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$L_p[-1,1]$ APPROXIMATION BY ALGEBRAIC POLYNOMIALS

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A characterization of the classical Lipschitz spaces in $L_p[-1,1]$ through approximation by algebraic polynomials is still not known for $1 \le p < \infty$. We show by examples that the characterization suggested from the L_∞ case is not valid for $p < \infty$. We also state some results of a more positive nature which give estimates for weighted approximation by algebraic polynomials in $L_p[-1,1]$ $1 \le p < \infty$.

1. Introduction

The problem is to characterize the Lipschitz spaces in $L_p[-1,1]$, $1 \le p < \infty$, through algebraic polynomial approximation. This is one of the more intriguing questions still unanswered in one variable constructive function theory. The case $p = \infty$ is of course settled through the work of A. F. Timan [6], V. Dzadyk [3], G. Freud [4], and Yu. Brudnyi [2]. Namely, if we let Lip (α,p,r), $r = 1,2,\ldots$ $0 < \alpha \le r$, $1 \le p \le \infty$, be the set of functions f for which $\omega_r(f,t)_p = O(t^\alpha)$, then $f \in$ Lip (α,∞,r), $0 < \alpha < r$ if and only if there are polynomials $p_n \in P_n$, $n = 1,2,\ldots$ so that

$$(1) \qquad ||\Delta_n^{-\alpha}(f-p_n)||_\infty = O(1) \qquad (n \to \infty)$$

with $\Delta_n(x) \equiv \sqrt{(1-x^2)}n^{-1} + n^{-2}$. Here, the key ingredient is the weight Δ_n which indicates improved approximation near the endpoints of the interval.

It might be expected that the condition (1.1) for the L_p norm instead of L_∞ norm would characterize Lip (α,p,r) for $1 \le p < \infty$ as well. This

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fails to be the case, although just barely, as we will show in this paper. Indeed, if $0 < \alpha < 1$, $1 \le p < \infty$, then for each n there is a function $f_n \in \mathrm{Lip}\,(\alpha,p,1)$ so that

$$(1.2)\qquad \inf_{p_n \in P_n} ||\Delta_n^{-\alpha}(f_n - p_n)||_p \ge \text{const}\ ||f_n||_{\alpha,p} \log n$$

with $||\ ||_{\alpha,p}$ the Lip $(\alpha,p,1)$ norm

$$(1.3)\qquad ||g||_{\alpha,p} \equiv \sup_{t>0} t^{-\alpha}||g(\cdot+t)-g(\cdot)||_p + ||g||_p$$

and the constant independent of n.

These functions can be used to show the existence of a function $f \in \mathrm{Lip}\,(\alpha,p,1)$ with

$$\inf_{p_n \in P_n} ||\Delta_n^{-\alpha}(f-p_n)||_p \neq O(1)$$

Actually (1.2) is a tight estimate in that it can be shown that if $f \in \mathrm{Lip}\,(\alpha,p,1)$ then there are polynomials $p_n \in P_n$ with

$$(1.4)\qquad ||\Delta_n^{-\alpha}(f-p_n)||_p \le \text{const}\ ||f||_{\alpha,p} \log n, \quad n = 1,2,\ldots$$

We give estimates of this type in Section 3, without proof. Details will appear elsewhere. Another interesting estimate of this type is that when $f \in B_p^{\alpha,p}$, $0 < \alpha < 1$, $1 \le p \le \infty$, then there are $p_n \in P_n$ with

$$(1.5)\qquad ||\Delta_n^{-\alpha}(f-p_n)||_p \le \text{const}\ ||f||_{B_p^{\alpha,p}}$$

where the $B_p^{\alpha,q}$ are the usual Besov spaces. This last result indicates better the scaling with p since the space $B^{\alpha,\infty}$ is Lip $(\alpha,\infty,1)$. The estimates (1.4) and (1.5) are derived using interpolation theory. In fact, they are both contained in inequality (3.3) which gives an estimate for weighted L_p approximation by algebraic polynomials through K functionals. This latter inequality may actually characterize the spaces Lip $(\alpha,p,1)$. Such a characterization would require an inverse theorem to (3.3).

2. The Counterexample

We want to construct functions in Lip (α,p) which cannot be approximated well by algebraic polynomials in the $L_p(\Delta_n^{-\alpha})$ norm. We will carry this out only for the case $p = 1$ since the case $p > 1$ is considerably more technical.

Let n be a positive integer which is a multiple of 4, $n = 4m$. Define $x_s = \cos s\pi(3n)^{-1}$ for any real number $0 \le s \le 3n/2$, set $J_j = [x_{2j}, x_{2j-1}]$, $I_j = [x_{2j-1}, x_{2j-2}]$ $j = 1,\ldots,3n$. The intervals I_j and J_j with $0 \le j \le 3m$ are all contained in $[0,1]$ and they get smaller as they get closer to 1.

In fact, there are constants A_1, A_2, B_1, $B_2 > 0$ such that

$$A_1 n^{-2} j \le |I_j| \le |J_j| \le A_2 n^{-2} j \qquad j = 1,2,\ldots,3m \tag{2.1}$$

$$B_1 \Delta_n(x) \le |I_j| \le |J_j| \le B_2 \Delta_n(x), \; x \in I_j \cup J_j, \; j = 1,\ldots,3m. \tag{2.2}$$

We will now define for each n a function f_n which can not be approximated well by polynomials in the $L_1(\Delta_n^{-\alpha})$ norm. Later we can take an appropriate combination of these f_n to get one function which is not approximated well for all n. Through the remainder of this section $0 < \alpha < 1$ is fixed and we do not indicate dependence of constants on α.

Define

$$f_n(x) = \begin{cases} j^{\alpha-2}\, n^{-2\alpha+2} \doteq \gamma_j, & x \in I_j,\ j = 1,\ldots,3m \\ 0 & \text{otherwise.} \end{cases} \tag{2.3}$$

In particular $f_n(x) = 0$ for $-\pi \le x \le 0$. For any function $g \in$ Lip $(\alpha,1,1)$ let

$$||g||_{\alpha,1} \equiv ||g||_1 + \sup_{t>0} t^{-\alpha} ||g(\cdot+t) - g(\cdot)||_1 [-1,1-t].$$

LEMMA 1. <u>There is a constant</u> $c > 0$ <u>so that</u>

$$||f_n||_{\alpha,1} \le c \tag{2.4}$$

$$||f_n(\cdot+t) - f_n(\cdot)||_1 \le c\, n^{-\alpha}, \; 0 < t \tag{2.5}$$

$$(2.6) \qquad ||f_n(\cdot+t) - f_n(\cdot)||_1 \le c\ t\ n^{-2\alpha+2}, \qquad t \le n^{-2}.$$

PROOF. The function $\phi_n(x) = |f_n(x+t) - f_n(x)|$ can assume only the values 0, γ_j or $\gamma_j - \gamma_k$ for some j and k. Each γ_j will appear at most four times and each time for an interval of length at most min $(|I_j|, t)$ because when x, x+t are both in I_j, $\phi_n(x) = 0$. Hence,

$$(2.7) \qquad \int_{-1}^{1-t} |\phi_n(x)| dx \le 4 \sum_{j=1}^{3m} \gamma_j \min(|I_j|, t)$$
$$\le \text{const} \sum_{j=1}^{3m} \gamma_j \min(t, jn^{-2})$$

because of (2.1). So if $in^{-1} < t \le (i+1)n^{-1}$ with $0 \le i \le 3m-1$ we get

$$(2.8) \qquad \int_{-1}^{1-t} |\phi_n(x)| dx \le \text{const} \left\{ \left(\frac{i}{n^2}\right)^{\alpha} + t \left(\frac{i+1}{n^2}\right)^{\alpha-1} \right\}$$

this together with the fact that

$$\int_{-1}^{1-t} |\phi_n(x)| dx \le 2\ ||f_n||_1 \le \text{const } n^{-\alpha}$$

gives (2.4) and (2.5). The inequality (2.6) follows from (2.8) when $i = 0$.

We turn now to showing that a polynomial p_n of degree n can not approximate f_n well in the weighted $L_1(\Delta_n^{-\alpha})$ space. The reason for this is that the polynomial would have to be close to the value γ_j on the interval I_j (at least for many j's) and close to 0 on the intervals J_j, a very ambitious polynomial!

Let $J = \bigcup_{k=1}^{6m} J_k$ and $I = \bigcup_{k=1}^{6m} I_k$. We want to show that the integral of a polynomial over I can be estimated by its integral over J.

LEMMA 2. <u>There is a constant</u> $c > 0$ <u>so that for each polynomial</u> p <u>of degree</u> $\le n$

$$\int_I \Delta_n^{-\alpha}(x) |p(x)| dx \le c \int_J \Delta_n^{-\alpha}(x) |p(x)| dx.$$

PROOF. This is proved by establishing similar inequalities for trigonometric polynomials. Let $\theta_k = 2\pi k(3n)^{-1}$, $k = 0,1,\dots,3n-1$ and $-\infty < s < \infty$. If T is any trigonometric polynomial of degree $\le n$, we have (see [7, p.33])

$$(2.9) \qquad T(\theta) = \frac{2}{3n} \sum_{k=0}^{3n-1} T(\theta_k+s)\, V_n(\theta-\theta_k-s)$$

where

$$V_n(t) = \frac{\sin(3nt/2)\sin(nt/2)}{2n\,\sin^2(t/2)} .$$

When $0 \le s \le \pi/n$, then $|V_n(\theta-\theta_k-s)| \le$ const. $n^{-1}\,[|\theta-\theta_k|+n^{-1}]^{-2}$. So, averaging with respect to s over $[0,\pi n^{-1}]$ in (2.9) gives

$$(2.10) \qquad |T(\theta)| \le \text{const. } n^{-2} \sum_{k=0}^{3n-1} (n\int_{J_k'} T(u)du)\, [|\theta-\theta_k|+n^{-1}]^{-2}$$

where $J_k' = [\theta_k, \theta_k+\pi n^{-1}]$.

Now for our algebraic polynomial p, let $T(\theta) = p(\cos\theta)$. Then (2.10) and the fact that $|\sin u| \ge \frac{k}{n}$ on J_k' gives

$$|P(x)| \le \text{const. } n^{-2} \sum_{k=0}^{3n-1} (\frac{n^2}{k} \int_{J_k} |p(t)|dt)\, [|\theta-\theta_k|+n^{-1}]^{-2}.$$

Now multiply by $\Delta_n^{-\alpha}$ and integrate this last inequality over I_j, use (2.1), (2.2) and the fact that $[|\theta-\theta_k|+n^{-1}] \ge$ const $(|j-k|+1)n^{-1}$ for $\cos\theta = x \in I_j$ to find

$$\int_{I_j} |P(x)|\Delta_n^{-\alpha}(x)dx \le \text{const. } \sum_{k=0}^{3n-1} (\frac{n^2}{k} \int_{J_k} |p(t)|dt)(\frac{j}{n^2})^{1-\alpha}(|k-j|+1)^{-2}.$$

Summing over j gives

$$\begin{aligned}\int_I |P(x)|\Delta_n^{-\alpha}(x)dx &\le \text{const. } \sum_{k=0}^{3n-1} (\int_{J_k} |p(t)|dt)[\frac{n^2}{k} \sum (\frac{j}{n^2})^{1-\alpha}(|k-j|+1)^{-2}] \\ &\le \text{const. } \sum_{k=0}^{3n-1} (\frac{k}{n^2})^{-\alpha} \int_{J_k} |p(t)|dt \\ &\le \text{const. } \sum_{k=0}^{3n-1} \int_{J_k} |p(t)|\Delta_n^{-\alpha}(t)dt \\ &= \text{const. } \int_J |p(t)|\Delta_n^{-\alpha}(t)dt\end{aligned}$$

as desired.

It is a simple matter to use Lemma 2 and show that f_n can not be approximated well by polynomials of degree n in the weighted $L_1(\Delta_n^{-\alpha})$ norm.

THEOREM 1. For each n, the function f_n defined by (2.3) has

$$\|f_n\|_{\alpha,1} \le c \tag{2.11}$$

$$\inf_{p\in P_n} \|\Delta_n^{-\alpha}(f_n-p)\|_1 \ge c \log n \tag{2.12}$$

PROOF. The estimate (2.11) was already given in Lemma 1. If p is any polynomial of degree ≤ n and $\delta_n = \|\Delta_n^{-\alpha}(f_n-p)\|_1$, then

$$\int_J |p(x)|\Delta_n^{-\alpha}(x)dx \le \int_J |f_n(x)-p(x)|\Delta_n^{-\alpha}(x)dx \le \delta_n$$

where J is defined as in Lemma 2. Note that $f_n(x) = 0$, $x \in J$. Using Lemma 2, we have

$$\int_I |p(x)|\Delta_n^{-\alpha}(x)dx \le \text{const.} \int_J |p(x)|\Delta_n^{-\alpha}(x)dx \le \text{const.}\ \delta_n.$$

Hence

$$\|(f_n-p)\Delta_n^{-\alpha}\|_1 \ge \|f_n\Delta_n^{-\alpha}\|_1 - \|p\Delta_n^{-\alpha}\|_1 \ge \|f_n\Delta_n^{-\alpha}\|_1 - \text{const.}\ \delta_n \tag{2.13}$$

On each interval I_j, we have from (2.1) and (2.2) that $\Delta_n^{-\alpha}(x) \ge \text{const}\ n^{2\alpha} j^{-\alpha}$. Hence

$$\begin{aligned}\|f_n\Delta_n^{-\alpha}\|_1 &\ge \text{const.} \sum_{j=1}^{6m} j^{\alpha-2}(n^2)^{-\alpha+1} n^{2\alpha} j^{-\alpha} \cdot |I_j| \\ &\ge \text{const.} \sum_{j=1}^{6m} j^{-2} n^2\ j\ n^{-2} \ge \text{const.} \sum_{j=1}^{6m} j^{-1} \\ &\ge \text{const.}\ \log 6m \ge \text{const.}\ \log n\end{aligned}$$

this combined with (2.13) and the definition of δ_n shows that

$$\|\Delta_n^{-\alpha}(f_n-p)\|_1 \ge \text{const.}\ \log n$$

as desired.

As we have mentioned, we would like to construct one function f which cannot be approximated well by polynomials of degree n for all n. This is easy to accomplish. We need only define $f = \sum_1^\infty f_{\phi_k}$ where $\phi_k = 2^{2^k}$. Since $||f_{\phi_k}|| \le \text{const. } \phi_k^{-\alpha}$ for each k, the series converges in L_1 to a function in $L^1[-1,1]$. Also, given any $0 < t < 1$, we can write

$$f(x) = \sum_{\phi_k \ge t^{-1}} f_{\phi_k}(x) + \sum_{\phi_k \le t^{-1/2}} f_{\phi_k}(x) + \sum_{t^{-1/2} < \phi_k < t^{-1}} f_{\phi_k}(x)$$

$$= s_1(x) + s_2(x) + s_3(x)$$

where s_3 has at most one term in it. Using (2.5) on s_1, (2.6) and (2.4) on the one term in s_3 we have

$$(2.14) \quad ||f(\cdot+t)-f(\cdot)||_1 \le \text{const. } \{ \sum_{\phi_k > t^{-1}} \phi_k^{-\alpha} + t \sum_{\phi_k < t^{-1/2}} \phi_k^{-2\alpha+2} + t^\alpha \}$$

$$\le \text{const. } t^\alpha$$

so that f is in Lip $(\alpha,1,1)$.

From Lemma 1, it follows that each function f_{ϕ_k} is in Lip $(1,1,1)$ and in fact $||f_{\phi_k}||_{1,1} \le \phi_k^{2-2\alpha}$. Hence, as we prove in Section 3 (Lemma 3), there is a polynomial p_k of degree n so that

$$||\Delta_n^{-1}(f_{\phi_k} - p_k)||_1 \le \text{const. } \phi_k^{2-2\alpha}.$$

Hence,

$$||\Delta_n^{-\alpha}(f_{\phi_k} - p_k)||_1 \le \text{const. } (\phi_k^2 n^{-1})^{1-\alpha}.$$

In particular for each $N = \phi_n$, there is a polynomial p of degree $\le N$ with

$$(2.15) \quad ||\Delta_N^{-\alpha}(\sum_1^{n-1} f_{\phi_k} - p)||_1 \le \text{const. } \sum_1^{n-1} \phi_k^2 \phi_n^{-1} \le \text{const.}$$

On the other hand

$$||\Delta_N^{-\alpha}(\sum_{n+1}^\infty f_{\phi_k})||_1 \le \text{const.} N^{2\alpha} \sum_{n+1}^\infty \phi_k^{-\alpha} \le \text{const. } N^{2\alpha} \phi_{n+1}^{-\alpha} \le \text{const.}$$

Here, we used the fact that $\Delta_N \geq N^{-2}$ and $||f_j|| \leq$ const. $j^{-\alpha}$ for each j.

For the function f_N, we know that for any polynomial p of degree $\leq N$, we have

$$||\Delta_N^{-\alpha}(f_N - p)||_1 \geq \text{const. } \log N$$

because of Theorem 1. Thus we have proved the following theorem.

THEOREM 2. There is a function f in Lip $(\alpha,1,1)$ for which

$$\inf_{p \in P_n} ||\Delta_n^{-\alpha}(f-p)||_1 \geq c \log n$$

for infinitely many n, in particular any n of the form 2^{2^k}.

3. Direct Theorems

Let us give some results of a more positive nature which estimate the degree of approximation by algebraic polynomials. Proof of these results are for the most part ommitted and will be given elsewhere. Recall that Lip $(1,p,1)$ is the set of functions f with $f' \in L_p$ if $1 < p \leq \infty$ and $f \in BV$, $p = 1$. Let $||\cdot||_{1,p}$ denote the corresponding norm on Lip $(1,p,1)$.

LEMMA 3. If $f \in$ Lip $(1,p,1)$ and $n \geq 1$, then there is a bounded linear operator L_n from L_p into $L_p(\Delta_n^{-1})$ such that for each n $L_n(f) = p \in P_n$ and

$$(3.1) \qquad ||\Delta_n^{-1}(f-p)||_p \leq c\ ||f||_{1,p}$$

where c is an absolute constant.

The proof of this lemma is simple when $p = 1$. Indeed if $g(\theta) = f(\cos\theta)$, then g is also of bounded variation on $[-\pi,\pi]$ and $\int_{-\pi}^{\pi} |dg(\theta)| = 2\int_{-1}^{1}|df(x)|$. Hence, if $\overline{L}_n$ is the classical Jackson operator [5, p. 55] then $T = \overline{L}_n\ (g)$ is a trigonometric polynomial of degree $\leq n$ with

$$||g-T||_1\ [-\pi,\pi] \leq \text{const. } ||g||_{1,1}\ n^{-1}.$$

Defining $L_n(f) = p(x) = T(\cos\theta)$, when $x = \cos\theta$ and changing to $[-1,1]$ gives

$$||(1-x^2)^{-1} \ (f(x)-p(x))||_1 = \tfrac{1}{2}||g-T||_1 \ [-\pi,\pi]$$

$$\le \text{const.} \ ||g||_{1,1} \ n^{-1} \le \text{const.} \ ||f||_{1,1} n^{-1}$$

which is stronger than (3.1) for $p = 1$.

In order to extend (3.1) to general α, we use interpolation theory. If we interpolate between L_p and Lip $(1,p,1)$ we have with the K method (see [1]);

$$(L_p, \text{Lip } (1,p,1))_{\theta,\infty} = \text{Lip } (\theta,p,1) \quad 0 < \theta < 1.$$

We need also to check interpolation between L_p and $L_p(\Delta_n^{-1})$. It is an easy estimate to show that there are constants c_1, $c_2 > 0$ such that

(3.2) $$c_1 \ K(f,t,L_p,L_p(\Delta_n^{-1})) \le \{\int_{-1}^{1} \min(1,t\Delta_n^{-1}(x))|f(x)|^p dx\}^{1/p}$$

$$c_2 \ K \ (f,t,L_p,L_p(\Delta_n^{-1})).$$

Hence Lemma 3 gives the following theorem

THEOREM 3. If $f \in$ Lip $(\alpha,p,1)$, $n \ge 1$ and L_n is the operator from Lemma 3, then $p = L_n(f)$ satisfies

(3.3) $$||\min \ (1,t\Delta_n^{-1}(\cdot))(f(\cdot)-p(\cdot))||_p \ [-1,1] \le \text{const.} \ t^{\alpha}||f||_{\alpha,p}$$

for all $t > 0$.

It is very important in this theorem that (3.3) holds for all $t > 0$. By varying t one obtains sharper inequalities. In fact, the inequalities (3.3) falls just short of giving

$$||\Delta_n^{-\alpha}(f-p)||_p \ [-1,1] \le \text{const.} \ ||f||_{\alpha,p}[-1,1].$$

Let us mention two corollaries of Theorem 3.

COROLLARY 1. If $0 < \alpha < 1$, $f \in$ Lip $(\alpha,p,1)$ and $n \ge 1$, then there is a polynomial $p \in P_n$ such that

(3.4) $$||\Delta_n^{-\alpha}(f - p)||_p \ [-1,1] \le c \ \log n$$

with c depending only on p.

This result can not be improved in the sense that Theorem 2 shows the existence of a function for which the opposite inequality in (3.4) holds for infinitely many n.

COROLLARY 2. *If* $0 < \alpha < 1$, $1 \le p \le \infty$, $n \ge 1$ *and* f *is in the Besov space* $B_p^{\alpha,p}$, *then there is a polynomial* $p \in P_n$ *such that*

$$||\Delta_n^{-\alpha}(f-p)||_p \le \text{const. } ||f||_{B_p^{\alpha,p}}. \tag{3.5}$$

This last result gives a natural seating of spaces, namely $B_p^{\alpha,p}$, $1 \le p \le \infty$, for which the case $p = \infty$ gives the classical result (1.1).

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