)$. The inverse results are given in terms of the lower mesh lengths $\underline{\Delta}_n = \min_{1\le i\le m_n} |x_i^{(n)} - x_{i-1}^{(n)}|$ with the $x_i^{(n)}$ the knots of Δ_n. The reason for the dependence of inverse theorems on the lower mesh lengths is dicussed somewhat in Section 4.

Our inverse theorem requires no added assumptions on the knot sequence in contrast to most of the known inverse theorems, e.g. [7], [9], [3], [5], [6] where some sort of mixing condition is assumed. Thus, we are dealing in essence with the worst situation. This turns out to be the case of nested knots. Some partial results for nested sequences of knots were given in [6]. The situation for nested knots (and therefore general knot sets as well) is complicated by the fact that there is no saturation phenomenon (see [6,10]).

The inverse theorems for general knot sequences depend on ρ and p as well as r. This is in contrast to the mixed knot sequences where the inverse theorems depend only on r. Let us mention only one interesting case. If the sequence of knot sets has bounded mesh ratios, i.e. $\underline{\Delta}_n^{-1}\bar{\Delta}_n \le M$, $n = 1,2,\ldots$ then we show that for $0 < \alpha < \rho+1+1/p$, there are splines $S_n \in \mathcal{S}_{r,\rho}(\Delta_n)$ with $\|f - S_n\|_p = O(\bar{\Delta}_n^\alpha)$ if and only if $\omega_r(f,t) = O(t^\alpha)$. When $\alpha = \rho+1+1/p$, one obtains only that if $\|f - S_n\|_p = O(\underline{\Delta}_n^\alpha)$ then $\omega_r(f,t)_p = O(t^\alpha |\log t|)$. This last result cannot be improved as we show with examples in Section 4. The limitation on α given by $\rho+1+1/p$ comes from the inherent smoothness that a spline $S \in \mathcal{S}_{r,\rho}(\Delta_n)$ has in $L_p[0,1]$.

2. Smoothness of splines. The proof of our inverse theorem rests

on estimating the smoothness of the approximating splines S_n. This will give in turn an estimate for the smoothness of f. The smoothness of a spline is controlled by its jumps and jumps in its derivatives. If $S \in \mathcal{S}_{r,\rho}(\Delta)$ then S has continuous derivatives up to order ρ on $[0,1]$. Of course, S has derivatives of all orders on each interval (x_{i-1},x_i), $i=1,2,\dots,m$. For a knot x_i, we define

$$[S^{(\nu)}]_i = S^{(\nu)}(x_i+) - S^{(\nu)}(x_i-). \tag{2.1}$$

Thus $[S^{(\nu)}]_i$ is the jump in $S^{(\nu)}$ at x_i. It becomes convenient in certain sums to let i range from 0 to m, so we define $[S^{(\nu)}]_i = 0$ when i=0 or m.

L_p estimates for the smoothness of $S^{(\nu)}$ are given in terms of the seminorms

$$J_p(S^{(\nu)}) = \left(\sum_{i=1}^{m-1} |[S^{(\nu)}]_i|^p\right)^{1/p} \qquad 1 \le p < \infty \tag{2.2}$$

$$J_\infty(S^{(\nu)}) = \max_{1\le i<m} |[S^{(\nu)}]_i| , \quad p = \infty.$$

Namely, we have the following lemma which estimates the smoothness of a spline in terms of its jumps.

Lemma 1. *If* S *is in* $\mathcal{S}_{r,\rho}(\Delta)$ and $1 \le p \le \infty$ then

$$\omega_r(S,h)_p \le C(1+h/\underline{\Delta})^{1/p'} h^{1/p} \sum_{\nu=\rho+1}^{r-1} h^\nu J_p(S^{(\nu)}) \tag{2.3}$$

with C *a constant depending only on* r *and with* p' *the conjugate index to* p, $1/p'+1/p = 1$.

Proof. We will establish this inequality by showing that for any $2 \le k \le r$ and $\rho < \nu < r$, we have for $h > 0$,

$$\omega_k(S^{(\nu)},h)_p \le C \{(1+h/\underline{\Delta})^{1/p'} h^{1/p} J_p(S^{(\nu)}) + \tag{2.4}$$

$$+ h\omega_{k-1}(S^{(\nu+1)},h)_p \}$$

If we apply (2.4) iteratively starting with $\nu=0$ and k=r then we arrive at (2.3).

Now, to show the inequality (2.4), we write $S^{(\nu)} = T_1 + T_2$ with T_1 the spline in $\mathcal{S}_{1,-1}(\Delta)$ which has the same jump at each x_i as $S^{(\nu)}$, i.e. $[T_1]_i = [S^{(\nu)}]_i$, $i=1,2,\ldots,m-1$. Then T_2 is absolutely continuous and $T_2'(x) = S^{(\nu+1)}(x)$, $x \notin \Delta$. Thus,

$$(2.5) \qquad \omega_k(S^{(\nu)},h)_p \le \omega_k(T_1,h)_p + \omega_k(T_2,h)_p$$

$$\le C\ (\omega_1(T_1,h)_p + h\omega_{k-1}(S^{(\nu+1)},h)_p\ )$$

with C a constant depending only on r. Here, we have used well known properties of moduli of smoothness.
We finish the proof for $1 \le p < \infty$. The case $p = \infty$ is handled similarily.

In order to estimate $\omega_1(T_1,h)_p$, we let $d\mu_p$ denote the purely atomic measure with mass $|[S^{(\nu)}]_i|^p$ at each point x_i. Then, from Holder's inequality we find

$$|T_1(x+h)-T_1(x)|^p \le \left(\sum_{x\le x_i\le x+h} |[S^{(\nu)}]_i|\right)^p$$

$$\le (1+h/\underline{\Delta})^{p/p'} \sum_{x\le x_i\le x+h} |[S^{(\nu)}]_i|^p$$

$$\le (1+h/\underline{\Delta})^{p/p'} \int_0^h d\mu_p(x+t)$$

where we used the fact that on $[x,x+h]$ there are at most $(1+h/\underline{\Delta})$ knots. Integrating this last inequality and interchanging integrals, we find

$$\|T_1(x+h)-T_1(x)\|_{p,[0,1-h]} \le (1+h/\underline{\Delta})^{1/p'} \left(\int_0^h \left[\int_0 |d\mu_p(x)|\right] dt\right)^{1/p}$$

$$\le (1+h/\underline{\Delta})^{1/p'}\, h^{1/p}\, J_p(S^{(\nu)}) \ .$$

Thus,

$$\omega_1(T_1,h)_p \le (1+h/\underline{\Delta})^{1/p'}\, h^{1/p}\, J_p(S^{(\nu)}) \ .$$

When this is placed in (2.5), we obtain the desired estimate (2.4) and the lemma is proved.

3. Inverse theorems. Let $(\Delta_n)_1^\infty$ be an arbitrary sequence of knot sets with $\lim \underline{\Delta}_n = 0$. By reindexing this sequence if necessary, we can assume that $\underline{\Delta}_{n+1} \le \underline{\Delta}_n$, $n=1,2,\ldots$. We also define $\Delta_o = \{0,1\}$. As we have remarked earlier, the key to proving inverse theorems is to estimate the jumps in the approximating splines and their derivatives. We consider first the case of smooth splines, i.e. $\mathcal{S}_{r,r-2}(\Delta_n)$. This is the simplest case for estimating the jumps and the arguments are the most transparent here.

Lemma 2. Let $1 \le p \le \infty$ and $r > 0$. If there are splines $S_n \in \mathcal{S}_{r,r-2}(\Delta_n)$ with

$$(3.1) \qquad \|f-S_n\|_p \le \varepsilon_n, \quad n=1,2,\ldots ,$$

then for each $n > 0$,

$$(3.2) \qquad J_p(S_n^{(r-1)}) \le C \sum_{k=1}^{n} \underline{\Delta}_k^{-\theta}(\varepsilon_k + \varepsilon_{k-1})$$

with C depending only on r, $\theta = r-1+1/p$ and $\varepsilon_o = |f|_p$

Proof. Let $1 \le k \le n$. We will show that

$$(3.3) \qquad J_p(S_k^{(r-1)}) \le C \underline{\Delta}_k^{-\theta}(\varepsilon_k + \varepsilon_{k-1}) + J_p(S_{k-1}^{(r-1)})$$

where for $k = 0$, $S_o = 0$. Then, (3.2) follows from repeated application of (3.3) and the fact that $J_p(S_o^{(r-1)}) = 0$.

Now to prove (3.3), suppose that $x_i^{(k)}$ is a knot from Δ_k, $1 \le i < m_k$. Determine j so that $x_j^{(k-1)} \le x_i^{(k)} < x_{j+1}^{(k-1)}$. When $|x_i^{(k)} - x_j^{(k-1)}| \ge \frac{1}{4} \underline{\Delta}_k$ we let $a_i = x_i^{(k)}$ otherwise we let $a_i = x_j^{(k-1)}$. In any case, a_i is the right end point of an open interval $I_{i,\ell}$ with $|I_{i,\ell}| \ge \frac{1}{4} \underline{\Delta}_k$ and the spline $T_k = S_k - S_{k-1}$ has no knots on $I_{i,\ell}$. Similarily, if $|x_i^{(k)} - x_{j+1}^{(k-1)}| \ge \frac{1}{4} \underline{\Delta}_k$, we let $b_i = x_i^{(k)}$, otherwise we let $b_i = x_{j+1}^{(k-1)}$. Then b_i is the left end point of an interval $I_{i,r}$ of length $\ge \frac{1}{4} \underline{\Delta}_k$ on which T_k has no knots. In our selection of a_i and b_i, either both of these are $x_i^{(k)}$ or one of these is $x_i^{(k)}$ and the other is a knot of S_{k-1}, namely $x_j^{(k-1)}$ or $x_{j+1}^{(k-1)}$. In the latter case, we let $x_{j_i}^{(k-1)}$

denote this knot.

To estimate the jump in $S_k^{(r-1)}$ at $x_i^{(k)}$, we write simply

(3.4) $$[S_k^{(r-1)}]_i = T_k^{(r-1)}(x_i^{(k)}+) - T_k^{(r-1)}(x_i^{(k)}-) + S_{k-1}^{(r-1)}(x_i^{(k)}+) - S_{k-1}^{(r-1)}(x_i^{(k)}-)$$

$$= T_k^{(r-1)}(b_i+) - T_k^{(r-1)}(a_i-) + [S_{k-1}^{(r-1)}]_{j_i} .$$

Here, we used the fact that $S_{k-1}^{(r-1)}$ and $T_k^{(r-1)}$ are constant on (a_i,b_i). Note that for a given value of i there may be no j_i and then it is understood that the jump in $S_{k-1}^{(r-1)}$ at $x_{j_i}^{(k-1)}$ does not appear. If such a situation would always appear, this would be a property of the knot sets which is essentially equivalent to a mixing condition, cf. the remarks of Section 1.

We complete the proof for the case $1 \le p < \infty$. The case $p = \infty$ is similar and somewhat simpler. Since $T_k^{(r-1)}$ is constant on $I_{i,\ell}$ and $|I_{i,\ell}| \ge \frac{1}{4} \underline{\Delta}_k$, we have

(3.5) $$|T_k^{(r-1)}(a_i-)|^p = |I_{i,\ell}|^{-1} \int_{I_{i,\ell}} |T_k^{(r-1)}(x)|^p \, dx$$

$$\le 4 \underline{\Delta}_k^{-1} \int_{I_{i,\ell}} |T_k^{(r-1)}(x)|^p \, dx$$

$$\le C \underline{\Delta}_k^{(1-r)p-1} \int_{I_{i,\ell}} |T_k(x)|^p \, dx$$

with C depending only on r. The last inequality is a Markov type inequality for polynomials, see [11,p.236] . The same estimate holds for $T_k^{(r-1)}(b_i+)$ except that now the integral is taken over $I_{i,\pi}$. Using these estimates back in (3.4) gives

$$J_p(S_k^{(r-1)}) \le \left(\sum_{i=1}^{m-1} |T_k^{(r-1)}(a_i-)|^p\right)^{1/p} + \left(\sum_{i=1}^{m-1} |T_k^{(r-1)}(b_i+)|^p\right)^{1/p} + J_p(S_{k-1}^{(r-1)})$$

$$\le C\, \underline{\Delta}_k^{-\theta} \| T_k \|_p + J_p(S_{k-1}^{(r-1)})$$

where we used the fact that the intervals $I_{i,\ell}$ and $I_{i,\kappa}$ are disjoint. Our desired inequality (3.3) now follows from the fact that

$$\| T_k \|_p \le \| f - S_k \|_p + \| f - S_{k-1} \|_p \le \varepsilon_k + \varepsilon_{k-1} .$$

This completes the proof of the lemma.

We can give a general version of Lemma 2 for arbitrary smoothness ρ. However, the details are more complicated.

Lemma 3. Let $1 \le p \le \infty$, $r > 0$ and $-1 \le \rho \le r-2$. If there are splines $S_n \in \mathcal{S}_{r,\rho}(\Delta_n)$ with

$$\| f - S_n \|_p \le \varepsilon_n, \quad n = 1, 2, \ldots ,$$

then for each $n > 0$ with $\underline{\Delta}_n \le \frac{1}{4}$, we have

$$(3.6) \qquad \sum_{\nu=\rho+1}^{r-1} \underline{\Delta}_n^{\nu+1/p} J_p(S_n^{(\nu)}) \le C\, \Delta_n^{\theta} \sum_{k=1}^{n} \underline{\Delta}_k^{-\theta} (\varepsilon_k + \varepsilon_{k-1})$$

where $\theta = \rho + 1 + 1/p$, $\varepsilon_o = \| f \|$ and C depends only on r.

Proof. Case I. We assume additionally that $\underline{\Delta}_{k+1} \le 1/2\, \underline{\Delta}_k$, for $1 \le k < n$. Let $S_o = 0$. We will show that for each $1 \le k \le n$, we have

$$(3.7) \qquad \sum_{\nu=1}^{r-1-\rho} \underline{\Delta}_k^{\nu-1} J_p(S_k^{(\rho+\nu)}) \le C\, \underline{\Delta}_k^{-\theta} (\varepsilon_k + \varepsilon_{k-1})$$

$$+ A_k \sum_{\nu=1}^{r-1-\rho} \underline{\Delta}_{k-1}^{\nu-1} J_p(S_{k-1}^{(\rho+\nu)})$$

where $A_k = 1$ for $2 \le k \le n$. Then, (3.6) follows from successive application of (3.7) and the fact that $J_p(S_o^{(\mu)}) = 0$, for all μ.

To prove (3.7), we argue in a similar manner to the proof of (3.3) in Lemma 2. For a knot $x_i^{(k)} \in \Delta_k$, we define a_i, b_i, $I_{i,\ell}$, $I_{i,\kappa}$ and j_i exactly as in the proof of Lemma 2. For each $0 \le i \le m_k$ we have looking at the jumps of T_k passing from a_i to b_i

$$(3.8)\quad [S_k^{(\rho+\nu)}]_i = [T_k^{(\rho+\nu)}(b_i+) - T_k^{(\rho+\nu)}(b_i-)] +$$

$$+ [T_k^{(\rho+\nu)}(a_i+) - T_k^{(\rho+\nu)}(a_i-)] + [S_{k-1}^{(\rho+\nu)}]_{j_i}$$

$$= T_k^{(\rho+\nu)}(b_i+) - T_k^{(\rho+\nu)}(a_i-) + [S_{k-1}^{(\rho+\nu)}]_{j_i}$$

$$- (T_k^{(\rho+\nu)}(b_i-) - T_k^{(\rho+\nu)}(a_i+))$$

Here the term $T_k^{(\rho+\nu)}(b_i-)-T_k^{(\rho+\nu)}(a_i+)$ will be 0 if $a_i = b_i$. In the other case, for $1 \leq \nu \leq r-2-\rho$, the last term in brackets is new compared to our proof of (3.3) and we now work to estimate it. The other terms will be handled similar to that in (3.3).

If Q is any polynomial of degree $\leq r$ and I is some interval then $\|Q\|_\infty(I) \leq C|I|^{-1/p}\|Q\|_\rho(I)$ with C a constant depending only on r. The notation indicates that the norms are taken over the interval I. Since the function T_k has no knots on the interval $I_{i,\ell}$, we have that for any $\nu \geq 1$

$$(3.9)\quad \underline{\Delta}_k^{\nu-1}\,|T_k^{(\rho+\nu)}(a_i-)| \leq C\,\underline{\Delta}_k^{-\rho-1}\,\|T_k\|_\infty(I_{i,\ell}) \leq C\,\underline{\Delta}_k^{-\theta}\|T_k\|_p(I_{i,\ell})$$

where we have also used Markov's inequality for polynomials. Similarily, we have

$$(3.10)\quad \underline{\Delta}_k^{\nu-1}\,|T_k^{(\rho+\nu)}(b_i+)| \leq \underline{\Delta}_k^{-\theta}\,\|T_k\|_p(I_{i,\kappa}).$$

Since $a_i \neq b_i$ one of the a_i or b_i is $x_i^{(k)}$ and the other is the knot $x_{j_i}^{(k-1)}$. Let us suppose that $b_i = x_i^{(k)}$. The other case gives the same estimate. We write $T_k^{(\rho+\nu)}$ in its Taylor expansion at a_i+ to find

$$(3.11)\quad |T_k^{(\rho+\nu)}(b_i-)-T_k^{(\rho+\nu)}(a_i+)| \leq \sum_{\mu=1}^{s_\nu} \frac{|T_k^{(\rho+\nu+\mu)}(a_i+)|}{\mu!}(b_i-a_i)^\mu$$

$$\leq \sum_{\mu=1}^{s_\nu} \frac{|T_k^{(\rho+\nu+\mu)}(a_i+)|}{\mu!}(\tfrac{1}{2}\,\underline{\Delta}_k)^\mu$$

with $s_\nu = r-1-\rho-\nu$. Here we used the fact that $|b_i - a_i| \le 1/2\ \underline{\Delta}_k$. From (3.9) we see that

$$(3.12)\quad \underline{\Delta}_k^{\nu+\mu-1}\ |T_k^{(\rho+\nu+\mu)}(a_i+)| \le \underline{\Delta}_k^{\nu+\mu-1}(|T_k^{(\rho+\nu+\mu)}(a_i-)| + \\ + |[S_{k-1}^{(\rho+\nu+\mu)}]_{j_i}|) \\ \le C\ \underline{\Delta}_k^{-\theta}|T_k|_p(I_{i,\ell}) + \underline{\Delta}_k^{\nu+\mu-1}\ |[S_{k-1}^{(\rho+\nu+\mu)}]_{j_i}|\ .$$

Now let us return to (3.8). If we apply an ℓ_p norm to all sequences as a function of i and multiply by $\underline{\Delta}_k^{\nu-1}$, we find

$$(3.13)\quad \underline{\Delta}_k^{\nu-1}\ J_p(S_k^{(\rho+\nu)}) \le C\ \underline{\Delta}_k^{-\theta}\ \|T_k\|_p + \underline{\Delta}_k^{\nu-1}\ J_p(S_{k-1}^{(\rho+\nu)}) \\ + C\ \underline{\Delta}_k^{-\theta}\ \|T_k\|_p \sum_{\mu=1}^{s_\nu} \frac{(1/2)^\mu}{\mu!} + \underline{\Delta}_k^{\nu-1} \sum_{\mu=1}^{s_\nu} \frac{(\underline{\Delta}_k/2)^\mu}{\mu!}\ J_p(S_{k-1}^{(\rho+\nu+\mu)})$$

where $T_k^{(\rho+\nu)}(b_i+)$ was estimated by (3.10); $T_k^{(\rho+\nu)}(a_i-)$ was estimated by (3.9); $S_{k-1}^{(\rho+\nu)}$ gives $J_p(S_{k-1}^{(\rho+\nu)})$; and $(T_k^{(\rho+\nu)}(b_i-) - T_k^{(\rho+\nu)}(a_i+))$ was estimated by using (3.11) and (3.12). Summing (3.13) over ν we have

$$\sum_{\nu=1}^{r-1-\rho} \underline{\Delta}_k^{\nu-1}\ J_p(S_k^{(\rho+\nu)}) \le C\ \underline{\Delta}_k^{-\theta}\|T_k\|_p + \sum_{\nu=1}^{r-1-\rho} \underline{\Delta}_k^{\nu-1}\ J_p(S_{k-1}^{(\rho+\nu)}) \\ + \sum_1^\infty \frac{(1/2)^\mu}{\mu!} \sum_{\nu=2}^{r-1-\rho} \underline{\Delta}_k^{\nu-1}\ J_p(S_{k-1}^{(\rho+\nu)})\ .$$

The desired result (3.7) for $k \ge 2$ follows from the facts that $\|T_k\|_p \le \varepsilon_k + \varepsilon_{k-1}$;

$$1 + \sum_1^\infty \frac{(1/2)^\mu}{\mu!} \le e^{1/2} \le 2$$

and

$$\underline{\Delta}_k^{\nu-1} \le (1/2\underline{\Delta}_{k-1})^{\nu-1} \le 1/2\ \underline{\Delta}_{k-1}^{\nu-1}\ ,\quad \nu > 1.$$

This last inequality is where we used the additional assumption of Case I.

Case II. In the general case, we choose for given n a subsequence $(\Delta_{n_k})_0^\lambda$ which satisfies Case I and some additional condition. We start with $n_\lambda = n$. Then

i) if $\underline{\Delta}_{n_\lambda - 1} \geq 2\,\underline{\Delta}_{n_\lambda}$ we choose $n_{\lambda-1} = n_\lambda - 1$,
or if this is not the case,

ii) let j be the largest index $\leq n_\lambda$ with $\underline{\Delta}_{n_\lambda - j} < 2\,\underline{\Delta}_{n_\lambda}$ and choose $n_{\lambda-1} = n_\lambda - j - 1$.

If we have $j = n_\lambda$ in ii) we terminate the sequence by setting $0 = n_{\lambda-1} = n_0$. Otherwise, and if the resulting $n_{\lambda-1}$ is positive, we replace n_λ by $n_{\lambda-1}$, select $n_{\lambda-2}$ by the above procedure and continue in this way. This selection guaratees not only that $(\Delta_{n_k})_0^\lambda$ satisfies the condition $\underline{\Delta}_{n_k} \leq (1/2)\,\underline{\Delta}_{n_{k-1}}$ but also in addition $\underline{\Delta}_{n_k} \geq (1/2)\,\underline{\Delta}_{n_{k-1}+1}$.

From Case I we have

$$(3.14) \qquad \sum_{\nu=\rho+1}^{r-1} \underline{\Delta}_n^{\nu+1/p}\, J_p(S_n^{(\nu)}) \leq C\,\underline{\Delta}_n^{\theta} \sum_{k=1}^{\lambda} \underline{\Delta}_{n_k}^{-\theta}(\varepsilon_{n_k} + \varepsilon_{n_{k-1}}).$$

Certainly, we have

$$(3.15) \qquad \sum_{k=1}^{\lambda} \underline{\Delta}_{n_k}^{-\theta}\,\varepsilon_{n_k} \leq \sum_{k=1}^{n} \underline{\Delta}_k^{-\theta}\varepsilon_k .$$

On the other hand, our selection guarantees that $\underline{\Delta}_{n_k}^{-\theta} \leq 2^\theta \underline{\Delta}_{n_{k-1}+1}^{-\theta}$, so that

$$\underline{\Delta}_{n_k}^{-\theta}\,\varepsilon_{n_{k-1}} \leq 2^\theta\,\underline{\Delta}_{n_{k-1}+1}^{-\theta}\,\varepsilon_{n_{k-1}} .$$

Summing, we find

$$(3.16) \qquad \sum_{k=1}^{\lambda} \underline{\Delta}_{n_k}^{-\theta}\,\varepsilon_{n_{k-1}} \leq 2^\theta \sum_{k=1}^{n} \underline{\Delta}_k^{-\theta}(\varepsilon_k + \varepsilon_{k-1}) .$$

The estimates (3.15) and (3.16) when used in (3.14) provide the desired estimate (3.6) and the lemma is proved.

It is now a simple matter to combine Lemmas 1 and 3 to obtain the following general inverse theorem.

Theorem 1. Let $(\Delta_n)_0^\infty$ be a sequence of knot sets with $\underline{\Delta}_{n+1} \le \underline{\Delta}_n$, $n = 0,1,\ldots$ and $\Delta_0 = \{0,1\}$. Suppose that $r > 0$, $-1 \le \rho \le r-2$ and $1 \le p \le \infty$. If for each $n \ge 0$ there is an $S_n \in \mathcal{S}_{r,\rho}(\Delta_n)$ such that $\|f-S_n\|_p \le \varepsilon_n$, then for each $n \ge 0$ with $\underline{\Delta}_n \le 1/4$, we have

$$(3.17)\qquad \omega_r(f,\underline{\Delta}_n)_p \le C\,\underline{\Delta}_n^\theta \sum_{k=1}^{n} \underline{\Delta}_k^{-\theta}(\varepsilon_k + \varepsilon_{k-1})$$

where $\theta = \rho+1+1/p$ and C is a constant depending only on r.

Proof. The estimates (2.3) and (3.6) show that

$$\omega_r(S_n,\underline{\Delta}_n)_p \le C\,\underline{\Delta}_n^\theta \sum_{k=1}^{n} \underline{\Delta}_k^{-\theta}(\varepsilon_k + \varepsilon_{k-1}).$$

Since $\|f-S_n\|_p \le \varepsilon_n$, we also have $\omega_r(f-S_n,\underline{\Delta}_n)_p \le 2^r\varepsilon_n$. These two inequalities and the fact that $\omega_r(f,\underline{\Delta}_n)_p \le \omega_r(f-S_n,\underline{\Delta}_n)_p + \omega_r(S_n,\underline{\Delta}_n)_p$ combine to give (3.17).

Because of the monotonicity of ω_r, the estimate in (3.17) gives an estimate of $\omega_r(f,t)_p$ for any value of t. In the case that the sequence (Δ_n) is too sparse we loose something. By too sparse we mean if $\underline{\Delta}_{n+1}^{-1}\,\underline{\Delta}_n$ is not uniformly bounded.

An important special case in Theorem 1 is when the sequence (Δ_n) also satisfies the bounded ratio condition, that is

$$(3.18)\qquad \overline{\Delta}_n\,\underline{\Delta}_n^{-1} \le C, \quad n = 0,1,\ldots \quad \text{with } C \text{ a constant.}$$

In this case, for example, we can characterize the $\mathrm{Lip}\,(\alpha,r,p) = \{f: \omega_r(f,t)_p = O(t^\alpha),\ t > 0\}$ in terms of the degree of approximation

$$E_{n,\rho}(f) = \inf_{S \in \mathcal{S}_{r,\rho}(\Delta_n)} \|f-S\|_p .$$

Of course we will still need the condition

$$(3.19)\qquad C\,\underline{\Delta}_n \le \underline{\Delta}_{n+1} \le \underline{\Delta}_n \quad \text{with } C > 0 \text{ a constant,}$$

which prevents the sequence (Δ_n) from being too sparse.

Corollary 1. *Let* (Δ_n) *be a sequence of knot sets with* $\lim \underline{\Delta}_n = 0$ *which satisfy* (3.18) *and* (3.19).

a) *If* $r > 0$, $-1 \le j \le r-2$, $1 \le p \le \infty$ *and* $0 < \alpha < \rho+1+1/p$, *then* $f \in \mathrm{Lip}\,(\alpha,r,p)$ *if and only if* $E_{n,\rho}(f) = O(\overline{\Delta}_n^{\alpha})$.

b) *For* $\alpha = \rho+1+1/p$, $f \in \mathrm{Lip}\,(\alpha,r,p)$ *implies that* $E_{n,\rho}(f) = O(\overline{\Delta}_n^{\alpha})$ *and* $E_{n,\rho}(f) = O(\overline{\Delta}_n^{\alpha})$ *implies that* $\omega_r(f,t)_p = O(t^{\alpha}|\log t|)$.

c) *For* $\alpha > \rho+1+1/p$, $E_{n,\rho}(f) = O(\overline{\Delta}_n^{\alpha})$ *implies that* $\omega_r(f,t)_p = O(t^{\theta})$.

Proof. If $0 < \alpha \le \rho+1+1/p$ then the direct estimate (1.2) gives that for each $f \in \mathrm{Lip}\,(\alpha,r,p)$, we have $E_{n,\rho}(f) = O(\overline{\Delta}_n^{\alpha})$.
For the converse direction, we use Theorem 1. Choose a subsequence (Δ_{n_k}) so that

$$C_1\,\underline{\Delta}_{n_k} \le \underline{\Delta}_{n_{k+1}} \le 1/2\,\underline{\Delta}_{n_k} \tag{3.20}$$

This is possible because $\lim \underline{\Delta}_n = 0$ and (Δ_n) satisfies (3.19). Let $\Delta_{n_0} = \{0,1\}$. If $0 < \alpha \le \rho+1+1/p$, then (3.17) gives that for any k

$$\omega_r(f,\underline{\Delta}_{n_k})_p \le C\,\underline{\Delta}_{n_k} \sum_{j=0}^{k} \underline{\Delta}_{n_k}^{\theta-\alpha}\,\underline{\Delta}_{n_j}^{\alpha-\theta} \le C\,\underline{\Delta}_{n_k}^{\alpha} \sum_{j=0}^{k} 2^{-j(\theta-\alpha)} \tag{3.21}$$

$$\le \begin{cases} C_\alpha\,\underline{\Delta}_{n_k}^{\alpha}, & \alpha < \rho+1+1/p = \theta \\ C_\alpha\,\underline{\Delta}_{n_k}^{\alpha}\,|\log \underline{\Delta}_{n_k}|, & \alpha = \theta \\ C_\theta\,\underline{\Delta}_{n_k}^{\theta}, & \alpha > \theta \end{cases}$$

where we used (3.18) and the fact that $k \le C|\log \underline{\Delta}_{n_k}|$ because of the lower inequality in (3.20). Using usual properties of the modulus of smoothness this last inequality extends to all values of $t > 0$ because of (3.20). This proves the corollary.

4. Final remarks. We wish to discuss the results of the preceeding section, in particular their sharpness.

Concerning assertion b) of Corollary 1 we construct examples which show that the log term is essential. We do this for $\nu = 2$,

$p = 0$ but this can also be shown in general.

Given $j > 0$ define the hat function $\varphi_j(x) = (2^{-j} - |x|)$ with support $(-2^{-j}, 2^{-j})$ and height 2^{-j}. Then set

$$(4.1) \qquad f_k(x) = \sum_{j=2k+2}^{4k} \varphi_j(x-2^{-2k-1}) \quad k=1,2,\ldots$$

This f_k is symmetric with respect to 2^{-2k-1}, has support on $[2^{-2k-1}-2^{-2k-2}, 2^{-2k-1}+2^{-2k-2}] \subset [2^{-2k-2}, 2^{-2k}]$.

Then for $p = \infty$ our example is to let $\Delta_n = \{k2^{-n}\}_{k=1}^{2^n}$, and

$$f(x) = \sum_{k=1}^{\infty} f_k(x)$$

It is readily seen that $f(x)$ is approximated with order $O(\underline{\Delta}_n)$ by splines from $\mathcal{S}_{2,1}(\Delta_n)$. Indeed, setting

$$s_n(x) = \sum_{k=1}^{\infty} \sum_{j=2k+2}^{\min(4k,n)} \varphi_j(x-2^{-2k-1})$$

we have obviously $s_n \in \mathcal{S}_{2,1}(\Delta_n)$ and for $x \in [2^{-2k-2}, 2^{-2k}]$

$$|f(x)-s_n(x)| \leq \sum_{\min(4k,n)+1}^{4k} \|\varphi_j(x)\|_\infty \leq \sum_{n+1}^{\infty} 2^{-j} = 2^{-n}.$$

On the other hand, with $t = 2^{-4k}$ and $x = 2^{-2k-1}$ it follows

$$\omega_2(f,t)_\infty \geq |f(x+t)+f(x-t)-2f(x)| = \sum_{2k+2}^{4k} |\varphi_j(t)+\varphi_j(-t)-2\varphi_j(0)|$$

$$= 2t(4k-(2k+2)) \geq kt \geq Ct|\log t|$$

For $1 \leq p < \infty$ we take $f(x) = \sum_{k=1}^{\infty} k^{-1/p} f_k(x)$ and as the approximating spline function

$$s_n'(x) = \sum_{k=1}^{\infty} k^{-1/p} \sum_{2k+2}^{\min(n,4k)} \varphi_j(x-2^{-2k-1}).$$

On each subinterval $I_k = [2^{-2k-2}, 2^{-2k}]$ $f(x)$ coincides with $k^{-1/p} f_k(x)$ so that

$$\left\{\int_{I_k}|f(x)-S_n(x)|^p dx\right\}^{1/p} = \begin{cases} \int_{I_k} 0\,dx = 0, & n \ge 4k \\ \le k^{-1/p}\sum_{n+1}^{4k}\left(\int_{I_k}|\varphi_j(x)|^p dx\right)^{1/p}, & 2k+2\le n<4k \\ \le k^{-1/p}\sum_{2k+2}^{4k}\left(\int_{I_k}|\varphi_j(x)|^p dx\right)^{1/p}, & n<2k+2 \end{cases}$$

$$\le \begin{cases} 0, & n\ge 4k \\ 2^{1/p}k^{-1/p}\sum_{n+1}^{4k}2^{-j(1+1/p)} \le k^{-1/p}2^{-n(1+1/p)}, & 2k+2\le n<4k \\ 2^{1/p}k^{-1/p}2^{-2k(1+1/p)}, & n>2k+2 \end{cases}$$

because each φ_j has support on an interval of length 2^{-j+1}. It follows that

$$\int_0^1 |f(x)-S_n(x)|^p dx \le 2\cdot\sum_{\frac{n}{4}<k\le\frac{n}{2}} k^{-1}\,2^{-n(p+1)} + \sum_{2k>n-2} k^{-1}2^{-2k(p+1)}$$

$$\le 2\cdot 2^{-n(p+1)} + 2^{-(n-2)(p+1)}$$

$$\le (2+4^{p+1})\,2^{-n(p+1)},$$

so that the approximation order of f is $O(2^{-n(1+1/p)}) = O(\Delta_n^{1+1/p})$.

On the other hand, we estimate for $t = 2^{-4n-3}$

$$\int_0^1|f(x+t)+f(x-t)-2f(x)|^p dx \ge \sum_{k=1}^{n} k^{-1}\int_{I_{k,t}}|f_k(x+t)+f_k(x-t)-2f_k(x)|^p dx$$

where $I_{k,t} = [2^{-2k-1}-t/2,\ 2^{-2k-1}+t/2]$ is contained in the support of $f_k(x\pm t)$ and $f_k(x)$, and even in the support of each $\varphi_j(x\pm t)$, $\varphi_j(x)$ in (4.1). It follows that

$$\int_{I_{k,t}}|f_k(x+t)+f_k(x-t)-2f(x)|^p dx = \int_{-t/2}^{t/2}\left|\sum_{j=2k+2}^{4k}[\varphi_j(x+t)+\varphi_j(x-t)-2\varphi_j(x)]\right|^p dx$$

$$= \int_{-t/2}^{t/2} \left| \sum_{j=2k+2}^{4k} \left[-2|x|+|x-t|+|x+t| \right] \right|^p dx =$$

$$= (4k-2k-2)^p \int_{-t/2}^{t/2} |2(t-|x|)|^p dx$$

Hence we have

$$\int_0^1 |f(x+t)-f(x-t)-2f(x)|^p dx \geq C \sum_{k=1}^{n} k^{-1} k^p t^{p+1} \geq C\, n^p t^{p+1}$$

and consequently $\omega_2(f,t)_p \geq C\, t^{1+1/p} |\log t|$, with C not depending on t.

This shows that the logarithm term in part b) of the corollary cannot be dropped when $\alpha = \theta = \rho+1+1/p$.

Concerning part c), an approximation order $E_{n,\rho}(f) = O(\underline{\Delta}_n{}^{\alpha})$, $\alpha > \theta$, cannot imply $\omega_r(t,f) = O(t^{\beta})$ for some $\beta > \theta$ since then e.g. for nested sequences of knot sets

$$\mathcal{S}_{r,\rho}(\Delta_k) \subset \mathrm{Lip}\,(\beta,r,p) \subset C^{\rho+1}\,[a,b]$$

which is a contradiction. However at least in the case of nested sequences there are nontrivial functions (i.e. not splines) for which the approximation order may be $O(\overline{\Delta}_n{}^{\beta})$ for any $\beta > \theta$. The characterization of these classes of functions is an open problem. The corollary shows that they cannot be described in terms of moduli of continuity. In any case they must be contained in $C^{\rho}[a,b]$ or $W_{\infty}^{\rho+1}(a,b)$ but classical smoothness will not always increase the order of approximation, in particular there is the barrier $W_p^r(a,b)$ for the order $O(\underline{\Delta}_n{}^r)$ given by a weak saturation theorem (see [1], [10]).

Finally we wish to discuss the assumption (3.18) on the sequences of knot sets. If we drop it we cannot further conclude assertion a) of the corollary (3.21) and only (3.17) holds. Thus, if $\overline{\lim_{n\to\infty}}\, \overline{\Delta}_n / \underline{\Delta}_n \longrightarrow \infty$, this inequality will give less smoothness (in term of the modulus of continuity). This is in agreement with results on best approximation by splines with optimal knots, where the sequences of partitions may depend on the function to be approximated and only the number of knots $n(1/n \leq \bar{\Delta}/(b-a))$ is prescribed. According to Rice [8], Burchard-Hale [2] even functions with singularities are approximated with order n by piecewise linear splines with optimal knots. In this case, the optimal partitions (meshes) corresponding to the best approximation cannot have uniformly bounded mesh ratio.

On the other hand, by our inverse results we may conclude that for all sequences of partitions with uniformly bounded mesh ratios the order of approximation is the same. Thus in this case the approximation by splines with optimal meshes has no essential advantage over approximation with a priori given sequence of partitions (satisfying (3.18)). According to Burchard-Hale this is e.g. the case when the function to be approximated has a n-th derivative which does not "oscillate too much".

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