

$L_p[-1,1]$ APPROXIMATION BY ALGEBRAIC POLYNOMIALS

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A characterization of the classical Lipschitz spaces in $L_p[-1,1]$ through approximation by algebraic polynomials is still not known for $1 \leq p < \infty$. We show by examples that the characterization suggested from the L_∞ case is not valid for $p < \infty$. We also state some results of a more positive nature which give estimates for weighted approximation by algebraic polynomials in $L_p[-1,1]$ $1 \leq p < \infty$.

1. Introduction

The problem is to characterize the Lipschitz spaces in $L_p[-1,1]$, $1 \leq p < \infty$, through algebraic polynomial approximation. This is one of the more intriguing questions still unanswered in one variable constructive function theory. The case $p = \infty$ is of course settled through the work of A. F. Timan [6], V. Dzadyk [3], G. Freud [4], and Yu. Brudnyi [2]. Namely, if we let $\text{Lip}(\alpha, p, r)$, $r = 1, 2, \dots$ $0 < \alpha \leq r$, $1 \leq p \leq \infty$, be the set of functions f for which $\omega_r(f, t)_p = O(t^\alpha)$, then $f \in \text{Lip}(\alpha, \infty, r)$, $0 < \alpha < r$ if and only if there are polynomials $p_n \in P_n$, $n = 1, 2, \dots$ so that

$$(1) \quad \|\Delta_n^{-\alpha}(f-p_n)\|_\infty = O(1) \quad (n \rightarrow \infty)$$

with $\Delta_n(x) \equiv \sqrt{(1-x^2)n^{-1} + n^{-2}}$. Here, the key ingredient is the weight Δ_n which indicates improved approximation near the endpoints of the interval.

It might be expected that the condition (1.1) for the L_p norm instead of L_∞ norm would characterize $\text{Lip}(\alpha, p, r)$ for $1 \leq p < \infty$ as well. This

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fails to be the case, although just barely, as we will show in this paper. Indeed, if $0 < \alpha < 1$, $1 \leq p < \infty$, then for each n there is a function $f_n \in \text{Lip}(\alpha, p, 1)$ so that

$$(1.2) \quad \inf_{p_n \in P_n} \|\Delta_n^{-\alpha}(f_n - p_n)\|_p \geq \text{const} \|f_n\|_{\alpha, p} \log n$$

with $\|\cdot\|_{\alpha, p}$ the $\text{Lip}(\alpha, p, 1)$ norm

$$(1.3) \quad \|g\|_{\alpha, p} \equiv \sup_{t>0} t^{-\alpha} \|g(\cdot+t) - g(\cdot)\|_p + \|g\|_p$$

and the constant independent of n .

These functions can be used to show the existence of a function $f \in \text{Lip}(\alpha, p, 1)$ with

$$\inf_{p_n \in P_n} \|\Delta_n^{-\alpha}(f - p_n)\|_p \neq o(1)$$

Actually (1.2) is a tight estimate in that it can be shown that if $f \in \text{Lip}(\alpha, p, 1)$ then there are polynomials $p_n \in P_n$ with

$$(1.4) \quad \|\Delta_n^{-\alpha}(f - p_n)\|_p \leq \text{const} \|f\|_{\alpha, p} \log n, \quad n = 1, 2, \dots$$

We give estimates of this type in Section 3, without proof. Details will appear elsewhere. Another interesting estimate of this type is that when $f \in B_p^{\alpha, p}$, $0 < \alpha < 1$, $1 \leq p \leq \infty$, then there are $p_n \in P_n$ with

$$(1.5) \quad \|\Delta_n^{-\alpha}(f - p_n)\|_p \leq \text{const} \|f\|_{B_p^{\alpha, p}}$$

where the $B_p^{\alpha, q}$ are the usual Besov spaces. This last result indicates better the scaling with p since the space $B^{\alpha, \infty}$ is $\text{Lip}(\alpha, \infty, 1)$. The estimates (1.4) and (1.5) are derived using interpolation theory. In fact, they are both contained in inequality (3.3) which gives an estimate for weighted L_p approximation by algebraic polynomials through K functionals. This latter inequality may actually characterize the spaces $\text{Lip}(\alpha, p, 1)$. Such a characterization would require an inverse theorem to (3.3).

2. The Counterexample

We want to construct functions in $\text{Lip}(\alpha, p)$ which cannot be approximated well by algebraic polynomials in the $L_p(\Delta_n^{-\alpha})$ norm. We will carry this out only for the case $p = 1$ since the case $p > 1$ is considerably more technical.

Let n be a positive integer which is a multiple of 4, $n = 4m$. Define $x_s = \cos s\pi(3n)^{-1}$ for any real number $0 \leq s \leq 3n/2$, set $J_j = [x_{2j}, x_{2j-1}]$, $I_j = [x_{2j-1}, x_{2j-2}]$ $j = 1, \dots, 3m$. The intervals I_j and J_j with $0 \leq j \leq 3m$ are all contained in $[0, 1]$ and they get smaller as they get closer to 1.

In fact, there are constants $A_1, A_2, B_1, B_2 > 0$ such that

$$(2.1) \quad A_1 n^{-2j} \leq |I_j| \leq |J_j| \leq A_2 n^{-2j} \quad j = 1, 2, \dots, 3m$$

$$(2.2) \quad B_1 \Delta_n(x) \leq |I_j| \leq |J_j| \leq B_2 \Delta_n(x), \quad x \in I_j \cup J_j, \quad j = 1, \dots, 3m.$$

We will now define for each n a function f_n which can not be approximated well by polynomials in the $L_1(\Delta_n^{-\alpha})$ norm. Later we can take an appropriate combination of these f_n to get one function which is not approximated well for all n . Through the remainder of this section $0 < \alpha < 1$ is fixed and we do not indicate dependence of constants on α .

Define

$$(2.3) \quad f_n(x) = \begin{cases} j^{\alpha-2} n^{-2\alpha+2} = \gamma_j, & x \in I_j, \quad j = 1, \dots, 3m \\ 0 & \text{otherwise.} \end{cases}$$

In particular $f_n(x) = 0$ for $-\pi \leq x \leq 0$. For any function $g \in \text{Lip}(\alpha, 1, 1)$ let

$$\|g\|_{\alpha, 1} \equiv \|g\|_1 + \sup_{t>0} t^{-\alpha} \|g(\cdot+t) - g(\cdot)\|_1[-1, 1-t].$$

LEMMA 1. There is a constant $c > 0$ so that

$$(2.4) \quad \|f_n\|_{\alpha, 1} \leq c$$

$$(2.5) \quad \|f_n(\cdot+t) - f_n(\cdot)\|_1 \leq c n^{-\alpha}, \quad 0 < t < 1/n$$

$$(2.6) \quad \|f_n(\cdot+t) - f_n(\cdot)\|_1 \leq c t n^{-2\alpha+2}, \quad t \leq n^{-2}.$$

PROOF. The function $\phi_n(x) = |f_n(x+t) - f_n(x)|$ can assume only the values 0, γ_j or $\gamma_j - \gamma_k$ for some j and k . Each γ_j will appear at most four times and each time for an interval of length at most $\min(|I_j|, t)$ because when $x, x+t$ are both in I_j , $\phi_n(x) = 0$. Hence,

$$(2.7) \quad \int_{-1}^{1-t} |\phi_n(x)| dx \leq 4 \sum_{j=1}^{3m} \gamma_j \min(|I_j|, t) \\ \leq \text{const} \sum_{j=1}^{3m} \gamma_j \min(t, jn^{-2})$$

because of (2.1). So if $in^{-1} < t \leq (i+1)n^{-1}$ with $0 \leq i \leq 3m-1$ we get

$$(2.8) \quad \int_{-1}^{1-t} |\phi_n(x)| dx \leq \text{const} \left\{ \left(\frac{i}{2}\right)^\alpha + t \left(\frac{i+1}{2}\right)^{\alpha-1} \right\}$$

this together with the fact that

$$\int_{-1}^{1-t} |\phi_n(x)| dx \leq 2 \|f_n\|_1 \leq \text{const} n^{-\alpha}$$

gives (2.4) and (2.5). The inequality (2.6) follows from (2.8) when $i = 0$.

We turn now to showing that a polynomial p_n of degree n can not approximate f_n well in the weighted $L_1(\Delta_n^{-\alpha})$ space. The reason for this is that the polynomial would have to be close to the value γ_j on the interval I_j (at least for many j 's) and close to 0 on the intervals J_j , a very ambitious polynomial!

Let $J = \bigcup_{k=1}^{6m} J_k$ and $I = \bigcup_{k=1}^{6m} I_k$. We want to show that the integral of a polynomial over I can be estimated by its integral over J .

LEMMA 2. There is a constant $c > 0$ so that for each polynomial p of degree $\leq n$

$$\int_I \Delta_n^{-\alpha}(x) |p(x)| dx \leq c \int_J \Delta_n^{-\alpha}(x) |p(x)| dx.$$

PROOF. This is proved by establishing similar inequalities for trigonometric polynomials. Let $\theta_k = 2\pi k(3n)^{-1}$, $k = 0, 1, \dots, 3n-1$ and $-\infty < s < \infty$. If T is any trigonometric polynomial of degree $\leq n$, we have (see [7, p.33])

$$(2.9) \quad T(\theta) = \frac{2}{3n} \sum_{k=0}^{3n-1} T(\theta_k + s) V_n(\theta - \theta_k - s)$$

where

$$V_n(t) = \frac{\sin(3nt/2)\sin(nt/2)}{2n \sin^2(t/2)}$$

When $0 \leq s \leq \pi/n$, then $|V_n(\theta - \theta_k - s)| \leq \text{const. } n^{-1} [|\theta - \theta_k| + \pi n^{-1}]^{-2}$. So, averaging with respect to s over $[0, \pi n^{-1}]$ in (2.9) gives

$$(2.10) \quad |T(\theta)| \leq \text{const. } n^{-2} \sum_{k=0}^{3n-1} (n \int_{J_k} T(u) du) [|\theta - \theta_k| + \pi n^{-1}]^{-2}$$

where $J_k = [\theta_k, \theta_k + \pi n^{-1}]$.

Now for our algebraic polynomial p , let $T(\theta) = p(\cos \theta)$. Then (2.10) and the fact that $|\sin u| \geq \frac{k}{n}$ on J_k gives

$$|P(x)| \leq \text{const. } n^{-2} \sum_{k=0}^{3n-1} \left(\frac{n^2}{k}\right) \int_{J_k} |p(t)| dt [|\theta - \theta_k| + \pi n^{-1}]^{-2}.$$

Now multiply by $\Delta_n^{-\alpha}$ and integrate this last inequality over I_j , use (2.1), (2.2) and the fact that $[|\theta - \theta_k| + \pi n^{-1}] \geq \text{const } (|j-k|+1)n^{-1}$ for $\cos \theta = x \in I_j$ to find

$$\int_{I_j} |P(x)| \Delta_n^{-\alpha}(x) dx \leq \text{const. } \sum_{k=0}^{3n-1} \left(\frac{n^2}{k}\right) \int_{J_k} |p(t)| dt \left(\frac{1}{n}\right)^{1-\alpha} (|k-j|+1)^{-2}.$$

Summing over j gives

$$\begin{aligned} \int_I |P(x)| \Delta_n^{-\alpha}(x) dx &\leq \text{const. } \sum_{k=0}^{3n-1} \left(\int_{J_k} |p(t)| dt\right) \left[\frac{n^2}{k}\right] \sum \left(\frac{1}{n}\right)^{1-\alpha} (|k-j|+1)^{-2} \\ &\leq \text{const. } \sum_{k=0}^{3n-1} \left(\frac{k}{n^2}\right)^{-\alpha} \int_{J_k} |p(t)| dt \\ &\leq \text{const. } \sum_{k=0}^{3n-1} \int_{J_k} |p(t)| \Delta_n^{-\alpha}(t) dt \\ &= \text{const. } \int_J |p(t)| \Delta_n^{-\alpha}(t) dt \end{aligned}$$

as desired.

It is a simple matter to use Lemma 2 and show that f_n can not be approximated well by polynomials of degree n in the weighted $L_1(\Delta_n^{-\alpha})$ norm.

THEOREM 1. For each n , the function f_n defined by (2.3) has

$$(2.11) \quad \|f_n\|_{\alpha,1} \leq c$$

$$(2.12) \quad \inf_{p \in P_n} \|\Delta_n^{-\alpha}(f_n - p)\|_1 \geq c \log n$$

PROOF. The estimate (2.11) was already given in Lemma 1. If p is any polynomial of degree $\leq n$, and $\delta_n = \|\Delta_n^{-\alpha}(f_n - p)\|_1$, then

$$\int_J |p(x)| \Delta_n^{-\alpha}(x) dx \leq \int_J |f_n(x) - p(x)| \Delta_n^{-\alpha}(x) dx \leq \delta_n$$

where J is defined as in Lemma 2. Note that $f_n(x) = 0$, $x \in J$. Using Lemma 2, we have

$$\int_I |p(x)| \Delta_n^{-\alpha}(x) dx \leq \text{const.} \int_J |p(x)| \Delta_n^{-\alpha}(x) dx \leq \text{const.} \delta_n.$$

Hence

$$(2.13) \quad \|(\Delta_n^{-\alpha}(f_n - p))\|_1 \geq \|f_n \Delta_n^{-\alpha}\|_1 - \|p \Delta_n^{-\alpha}\|_1 \geq \|f_n \Delta_n^{-\alpha}\|_1 - \text{const.} \delta_n$$

On each interval I_j , we have from (2.1) and (2.2) that $\Delta_n^{-\alpha}(x) \geq \text{const.} n^{2\alpha} j^{-\alpha}$. Hence

$$\begin{aligned} \|f_n \Delta_n^{-\alpha}\|_1 &\geq \text{const.} \sum_{j=1}^{6m} j^{\alpha-2} (n^2)^{-\alpha+1} n^{2\alpha} j^{-\alpha} \cdot |I_j| \\ &\geq \text{const.} \sum_{j=1}^{6m} j^{-2} n^2 j^{-2} \geq \text{const.} \sum_{j=1}^{6m} j^{-1} \\ &\geq \text{const.} \log 6m \geq \text{const.} \log n \end{aligned}$$

this combined with (2.13) and the definition of δ_n shows that

$$\|\Delta_n^{-\alpha}(f_n - p)\|_1 \geq \text{const.} \log n$$

as desired.

As we have mentioned, we would like to construct one function f which cannot be approximated well by polynomials of degree n for all n . This is easy to accomplish. We need only define $f = \sum_1^{\infty} f_{\phi_k}$ where $\phi_k = 2^{2^k}$. Since $\|f_{\phi_k}\| \leq \text{const. } \phi_k^{-\alpha}$ for each k , the series converges in L_1 to a function in $L^1[-1,1]$. Also, given any $0 < t < 1$, we can write

$$\begin{aligned} f(x) &= \sum_{\phi_k \geq t^{-1}} f_{\phi_k}(x) + \sum_{\phi_k \leq t^{-1/2}} f_{\phi_k}(x) + \sum_{t^{-1/2} < \phi_k < t^{-1}} f_{\phi_k}(x) \\ &= s_1(x) + s_2(x) + s_3(x) \end{aligned}$$

where s_3 has at most one term in it. Using (2.5) on s_1 , (2.6) and (2.4) on the one term in s_3 we have

$$\begin{aligned} (2.14) \quad \|f(\cdot+t) - f(\cdot)\|_1 &\leq \text{const.} \left\{ \sum_{\phi_k > t^{-1}} \phi_k^{-\alpha} + t \sum_{\phi_k < t^{-1/2}} \phi_k^{-2\alpha+2+t\alpha} \right\} \\ &\leq \text{const. } t^{\alpha} \end{aligned}$$

so that f is in $\text{Lip}(\alpha, 1, 1)$.

From Lemma 1, it follows that each function f_{ϕ_k} is in $\text{Lip}(1, 1, 1)$ and in fact $\|f_{\phi_k}\|_{1,1} \leq \phi_k^{2-2\alpha}$. Hence, as we prove in Section 3 (Lemma 3), there is a polynomial p_k of degree n so that

$$\|\Delta_n^{-1}(f_{\phi_k} - p_k)\|_1 \leq \text{const. } \phi_k^{2-2\alpha}.$$

Hence,

$$\|\Delta_n^{-\alpha}(f_{\phi_k} - p_k)\|_1 \leq \text{const. } (\phi_k^{2n-1})^{1-\alpha}.$$

In particular for each $N = \phi_n$, there is a polynomial p of degree $\leq N$ with

$$(2.15) \quad \|\Delta_N^{-\alpha}(\sum_1^{n-1} f_{\phi_k} - p)\|_1 \leq \text{const. } \sum_1^{n-1} \phi_k^{2\alpha-1} \leq \text{const.}$$

On the other hand

$$\|\Delta_N^{-\alpha}(\sum_{n+1}^{\infty} f_{\phi_k})\|_1 \leq \text{const. } N^{2\alpha} \sum_{n+1}^{\infty} \phi_k^{-\alpha} \leq \text{const. } N^{2\alpha} \phi_{n+1}^{-\alpha} \leq \text{const.}$$

Here, we used the fact that $\Delta_N \geq N^{-2}$ and $\|f_j\| \leq \text{const. } j^{-\alpha}$ for each j .

For the function f_N , we know that for any polynomial p of degree $\leq N$, we have

$$\|\Delta_N^{-\alpha}(f_N - p)\|_1 \geq \text{const. } \log N$$

because of Theorem 1. Thus we have proved the following theorem.

THEOREM 2. There is a function f in $\text{Lip}(\alpha, 1, 1)$ for which

$$\inf_{p \in P_n} \|\Delta_n^{-\alpha}(f-p)\|_1 \geq c \log n$$

for infinitely many n , in particular any n of the form 2^{2^k} .

3. Direct Theorems

Let us give some results of a more positive nature which estimate the degree of approximation by algebraic polynomials. Proof of these results are for the most part omitted and will be given elsewhere. Recall that $\text{Lip}(1, p, 1)$ is the set of functions f with $f' \in L_p$ if $1 < p \leq \infty$ and $f \in \text{BV}$, $p = 1$. Let $\|\cdot\|_{1,p}$ denote the corresponding norm on $\text{Lip}(1, p, 1)$.

LEMMA 3. If $f \in \text{Lip}(1, p, 1)$ and $n \geq 1$, then there is a bounded linear operator L_n from L_p into $L_p(\Delta_n^{-1})$ such that for each n , $L_n(f) = p \in P_n$ and

$$(3.1) \quad \|\Delta_n^{-1}(f-p)\|_p \leq c \|f\|_{1,p}$$

where c is an absolute constant.

The proof of this lemma is simple when $p = 1$. Indeed if $g(\theta) = f(\cos\theta)$, then g is also of bounded variation on $[-\pi, \pi]$ and $\int_{-\pi}^{\pi} |dg(\theta)| = 2 \int_{-1}^1 |df(x)|$. Hence, if \bar{L}_n is the classical Jackson operator [5, p. 55] then $T = \bar{L}_n(g)$ is a trigonometric polynomial of degree $\leq n$ with

$$\|g-T\|_1[-\pi, \pi] \leq \text{const. } \|g\|_{1,1} n^{-1}.$$

Defining $L_n(f) = p(x) = T(\cos \theta)$, when $x = \cos \theta$ and changing to $[-1, 1]$ gives

$$\begin{aligned} \left\| (1-x^2)^{-1} (f(x)-p(x)) \right\|_1 &= \frac{1}{2} \|g-T\|_1 [-\pi, \pi] \\ &\leq \text{const.} \|g\|_{1,1} n^{-1} \leq \text{const.} \|f\|_{1,1} n^{-1} \end{aligned}$$

which is stronger than (3.1) for $p = 1$.

In order to extend (3.1) to general α , we use interpolation theory. If we interpolate between L_p and $\text{Lip}(1, p, 1)$ we have with the K method (see [1]);

$$(L_p, \text{Lip}(1, p, 1))_{\theta, \infty} = \text{Lip}(\theta, p, 1) \quad 0 < \theta < 1.$$

We need also to check interpolation between L_p and $L_p(\Delta_n^{-1})$. It is an easy estimate to show that there are constants $c_1, c_2 > 0$ such that

$$\begin{aligned} (3.2) \quad c_1 K(f, t, L_p, L_p(\Delta_n^{-1})) &\leq \left\{ \int_{-1}^1 \min(1, t\Delta_n^{-1}(x)) |f(x)|^p dx \right\}^{1/p} \\ & c_2 K(f, t, L_p, L_p(\Delta_n^{-1})). \end{aligned}$$

Hence Lemma 3 gives the following theorem

THEOREM 3. If $f \in \text{Lip}(\alpha, p, 1)$, $n \geq 1$ and L_n is the operator from Lemma 3, then $p = L_n(f)$ satisfies

$$(3.3) \quad \left\| \min(1, t\Delta_n^{-1}(\cdot)) (f(\cdot) - p(\cdot)) \right\|_p [-1, 1] \leq \text{const.} t^\alpha \|f\|_{\alpha, p}$$

for all $t > 0$.

It is very important in this theorem that (3.3) holds for all $t > 0$. By varying t one obtains sharper inequalities. In fact, the inequalities (3.3) falls just short of giving

$$\left\| \Delta_n^{-\alpha} (f-p) \right\|_p [-1, 1] \leq \text{const.} \|f\|_{\alpha, p} [-1, 1].$$

Let us mention two corollaries of Theorem 3.

COROLLARY 1. If $0 < \alpha < 1$, $f \in \text{Lip}(\alpha, p, 1)$ and $n \geq 1$, then there is a polynomial $p \in P_n$ such that

$$(3.4) \quad \left\| \Delta_n^{-\alpha} (f-p) \right\|_p [-1, 1] \leq c \log n$$

with c depending only on p .

This result can not be improved in the sense that Theorem 2 shows the existence of a function for which the opposite inequality in (3.4) holds for infinitely many n .

COROLLARY 2. If $0 < \alpha < 1$, $1 \leq p \leq \infty$, $n \geq 1$ and f is in the Besov space $B_p^{\alpha,p}$, then there is a polynomial $p \in P_n$ such that

$$(3.5) \quad \|\Delta_n^{-\alpha}(f-p)\|_p \leq \text{const.} \|f\|_{B_p^{\alpha,p}}.$$

This last result gives a natural seating of spaces, namely $B_p^{\alpha,p}$, $1 \leq p \leq \infty$, for which the case $p = \infty$ gives the classical result (1.1).

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