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MULTI-DIMENSIONAL SPLINE APPROXIMATION*

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Abstract. We give direct and inverse estimates for multivariate spline approximation. The direct estimates rest on new results for local polynomial approximation which generalize the work of Brudnyi and Bramble–Hilbert. The inverse estimates are multivariate extensions of one variable ideas.

1. Introduction. We are interested in developing a constructive function theory for spline approximation in several variables. Here, we mean spline approximation in the broad sense, thereby including piecewise polynomials. Such a constructive function theory is well established in one variable; see e.g., [7], [8], [9] and [13]. The obvious difficulties in moving to the multi-dimensional case are in handling various domains and partitions of these domains, as well as the inherent difficulty in generating smooth spline approximants. Also, we now have a variety of different degrees for polynomials, e.g., total degree or coordinate degree.

It is well understood that the heart of proving direct theorems for spline approximation is to give estimates for local polynomial approximation. Here, the classical results in one variable go back to H. Whitney [17]. In the multivariate case, the fundamental ideas behind such estimates were already given by L. Sobolev [15, p. 60] and used in his establishment of embedding theorems for Sobolev spaces. The paper of J. Bramble and S. Hilbert [3] reformulates Sobolev's ideas and shows their applicability to the finite element method error analysis. In a less known paper, Ju. Brudnyi [5] gives estimates for the approximation by polynomials, both in the case of total degree as well as coordinate degree and even in the preferable form of Whitney by using appropriate moduli of smoothness.

Consider the case of polynomials of total order r (degree $<r$), $\mathbb{P}_r = \operatorname{span}\{\mathbf{x}^\alpha: |\alpha|<r\}$. It is shown in [3] that for each $f \in L_p(\Omega)$, $1 \leqq p \leqq \infty$ there is a $P \in \mathbb{P}_r$ such that

$$\|f-P\|_p(\Omega) \leqq C \sum_{|\alpha|=r} \|D^\alpha f\|_p(\Omega), \tag{1.1}$$

whenever the right-hand side is finite (all derivatives in the distributional sense). Here, Ω is any suitably smooth domain (uniform cone property). In particular if Ω is a cube with side length δ, then (1.1) gives

$$\|f-P\|_p(\Omega) \leqq C\Big(\sum_{|\alpha|=r} \|D^\alpha f\|_p(\Omega)\Big)\delta^r \tag{1.2}$$

with C depending only on r and Ω. Using the equivalence [10] of the K-functional for interpolation between L_p and W_p^r with the rth order modulus of smoothness ω_r, it follows that for each $f \in L_p(\Omega)$ there is a $P \in \mathbb{P}_r$ with

$$\|f-P\|_p(\Omega) \leqq C\omega_r(f, \delta, \Omega)_p. \tag{1.2'}$$

This latter form is preferable to (1.2) since it applies to any f regardless of its smoothness, and, of course, (1.2′) includes (1.2). Estimates of the form (1.2′) go back to H. Whitney [17], who proved such results in the case of one variable.

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A similar result can be given for coordinate degree $\mathbf{r}=(r_1, r_2, \cdots, r_n)$. Let $\mathbb{P}_\mathbf{r}=\operatorname{span}\{\mathbf{x}^\alpha; \alpha_i<r_i, 1\leqq i\leqq n\}$. If Q is a rectangle with side length vector $\boldsymbol{\delta}=(\delta_1, \cdots, \delta_n)$, where $Q=\prod_1^n [a_i, b_i]$ and $\delta_i=b_i-a_i>0$, then for each $f\in L_p(Q)$ there is a polynomial $P\in\mathbb{P}_\mathbf{r}$ such that

$$\|f-P\|_p(Q)\leqq C\sum_1^n \|D^{r_i e_i}f\|_p(Q)\delta_i^{r_i}, \tag{1.3}$$

whenever the right-hand side is finite. Here $\mathbf{e}_i=(\delta_{ij})$, Kronecker notation. The interesting point in (1.3) is that only pure derivatives appear on the right-hand side. Hence, when $\mathbf{r}=r\mathbf{e}$ with $\mathbf{e}=(1, 1, \cdots, 1)$, there is a "gain" over the total degree r case due to the inclusion of extra monomials $\mathbf{x}^\alpha$, $\alpha<\mathbf{r}$, $|\alpha|>r$.

Again, (1.3) can be put into a more preferable form by using a coordinatewise modulus of smoothness $\omega_\mathbf{r}$ (see § 3). Namely, Brudnyi [5] has shown that if $f\in L_p(Q)$, there is a polynomial $P\in\mathbb{P}_\mathbf{r}$ such that

$$\|f-P\|_p(Q)\leqq C\omega_\mathbf{r}(f, \boldsymbol{\delta}, Q)_p. \tag{1.3'}$$

Both (1.3) and (1.3') are stated for rectangles in R^n. They can be established for more general domains; however, as we will see in §§ 2 and 3, estimates like (1.3') require stronger smoothness of the domain than estimates like (1.3).

In § 2, we shall give a general setting for estimates of the form (1.2) and (1.3) by establishing such estimates for general polynomial spaces $\mathbb{P}_\Lambda$, where $\Lambda\subseteq Z_+^n$ is a finite set with the property $\boldsymbol{\beta}\leqq\boldsymbol{\alpha}$, and $\boldsymbol{\alpha}\in\Lambda$ implies that $\boldsymbol{\beta}\in\Lambda$. Sets of this type occur at times in the finite element method, one class being the so-called serendipity elements [16]. The key to generalizing (1.2) and (1.3) to the spaces $\mathbb{P}_\Lambda$ is to decide what are the appropriate derivatives to appear on the right-hand side of such estimates. This turns out to be described in terms of $\partial\Lambda=\{\boldsymbol{\alpha}: \boldsymbol{\alpha}\notin\Lambda$ and $\boldsymbol{\beta}<\boldsymbol{\alpha}$ implies $\boldsymbol{\beta}\in\Lambda\}$. If Q is a rectangle in R^n, we show in § 2 that for any $f\in L_p(Q)$ there is a polynomial $P\in\mathbb{P}_\Lambda$ such that

$$\|f-P\|_p(Q)\leqq C\sum_{\alpha\in\partial\Lambda}\|D^\alpha f\|_p(Q)\boldsymbol{\delta}^\alpha, \tag{1.4}$$

whenever the right-hand side is finite. This result includes (1.2) and (1.3). It is also possible to include estimates for derivatives on the left-hand side of (1.4), and to handle more general domains.

In § 3, we examine to what extent we can establish estimates of the form (1.2') and (1.3') for approximation by the polynomials $\mathbb{P}_\Lambda$. We introduce an appropriate modulus of smoothness ω_Λ and show among other things that for $f\in L_p(\Omega)$ there is a polynomial $P\in\mathbb{P}_\Lambda$ with

$$\|f-P\|_p(\Omega)\leqq C\omega\Lambda(f, \boldsymbol{\delta}, \Omega)_p. \tag{1.4'}$$

As we have already noted for the case of total degree polynomials, one approach to deriving estimates of the form (1.4') is to introduce an appropriate K-functional and show its equivalence to ω_Λ. In this way, (1.4') follows directly from (1.4). This technique is discussed in § 3. However, we also give a second approach which has the advantage of being applicable to a larger class of domains Ω. We should note however that the conditions on the domain Ω which guarantee that (1.4') holds are quite strong and coordinatewise in nature.

The estimates for local polynomial approximation given in §§ 2 and 3 can be used to give estimates for the approximation by splines. We give two types of results. First, we estimate the approximation by piecewise Λ-polynomials, i.e., the splines satisfy no smoothness condition. This can be done in a rather general setting working with rather general domains and general rectangular partitions. Our second type of results are for smooth spline approximation. This is a substantial problem, and there are no general methods known for giving smooth spline approximants except for the tensor product case. We deal only with the case of splines of coordinate order $\mathbf{r}$ on tensor product grids. This enables us to use the quasi-interpolant approach introduced by C. de Boor and G. Fix [2]. Our main contribution to this problem is that we use coordinatewise estimates like (1.3) for our local polynomial estimates. Earlier works [2], [12] have used the local estimates for approximation by polynomials of *total* order r which in turn give the "wrong" estimates for the spline approximation since they include the mixed derivatives of order r on the right-hand side. The point is that the de Boor–Fix method gives locally polynomials of coordinate degree, not total degree. We also give in § 4 an approach to handling more general domains in the case of smooth spline approximation.

In § 5, we examine in what sense the estimates of § 4 are best possible. This is done by establishing various inverse theorems for piecewise polynomial and spline approximation. These in turn give a characterization of the smoothness of a function in terms of its degree of approximation by splines. There are essentially two types of results. The first applies to approximation by piecewise polynomials (and hence to smoother splines as well). Results of this type require some sort of mixing conditions on the partitions. The second type of result applies to splines which are assumed to have some a priori smoothness. These then apply to general partitions, at least in the sense of not requiring mixing. Among other things, our results can characterize the Besov spaces in terms of approximation by splines of coordinate order $\mathbf{r}$. Results of this type for rectangular domains were given by M. Munteanu and L. Schumaker [12]. Their results do not cover the full range because, as we mentioned earlier, they worked with total order polynomials in their direct estimates.

2. Local approximation by Λ-polynomials. We want to give a general setting for estimates of the type (1.2) and (1.3). Let $\Lambda \subseteq Z_+^n$ be bounded and satisfy

$$\text{(2.1)} \qquad \textit{if } \boldsymbol{\alpha} \in \Lambda \textit{ and } \boldsymbol{\beta} \leqq \boldsymbol{\alpha}, \textit{ then } \boldsymbol{\beta} \in \Lambda.$$

Define $\mathbb{P}_\Lambda = \operatorname{span}\{\mathbf{x}^{\boldsymbol{\alpha}} : \boldsymbol{\alpha} \in \Lambda\}$. In order to generalize (1.2) and (1.3) to $\mathbb{P}_\Lambda$, we must first decide which derivatives should appear on the right-hand side of such estimates. This turns out to be those $D^{\boldsymbol{\alpha}}$ with $\boldsymbol{\alpha} \in \partial\Lambda$, where $\partial\Lambda$ is the set of the minimal elements of $Z_+^n \backslash \Lambda$; i.e.,

$$\text{(2.2)} \qquad \partial\Lambda = \{\boldsymbol{\alpha} : \boldsymbol{\alpha} \notin \Lambda \textit{ and if } \boldsymbol{\beta} < \mathbf{a} \textit{ then } \boldsymbol{\beta} \in \Lambda\}.$$

For example, when $\Lambda = \{\boldsymbol{\alpha} : |\boldsymbol{\alpha}| < r\}$—the case of total degree $< r$—then $\partial\Lambda = \{\boldsymbol{\alpha} : |\boldsymbol{\alpha}| = r\}$, and if $\Lambda = \{\boldsymbol{\alpha} : \boldsymbol{\alpha} < \mathbf{r}\}$—the case of coordinate degree $< \mathbf{r}$—then $\partial\Lambda = \{r_i \mathbf{e}_i : 1 \leqq i \leqq n\}$. Here, $\mathbf{e}_i$ is the unit vector in the ith direction.

We begin with the following simple result.

LEMMA 2.1. *If Q is a rectangle in R^n with side length δ_i in the i-th direction and $f \in L_p(Q)$, $D^{r\mathbf{e}_i} f \in L_p(Q)$, then for any $0 \leqq k \leqq r$, we have $D^{k\mathbf{e}_i} f$ is in $L_p(Q)$, and for any $0 < \varepsilon < \delta_i$,*

$$\text{(2.3)} \qquad \varepsilon^k \|D^{k\mathbf{e}_i} f\|_p(Q) \leqq C[\|f\|_p(Q) + \varepsilon^r \|D^{r\mathbf{e}_i} f\|_p(Q)],$$

where[1] *C depends only on r.*

[1] Throughout the paper, we use the convention that C is a constant whose value may change at each occurrence, even on the same line.

Proof. For smooth functions f, say f in $C^r(Q)$, (2.3) follows simply by integrating the corresponding one variable inequality (cf. e.g., [1, Lemma 4.10]), The general case then follows from a limiting argument.

The following lemma characterizes $\partial\Lambda$.

LEMMA 2.2. *If Ω is a domain in R^n, and $D^\alpha f = 0$ (distributionally) for all $\boldsymbol{\alpha} \in \partial\Lambda$, then $f \in \mathbb{P}_\Lambda$. Moreover, if A is any set $\subset Z_+^n \setminus \Lambda$ with the property that $D^\alpha f = 0$, $\boldsymbol{\alpha} \in A$ implies that $f \in \mathbb{P}_\Lambda$, then $\partial\Lambda \subseteq A$.*

Proof. For each $1 \leqq i \leqq n$, there is an r_i so that $r_i\mathbf{e}_i \in \partial\Lambda$. Let $\mathbf{r} = (r_1, \cdots, r_n)$. Suppose first that f is in $C^{|\mathbf{r}|}(\Omega)$, and $D^\alpha f = 0$ for all $\boldsymbol{\alpha} \in \partial\Lambda$. If $|\boldsymbol{\beta}| \geqq |\mathbf{r}|$, then for some $\boldsymbol{\gamma} = r_i\mathbf{e}_i$, we have $\beta_i \geqq r_i$ so that $D^\beta f = D^{\beta-\gamma}D^\gamma f = 0$. Hence, f is a polynomial of total degree $<|\mathbf{r}|$. Now, if $\boldsymbol{\alpha} \notin \Lambda$, then there is a $\boldsymbol{\beta} \in \partial\Lambda$ with $\boldsymbol{\beta} \leqq \boldsymbol{\alpha}$ (from the definition of $\partial\Lambda$), and so, as above, $D^\alpha f = 0$ showing that the coefficient of $\mathbf{x}^\alpha$ in the expansion of f is 0. This means that $f \in \mathbb{P}_\Lambda$.

Now consider the case of general f. Take any n-cube Q contained strictly in Ω. If $\phi \in C_0$ has integral one and is supported on the unit sphere in R^n, then, for ε sufficiently small,

$$f_\varepsilon(\mathbf{x}) = \int_{R^n} f(\mathbf{x}+\mathbf{t})\phi_\varepsilon(\mathbf{t})\, d\mathbf{t}$$

is defined for $\mathbf{x} \in Q$, where $\phi_\varepsilon(\mathbf{t}) = \varepsilon^{-n}\phi(\varepsilon^{-1}\mathbf{t})$. Now, differentiating inside the integral (in the Sobolev sense) shows that $D^\alpha f_\varepsilon = 0$ on Q, $\boldsymbol{\alpha} \in \partial\Lambda$, and hence, by what we have already proved, $f_\varepsilon \in \mathbb{P}_\Lambda$. Taking a limit as $\varepsilon \to 0$ shows that f is in $\mathbb{P}_\Lambda$ on Q. This is true for all Q; and by the connectivity of Ω, f is the same polynomial on each Q. This proves the first statement of Lemma 2.2.

To prove the second statement, observe that if $\boldsymbol{\alpha} \in \partial\Lambda$, then $\boldsymbol{\alpha} \notin \Lambda$. Hence there must be a $\boldsymbol{\beta} \in A$ with $D^\beta(\mathbf{x}^\alpha) \neq 0$. That means $\boldsymbol{\beta} \leqq \boldsymbol{\alpha}$. But then $\boldsymbol{\beta} = \boldsymbol{\alpha}$, since $\boldsymbol{\beta} < \boldsymbol{\alpha}$ would imply $\boldsymbol{\beta} < \Lambda$ by the definition of $\partial\Lambda$ (2.2). This contradicts the fact that $\boldsymbol{\beta} \in A \subset Z_+^n \setminus \Lambda$. Therefore $\boldsymbol{\alpha} \in A$, as desired.

We can now prove a general estimate for approximation by the polynomials $\mathbb{P}_\Lambda$. For $f \in L_p(\Omega)$, we say that $f \in W_p^\Lambda(\Omega)$ if $D^\alpha f \in L_p(\Omega)$ for all $\boldsymbol{\alpha} \in \partial\Lambda$. As usual, W_p^k denotes the Sobolev space of those functions $f \in L_p(\Omega)$ for which $D^\alpha f \in L_p(\Omega)$, whenever $|\boldsymbol{\alpha}| \leqq k$, and $W_p^s(\Omega)$ denotes the Sobolev spaces of general order s. We will need the following critical lemma.

LEMMA 2.3. *Suppose that Ω is a bounded Lipschitz graph domain (cf. [14, (3), p. 63]) and $s < l$, where s is not necessarily an integer and $l := \min\{|\boldsymbol{\alpha}| : \boldsymbol{\alpha} \in \partial\Lambda\}$ is the maximal total order in $\mathbf{P}_\Lambda$. Then whenever $f \in W_p^\Lambda(\Omega)$, we have $f \in W_p^s(\Omega)$. In fact, there is a constant $C > 0$ such that for each $f \in W_p^\Lambda(\Omega)$, we have*

$$\|f\|_{W_p^s(\Omega)} \leqq C\|f\|_{p,\Lambda}(\Omega), \tag{2.4}$$

where $\|f\|_{p,\Lambda}(\Omega) = \|f\|_p(\Omega) + \sum_{\alpha\in\partial\Lambda}\|D^\alpha f\|_p(\Omega)$.

Proof. For each $1 \leqq i \leqq n$, there is an $r_i > 0$ with $r_i\mathbf{e}_i \in \partial\Lambda$. Hence, when $f \in W_p^\Lambda(\Omega)$, then $D^{r_i\mathbf{e}_i} f \in L_p(\Omega)$, $1 \leqq i \leqq n$ and $s < \min_i r_i$. The result therefore follows from the inequality

$$\|f\|_{W_p^s(\Omega)} \leqq C\Big(\sum_{i=1}^n \|D^{r_i\mathbf{e}_i} f\|_p(\Omega) + \|f\|_p(\Omega)\Big),$$

which can be found in the paper of K. Smith [14, p. 66].

For $f \in L_p(\Omega)$, let

$$E_\Lambda(f, \Omega)_p := \inf_{P\in\mathbb{P}_\Lambda} \|f - P\|_p(\Omega), \tag{2.5}$$

and when k is the largest integer less than l, then set

$$\tilde{E}_\Lambda(f,\Omega)_p := \inf_{P\in\mathbb{P}_\Lambda} \|f-P\|_{W_p^k(\Omega)}. \tag{2.6}$$

The following theorem is the main result of this section.

THEOREM 2.1. *If Ω is a Lipschitz graph domain and $f\in W_p^\Lambda(\Omega)$, then*

$$E_\Lambda(f,\Omega)_p \leqq \tilde{E}_\Lambda(f,\Omega)_p \leqq C \sum_{\alpha\in\partial\Lambda} \|D^\alpha f\|_p(\Omega), \tag{2.7}$$

where C depends at most on p, Λ, and Ω.

Proof. Suppose that (2.7) is not valid. Then, for each m, there is a function $f_m\in W_p^\Lambda(\Omega)$ with

$$1=\tilde{E}_\Lambda(f_m,\Omega)_p \geqq m \sum_{\alpha\in\partial\Lambda} \|D^\alpha f_m\|_p(\Omega). \tag{2.8}$$

Furthermore, by subtracting an appropriate polynomial from $\mathbb{P}_\Lambda$ if necessary, we can assume that $\tilde{E}_\Lambda(f_m,\Omega)_p=\|f_m\|_{W_p^k(\Omega)}$.

Now, (2.8) gives that $\sum_{\alpha\in\partial\Lambda}\|D^\alpha f_m\|_p(\Omega)\leqq 1$. Taking $k<s<l$, (2.4) says that $\{f_m\}$ is precompact in $W_p^k(\Omega)$. Let (m_j) be a subsequence and let $f\in W_p^k(\Omega)$ be such that $\|f_{m_j}-f\|_{W_p^k(\Omega)}\to 0$, $j\to\infty$. For any $\alpha\in\partial\Lambda$, and $\phi\in C_0^\infty(\Omega)$, we have from (2.8) that

$$0=\lim_{j\to\infty}(-1)^{|\alpha|}\int_\Omega D^\alpha f_{m_j}\phi=\lim_{j\to\infty}\int_\Omega f_{m_j}D^\alpha\phi=\int_\Omega fD^\alpha\phi.$$

Hence, $D^\alpha f=0$ for all $\alpha\in\partial\Lambda$, so that from Lemma 2.2, $f\in\mathbb{P}_\Lambda$. However, this means that

$$1=\tilde{E}_\Lambda(f_{m_j},\Omega)_p\leqq\tilde{E}_\Lambda(f,\Omega)_p+\|f-f_{m_j}\|_{W_p^k(\Omega)}=\|f-f_{m_j}\|_{W_p^k(\Omega)},$$

and this is the desired contradiction because $\|f-f_{m_j}\|_{W_p^k(\Omega)}\to 0$, $j\to\infty$.

COROLLARY 2.1. *If Q is a rectangle in R^n with side length vector $\boldsymbol{\delta}=(\delta_1,\cdots,\delta_n)$, then for each $f\in W_p^\Lambda(Q)$, we have*

$$E_\Lambda(f,Q)_p\leqq\tilde{E}_\Lambda(f,Q)_p\leqq C\sum_{\alpha\in\partial\Lambda}\boldsymbol{\delta}^\alpha\|D^\alpha f\|_p(Q) \tag{2.9}$$

with a constant C independent of f or Q.

Proof. Theorem 2.1 gives this result for the unit rectangle, for example. The general result then follows by simply changing scale.

3. The modulus of smoothness ω_Λ. As we have mentioned in the Introduction, it is preferable to estimate $E_\Lambda(f,\Omega)_p$ in terms of a modulus of smoothness rather than derivatives. This turns out to be a rather difficult task in that the possibility of such estimates depends quite strongly on the domain.

For an $\alpha\in Z_+^n$ and $\mathbf{t}>\mathbf{0}$, define

$$\Delta_\mathbf{t}^\alpha(f,\mathbf{x})=\Delta_{t_1,1}^{\alpha_1}\cdots\Delta_{t_n,n}^{\alpha_n}(f,\mathbf{x}),$$

where $\Delta_{t_i,i}^{\alpha_i}$ is the usual α_ith forward difference of step length t_i with respect to x_i. For $\Omega(\alpha,\mathbf{h})=\{\mathbf{x}\colon (x_i+s_i\alpha_i)_1^n\in\Omega \text{ for all } \mathbf{s}\leqq\mathbf{h},\ \mathbf{s}\in R_+^n\}$, define

$$\omega_\alpha(f,\mathbf{h},\Omega)_p=\sup_{0<\mathbf{t}<\mathbf{h}}\|\Delta_\mathbf{t}^\alpha(f,\cdot)\|_p(\Omega(\alpha,\mathbf{h}))$$

and

$$\omega_\Lambda(f, \mathbf{h}, \Omega)_p = \sum_{\alpha \in \partial\Lambda} \omega_\alpha(f, \mathbf{h}, \Omega)_p. \tag{3.1}$$

So ω_Λ is a multivariate modulus of smoothness. Some simple properties of ω_Λ are:

$$\begin{aligned} &\text{(i)} \quad \textit{if } \mathbf{h} \leqq \mathbf{h}' \quad \textit{then} \quad \omega_\Lambda(f, \mathbf{h}, \Omega)_p \leqq \omega_\Lambda(f, \mathbf{h}', \Omega)_p; \\ &\text{(ii)} \quad \omega_\Lambda(f+g, \mathbf{h}, \Omega)_p \leqq \omega_\Lambda(f, \mathbf{h}, \Omega)_p + \omega_\Lambda(g, \mathbf{h}, \Omega)_p; \\ &\text{(iii)} \quad \omega_\Lambda(f, \lambda\mathbf{h}, \Omega)_p \leqq (1+\lambda)^m \omega_\Lambda(f, \mathbf{h}, \Omega)_p \quad \textit{with } m = \max_{\alpha \in \partial\Lambda} |\alpha|. \end{aligned} \tag{3.2}$$

These properties are easily derived from their one dimensional counterparts.

A deeper property of moduli of smoothness which is very important in approximation theory and the study of function spaces is the equivalence of the moduli of smoothness with certain K-functionals. Given $f \in L_p(\Omega)$ define

$$K_\Lambda(f, \mathbf{h}, \Omega)_p = \inf_{g \in W_p^\Lambda(\Omega)} \left\{ \|f-g\|_p(\Omega) + \sum_{\alpha \in \partial\Lambda} \mathbf{h}^\alpha \|D^\alpha g\|_p(\Omega) \right\}, \tag{3.3}$$

where $W_p^\Lambda(\Omega)$ is the set of functions $g \in L_p(\Omega)$ for which $D^\alpha g \in L_p(\Omega)$ for all $\alpha \in \partial\Lambda$.

We will show the equivalence of ω_Λ and K_Λ for certain domains. The ideas involved in the proof trace back to the case of total degree treated by H. Johnen and K. Scherer [10]. Let M_k denote the B-spline (of one variable) of order k (degree $k-1$) with knots $0, 1, \cdots, k$ normalized to have integral one on $(-\infty, \infty)$. Given α define

$$M_\alpha(\mathbf{x}) = M_{\alpha_1}(x_1) \cdots M_{\alpha_n}(x_n),$$

and for $\mathbf{t} > \mathbf{0}$ let

$$M_\alpha(\mathbf{t}, \mathbf{x}) = (t_1 \cdots t_n)^{-1} M_{\alpha_1}(t_1^{-1}x_1) \cdots M_{\alpha_n}(t_1^{-1}x_n).$$

It follows from the one-dimensional formula that

$$\Delta_\mathbf{t}^\alpha(f, \mathbf{x}) = \mathbf{t}^\alpha \int_{R^n} D^\alpha f(\mathbf{x}+\mathbf{u}) M_\alpha(\mathbf{t}, \mathbf{u})\, d\mathbf{u}. \tag{3.4}$$

Thus, applying norms and observing that $M_\alpha(\mathbf{t}, \cdot)$ has integral one, we have

$$\omega_\alpha(f, \mathbf{t}, \Omega)_p \leqq \mathbf{t}^\alpha \|D^\alpha f\|_p(\Omega).$$

Summing over $\alpha \in \partial\Lambda$, we get

$$\omega_\Lambda(f, \mathbf{t}, \Omega)_p \leqq \sum_{\alpha \in \partial\Lambda} \mathbf{t}^\alpha \|D^\alpha f\|_p(\Omega) \tag{3.5}$$

whenever $f \in W_p^\Lambda(\Omega)$.

For any $f \in L_p(\Omega)$, we have

$$\omega_\Lambda(f, \mathbf{t}, \Omega)_p \leqq C\|f\|_p(\Omega). \tag{3.6}$$

Hence, writing $f = (f-g)+g$ and using (3.6) on $f-g$ and using (3.5) on g will give

$$\omega_\Lambda(f, \mathbf{t}, \Omega)_p \leqq C K_\Lambda(f, \mathbf{t}, \Omega)_p, \qquad \mathbf{t} > \mathbf{0}. \tag{3.7}$$

The converse to (3.7) holds only for certain domains Ω.

We begin with the simple case of Ω, a rectangle in R^n.

LEMMA 3.1. *There is a constant $C>0$ such that for any rectangle Q, we have*

$$K_\Lambda(f,\mathbf{t},\Omega)_p \leqq C\omega_\Lambda(f,\mathbf{t},\Omega)_p \tag{3.8}$$

for each $f\in L_p(Q)$ and $\mathbf{t}>\mathbf{0}$.

Proof. We prove only the case $n=2$ which is typical. Let $Q=I\times I$ with $I=[0,1]$. The general case follows by a change of variable. We denote by Q_{ij}, with $i=0$ or 1 and $j=0$ or 1, the rectangle with sides of length $\frac{2}{3}$ which has (i,j) as one of its vertices, and $Q_{ij}\subset Q$. We first show that if $f\in L_p(Q)$ and $\mathbf{t}>\mathbf{0}$, then there are functions g_{ij} such that

$$\|f-g_{ij}\|_p(Q_{ij})+\sum_{\alpha\in\partial\Lambda}\mathbf{t}^\alpha\|D^\alpha g_{ij}\|_p(Q_{ij})\leqq C\omega_\Lambda(f,\mathbf{t},Q)_p. \tag{3.9}$$

The case $i,j=0$ is typical. For any function h and for $k=1,2$, let

$$I_k(h)(\mathbf{x})=\int_{-\infty}^{\infty}[(-1)^{r_k+1}\Delta^{r_k}_{u_k,h}(h,\mathbf{x})+h(\mathbf{x})]M_{r_k}(t_k^{-1}u_k)t_k^{-1}\,du_k, \tag{3.10}$$

where $(r_1,0)$ and $(0,r_2)$ are in $\partial\Lambda$. Then, we can define

$$g_{00}(\mathbf{x})=I_1(I_2(f))(\mathbf{x}),\qquad \mathbf{x}\in Q_{00}. \tag{3.11}$$

This function is well defined if $\mathbf{t}\leqq(1/(3r_1),1/(3r_2))$. It is easy to see that

$$\begin{aligned}\|f-g_{00}\|_p(Q_{00})&\leqq\|I_1(I_2(f))-I_2(f)\|_p(Q_{00})+\|I_2(f)-f\|_p(Q_{00})\\&\leqq C\|\Delta^{r_1}_{t_1,1}(f)\|_p(Q)+\|\Delta^{r_2}_{t_2,2}(f)\|_p(Q)\\&\leqq C\omega_\Lambda(f,\mathbf{t},Q)_p.\end{aligned} \tag{3.12}$$

Now

$$\begin{aligned}g_{00}(\mathbf{x})&=-\sum{}'(-1)^{i+j}\binom{r_1}{i}\binom{r_2}{j}\int\int_{R^2}f(x_1+iu_1,x_2+ju_2)M_{(r_1,r_2)}(\mathbf{t},\mathbf{u})\,du_1\,du_2\\&=\sum{}'c_{ij}L_{ij}(f),\end{aligned}$$

where $\sum'$ indicates that the sum is taken over all $1\leqq i\leqq r_1$, $1\leqq j\leqq r_2$, and the L_{ij} are the appropriately defined linear operators. We will show that, for each $\alpha\in\partial\Lambda$,

$$\mathbf{t}^\alpha\|D^\alpha(L_{ij}(f))\|_p(Q_{00})\leqq C\omega_\alpha(f,\mathbf{t},Q)_p. \tag{3.13}$$

Observe that if $\alpha\in\partial\Lambda$, then $\alpha_1\leqq r_1$ and $\alpha_2\leqq r_2$ because of property (2.1) for Λ. We will now verify (3.13) in the case that f is in $C^m(Q)$. The general case then follows by a passage to the limit. Let F satisfy $D^{\mathbf{r}}F=f$ with $\mathbf{r}=(r_1,r_2)$. Then using (3.4), we find after a simple change of variables that

$$L_{ij}(f,\mathbf{x})=(it_1)^{-r_1}(jt_2)^{-r_2}\Delta^{\mathbf{r}}_{\mathbf{t}}(F,\mathbf{x}).$$

Now, we differentiate this last identity and estimate

$$\begin{aligned}\|D^\alpha L_{ij}(f)\|_p(Q_{00})&\leqq C\mathbf{t}^{-\mathbf{r}}\omega_{\mathbf{r}}(D^\alpha F,\mathbf{t},Q)_p\leqq C\mathbf{t}^{-\alpha}\omega_\alpha(D^{\mathbf{r}}F,\mathbf{t},Q)_p\\&\leqq C\mathbf{t}^{-\alpha}\omega_\alpha(f,\mathbf{t},Q)_p,\end{aligned}$$

where the last inequality follows from the corresponding one-dimensional results. This establishes the inequality (3.13).

It follows from (3.13) that

$$\sum_{\alpha\in\partial\Lambda}\mathbf{t}^\alpha\|D^\alpha g_{00}\|_p(Q_{00})\leqq C\sum_{\alpha\in\partial\Lambda}\omega_\alpha(f,\mathbf{t},Q)_p\leqq C\omega_\Lambda(f,\mathbf{t},Q)_p. \tag{3.14}$$

Thus, (3.14) and (3.12) combine to give (3.9) in the case $i, j = 0$. The other cases are proved in the same way.

Now, we will show how to use the estimates (3.9) to give a proof of (3.8). Let $\phi \in C^{\infty}[0, 1]$ be such that ϕ is 1 on $[0, \frac{1}{3}]$, monotone decreasing on $[\frac{1}{3}, \frac{2}{3}]$, and 0 on $[\frac{2}{3}, 1]$. Set $\phi_0 = \phi$, $\phi_1 = 1 - \phi$ and $\phi_{ij}(x_1, x_2) = \phi_i(x_1)\phi_j(x_2)$ when $i, j = 0$ or 1. We can show that the function

$$g = \sum_{i,j=0}^{1} \phi_{ij} g_{ij}$$

gives the estimate (3.8). In fact, the estimate

$$\|f - g\|_p(Q) \leqq C\omega_\Lambda(f, \mathbf{t}, Q)_p \tag{3.15}$$

follows immediately from (3.9).

To estimate the derivatives of g, we decompose Q into nine congruent cubes labeled R_μ where μ is the lower left corner of R_μ. Estimates for $\|D^\alpha g\|_p(R_\mu)$, $\alpha \in \partial\Lambda$ and $\mu = (0, 0), (\frac{2}{3}, 0), (0, \frac{2}{3}), (\frac{2}{3}, \frac{2}{3})$, follow from (3.9) because $g = g_{ij}$ with appropriate i, j on such cubes. To estimate $D^\alpha g$ on the other cubes takes a little more care. The case $R = R_\mu$ with $\mu = (\frac{1}{3}, \frac{1}{3})$ is typical of what needs to be done.

We write

$$\begin{aligned} g(x, y) = g_{11}(x, y) + \phi(x)(g_{01}(x, y) - g_{11}(x, y)) + \phi(y)(g_{10}(x, y) - g_{11}(x, y)) \\ + \phi(x)\phi(y)(g_{11}(x, y) - g_{01}(x, y) - g_{10}(x, y) + g_{00}(x, y)). \end{aligned}$$

Differentiating this identity for $\alpha \in \partial\Lambda$, we have

$$\|D^\alpha g\|_p(R) \leqq \|D^\alpha g_{11}\|_p(R) + C \sum{}' \sum_{0 \leqq \nu \leqq \alpha} \|D^\nu(g_{ij} - g_{i'j'})\|_p(R), \tag{3.16}$$

where $\sum'$ is taken over all $(i, j), (i', j')$ with $i, j, i', j' = 0$ or 1, and $(i, j) \neq (i', j')$. Now, it is easy to see that, for any h, $\mathbf{0} \leqq \nu \leqq \alpha$ and $\mathbf{0} < \mathbf{t} < (\frac{2}{3}, \frac{2}{3})$,

$$\mathbf{t}^\nu \|D^\nu h\|_p(R) \leqq C(\|h\|_p(R) + \sum_{\beta \in \partial\Lambda} \mathbf{t}^\beta \|D^\beta h\|_p(R)). \tag{3.17}$$

Indeed, $D^\nu h$ can be estimated by $D^{(\alpha_1, \nu_2)}h$ and $D^{(0, \nu_2)}h$ because of Lemma 2.1. Similarly, $D^{(\alpha_1, \nu_2)}h$ is estimated by $D^{(\alpha_1, \alpha_2)}h$ and $D^{(\alpha_1, 0)}h$. Finally, $D^{(\alpha, \nu_2)}h$ and $D^{(\alpha_1, 0)}h$ can be estimated by $D^{(0, r_2)}h$ and $D^{(r_1, 0)}h$ together with h, where $(r_1, 0)$, $(0, r_2)$ are in $\partial\Lambda$.

Let us now use (3.16) and (3.17) to find

$$\begin{aligned} \mathbf{t}^\alpha \|D^\alpha g\|_p(R) &\leqq \mathbf{t}^\alpha \|D^\alpha g_{11}\|_p(R) + \mathbf{t}^\alpha \sum{}' \sum_{0 \leqq \nu \leqq \alpha} \|D^\nu(g_{ij} - g_{i'j'})\|_p(R) \\ &\leqq \mathbf{t}^\alpha \|D^\alpha g_{11}\|_p(R) + C \sum{}' \|g_{ij} - g_{i'j'}\|_p(R) \\ &\quad + \sum_{\beta \in \partial\Lambda} \mathbf{t}^\beta \|D^\beta(g_{ij} - g_{i'j'})\|_p(R) \\ &\leqq \mathbf{t}^\alpha \|D^\alpha g_{11}\|_p(R) + C \sum_{i,j=0}^{1} (\|f - g_{ij}\|_p(R) + \sum_{\beta \in \partial\Lambda} \mathbf{t}^\beta \|D^\beta g_{ij}\|_p(R)), \end{aligned} \tag{3.18}$$

where we used the fact that $\mathbf{t}^\alpha \leqq \mathbf{t}^\nu$ if $\nu \leqq \alpha$. Now $R \subset Q_{ij}$ for all i, j so each norm on the right-hand side of (3.18) can be enlarged to the appropriate Q_{ij}. Hence, using (3.9) gives

$$\sum_{\alpha \in \partial\Lambda} \mathbf{t}^\alpha \|D^\alpha g\|_p(R) \leqq C\omega_\Lambda(f, \mathbf{t}, Q)_p.$$

As previously mentioned, the same estimate holds for the other R_μ. Thus,

(3.19)
$$\sum_{\alpha\in\partial\Lambda} \mathbf{t}^\alpha \|D^\alpha g\|_p(Q) \leqq C\omega_\Lambda(f, \mathbf{t}, Q)_p.$$

When this is combined with (3.15), we get

$$K_\Lambda(f, \mathbf{t}, Q)_p \leqq \|f-g\|_p(Q) + \sum_{\alpha\in\partial\Lambda} \mathbf{t}^\alpha \|D^\alpha g\|_p(Q) \leqq C\omega_\Lambda(f, \mathbf{t}, Q)_p$$

and the lemma is proved.

It is possible to prove Lemma 3.1 for more general domains. Before doing this, we want to observe that the conditions on Ω have to be quite stringent and of a coordinatewise nature. Given any $0<a<1<b$, let $\Omega:=\{(x, y): ax<y<bx, 0<x<1\}$ and for any $0<\delta<1$ define

$$f_\delta(x, y) = \begin{cases} \ln x, & x \geqq \delta, \\ \ln \delta + \delta^{-1}(x-\delta), & x<\delta, \end{cases}$$

which is in $C^1(\Omega)$.

Then for $\partial\Lambda = \{(0, 2), (2, 0)\}$ (so Λ is of coordinate degree 1), we have

$$\omega_{(0,2)}(f_\delta, \mathbf{t}, \Omega)_\infty = 0$$

and

$$\omega_{(2,0)}(f_\delta, \mathbf{t}, \Omega)_\infty \leqq \sup_{h\leqq t_2} \sup\{h^2|D^{(2,0)}f_\delta(x, y)|: (x, y), (x+2h, y)\in\Omega\}$$

$$\leqq \sup_{h\leqq t_2} \sup\{h^2x^{-2}: (x, y), (x+2h, y)\in\Omega\} \leqq C$$

with $C=((b-a)/2a)^2$. Here, we used the fact that $(x, y), (x+2h, y)\in\Omega$ implies that $2ah \leqq (b-a)x$. If Lemma 3.1 were valid for Ω then taking $\mathbf{t}=(1, 1)$, there would be a function g_δ with

(3.20)
$$\|f_\delta - g_\delta\|_\infty + \sum_{\alpha\in\partial\Lambda} \|D^\alpha g_\delta\|_\infty \leqq C$$

with C independent of δ.

Now fix $\varepsilon>0$ (to be prescribed later) and define $\Omega_\varepsilon := \{(x, y)\in\Omega: \varepsilon<x\}$. Then we get from (3.20) that

(3.20′)
$$\|g_\delta\|_{\infty,\Lambda}(\Omega_\varepsilon) \leqq C + \|f_\delta\|_\infty(\Omega_\varepsilon) \leqq C|\ln \varepsilon|.$$

Therefore it follows from Lemma 2.3 that there is a subsequence $\delta_m \to 0$ and a function $g\in W^1_\infty(\Omega_\varepsilon)$ for which $\|g-g_{\delta_m}\|_{W^1_\infty}(\Omega_\varepsilon)\to 0$. So we obtain by (3.20′), and again by Lemma 2.3, the estimate

(3.21)
$$\|g\|_{W^1_\infty}(\Omega_\varepsilon) \leqq \limsup_{m\to\infty} \|g_{\delta_m}\|_{W^1_\infty}(\Omega_\varepsilon) \leqq C|\ln \varepsilon|$$

and

(3.21′)
$$\|f_0 - g\|_\infty \leqq C.$$

On the other hand, taking $\mathbf{x}=(2\varepsilon, 2\varepsilon)\in\Omega$ and $\mathbf{h}=(\varepsilon^{1/2}, \varepsilon^{1/2})$, we find, using (3.21′) and

$\|\nabla g\|_\infty$ denoting the L_∞ norm of the function $|\nabla g(\cdot)|$, that

$$\|\nabla g\|_\infty(\Omega_\varepsilon) \geqq |\mathbf{h}|^{-1}|g(\mathbf{x}+\mathbf{h})-g(\mathbf{x})| \geqq |\mathbf{h}|^{-1}(|f_0(\mathbf{x}+\mathbf{h})-f_0(\mathbf{x})|-C)$$
$$\geqq C|\mathbf{h}|^{-1}\ln(1+\tfrac{1}{2}\varepsilon^{-1/2}) \geqq C\varepsilon^{-1/2}|\ln \varepsilon|.$$

This contradicts the estimate (3.21) provided ε is chosen suitably small because the constants above do not depend on ε.

We have just shown that Lemma 3.1 is not valid even for the simple domains of cones in R^n unless the cones contain a complete set of coordinate vectors. We will now introduce some conditions on Ω under which Lemma 3.1 is valid. It will be convenient to assume that Ω is coordinatewise convex, i.e., we suppose that

(3.22) *if* $\mathbf{x}, \mathbf{x}+h\mathbf{e}_i \in \Omega$, *for some* $h>0$ *and some* i, *then* $\mathbf{x}+t\mathbf{e}_i \in \Omega$ *for all* $0 \leqq t \leqq h$.

Our next theorem will assume a uniform "rectangle property" for Ω, namely,

(3.23) Ω *satisfies* (3.22) *and* $\Omega \subseteq \bigcup_1^m Q_i$, *where each* Q_i *is a rectangle and for each* i, *there is a rectangle* R_i *with one of its vertices* $\mathbf{0}$ *such that* $\mathbf{x} \in Q_i \cap \Omega$ *implies* $\mathbf{x}+R_i \subset \Omega$.

Property (3.23) is a finite cone property where each cone is required to contain a full set of coordinate vectors.

THEOREM 3.1. *If* Ω *satisfies* (3.23), *then there are constants* $C_1, C_2>0$ *depending only on* Λ *and* Ω, *such that for each* $\mathbf{h}>\mathbf{0}$,

$$C_1\omega_\Lambda(f,\mathbf{h},\Omega)_p \leqq K_\Lambda(f,\mathbf{h},\Omega)_p \leqq C_2\omega_\Lambda(f,\mathbf{h},\Omega)_p.$$

Proof. We have already shown (cf. (3.7)) the lower inequality. We do not want to give the complete details for the proof of the upper inequality since the proof is very similar to that of Lemma 3.1. Indeed, for each i, one constructs a function g_i by smoothing in the coordinate directions (this is where (3.23) is used), so that g_i satisfies

$$\|f-g_i\|_p(\Omega\cap Q_i)+\sum_{\alpha\in\partial\Lambda}\mathbf{h}^\alpha\|D^\alpha g_i\|_p(\Omega\cap Q_i) \leqq C\omega_\Lambda(f,\mathbf{h},\Omega)_p.$$

These estimates are then put together by taking an appropriate partition of unity similar to the construction in Lemma 3.1. The result is a function g with

$$K_\Lambda(f,\mathbf{h},\Omega)_p \leqq \|f-g\|_p(\Omega)+\sum_{\alpha\in\partial\Lambda}\mathbf{h}^\alpha\|D^\alpha g\|_p(\Omega) \leqq C\omega_\Lambda(f,\mathbf{h},\Omega)_p$$

as desired.

COROLLARY 3.1. *If* Ω *satisfies* (3.23) *and* $\Omega \subset Q$ *is a rectangle with side length vector* $\boldsymbol{\delta}$, *then for each* $f \in L_p(\Omega)$

$$E_\Lambda(f,\Omega)_p \leqq C\omega_\Lambda(f,\boldsymbol{\delta},\Omega)_p, \tag{3.24}$$

where C *depends only on* Λ, Ω *and* p.

Proof. The domain Ω satisfies the conditions of Theorem 2.1. Hence, for any $g \in W_p^\Lambda(\Omega)$ we have

$$E_\Lambda(g,\Omega)_p \leqq C\sum_{\alpha\in\partial\Lambda}\boldsymbol{\delta}^\alpha\|D^\alpha g\|_p(\Omega).$$

According to Theorem 3.1, there is a $g \in W_p^\Lambda(\Omega)$ with

$$\|f-g\|_p(\Omega)+\sum_{\alpha\in\partial\Lambda}\boldsymbol{\delta}^\alpha\|D^\alpha g\|_p(\Omega) \leqq C\omega_\Lambda(f,\boldsymbol{\delta},\Omega)_p.$$

Hence,

$$E_\Lambda(f,\Omega)_p \leqq E_\Lambda(f-g,\Omega)_p + E_\Lambda(g,\Omega)_p$$

$$\leqq \|f-g\|_p(\Omega) + \sum_{\alpha\in\partial\Lambda} \delta^\alpha \|D^\alpha g\|_p(\Omega) \leqq C\omega_\Lambda(f,\delta,\Omega)_p$$

as desired.

It is possible to prove this corollary under somewhat weaker conditions than (3.23). We want to indicate how to do this since some of the ideas are useful to us in § 4. We need to introduce another (coordinate) property for domains. Suppose $Q = \prod_{i=1}^n [a_i, b_i]$ is a rectangle in R^n with side length vector δ. For $\lambda > 0$, $1 \leqq j \leqq n$, let $Q_{\lambda,j}$ be the extended rectangle $\prod_{i=1}^n [a_i, \tilde{b}_i]$ with $\tilde{b}_i = b_i$, $i \neq j$, $\tilde{b}_j = b_j + \lambda\delta_j$; and similarly when $\lambda < 0$, $Q_{\lambda,j} := \prod_{i=1}^n [\tilde{a}_i, b_i]$ with $\tilde{a}_i = a_i$, $i \neq j$, $\tilde{a}_j = a_j + \lambda\delta_j$.

LEMMA 3.2. *Suppose Ω is coordinatewise convex* (3.22) *and $Q \cap \Omega \neq \phi$, where Q is a rectangle with side length vector δ. If $f \in L_p(\Omega)$, and $Q_{\lambda,j}$ is as above, then*

$$\|f\|_p(\Omega \cap Q_{\lambda,j}) \leqq C(\|f\|_p(Q\cap\Omega) + \omega_\Lambda(f,\delta,\Omega\cap Q_{\lambda,j})_p), \tag{3.25}$$

where C depends only on λ and the cardinality of Λ.

Proof. Let r_j be chosen so that $r_j\mathbf{e}_j$ is in Λ. We shall prove (3.25) in the case $\lambda \leqq 1/r_j$. The general case of any $\lambda > 1/r_j$ then follows from a repeated application of this estimate. With $\Delta_{h,j}^k(f,\mathbf{x}) = \sum_{\nu=0}^k \binom{k}{\nu}(-1)^{k-\nu} f(\mathbf{x}+\nu h\mathbf{e}_j)$, we have, whenever $[\mathbf{x}, \mathbf{x}+r_jh\mathbf{e}_j] \subseteq \Omega$,

$$f(\mathbf{x}+r_jh e_j) = \Delta_{h,j}^{r_j}(f,\mathbf{x}) - (\Delta_{h,j}^{r_j}(f,\mathbf{x}) - f(\mathbf{x}+r_jh\mathbf{e}_j)). \tag{3.26}$$

The bracketed term on the right-hand side of (3.26) involves only values of f at the points $\mathbf{x}+kh\mathbf{e}_j$, $0 \leqq k \leqq r_j$. Take $h = \delta_j/r_j$. On account of the coordinatewise convexity of Ω, these points are in $Q\cap\Omega$ whenever $\mathbf{x} \in S = \{\mathbf{x} \in Q\cap\Omega,\ a_j \leqq x_j \leqq a_j + h\}$ and $\mathbf{x} + r_jh\mathbf{e}_j \in \Omega$. Furthermore, $(Q_{\lambda,j}\backslash Q)\cap\Omega = \{\mathbf{y}\in\Omega: \mathbf{y} = \mathbf{x}+r_jh\mathbf{e}_j,\ \mathbf{x}\in S\}$, so returning to (3.26) gives

$$\begin{aligned}\|f\|_p((Q_{\lambda,j}\backslash Q)\cap\Omega) &\leqq \|\Delta_{h,j}^{r_j}(f,\cdot)\|_p(S) + 2^{r_j}\|f\|_p(Q\cap\Omega)\\ &\leqq \omega_\Lambda(f,\delta,Q_{\lambda,j}\cap\Omega)_p + 2^{r_j}\|f\|_p(Q\cap\Omega).\end{aligned} \tag{3.27}$$

A repeated application of (3.27) gives the general case of $\lambda > 0$. The case $\lambda < 0$ is handled similarly.

We now introduce a new property for domains.

(3.28) *We say the rectangle Q covers the rectangle $\tilde{Q}$ (with respect to Ω) in a distance K if there are vectors $\boldsymbol{\eta}_1, \cdots, \boldsymbol{\eta}_m$ with $\boldsymbol{\eta}_i = \lambda_i\mathbf{e}_j$ and the rectangles $Q_i = Q + \sum_{\nu=1}^i \boldsymbol{\eta}_\nu$, $i = 0, 1, \cdots, m$ satisfy*

(i) $Q_{i+1}\cap\Omega \subseteq (Q_i\cap\Omega) + \boldsymbol{\eta}_{i+1}$, $i = 0, 1, \cdots, m-1$,

(ii) $\tilde{Q}\cap\Omega \subseteq \bigcup_{i=1}^m (Q_i\cap\Omega)$,

(iii) *$m \leqq K$ and the vector $\boldsymbol{\eta} = \sum_{i=1}^m (\operatorname{sgn}\lambda_i)\boldsymbol{\eta}_i$ satisfies $\boldsymbol{\eta} \leqq K\delta$,*

where δ is the side length vector of Q. More generally, we say that a rectangle Q covers a domain Ω in a distance K if Ω is contained in a union of a finite number of rectangles each of which is covered by Q in a distance K.

We can establish (3.24) for domains with the following property:

(3.29) *Ω satisfies (3.22) and there is a constant $K>0$ and a rectangle $Q \subseteq \Omega$ such that Q covers Ω in a distance K.*

Note that (3.29) is much weaker than (3.23). For example, any bounded cone with vertex at the origin which contains just one coordinate vector will satisfy (3.29). Also Ω need not even satisfy a cone condition to satisfy (3.29).

THEOREM 3.2. *If Ω satisfies (3.29) and $\mathbf{\delta}$ is the side length vector of the rectangle Q appearing in (3.29), then for each $f \in L_p(\Omega)$ there is a polynomial $P \in \mathbb{P}_\Lambda$ such that*

(3.30) $$\|f-P\|_p(\Omega) \leqq C\omega_\Lambda(f, \mathbf{\delta}, \Omega)_p,$$

where C depends only on the constant K of (3.28), Λ and the total number of rectangles involved in the covering.

Proof. The proof proceeds by a repeated application of Lemma 3.2. According to Corollary 3.1, we can choose $P \in \mathbb{P}_\Lambda$ so that

(3.31) $$\|f-P\|_p(Q) \leqq C\omega_\Lambda(f, \mathbf{\delta}, Q)_p.$$

Let $\tilde{Q}$ be any rectangle covered by Q in a distance K and let $Q_1, \cdots, Q_m$ be the rectangles appearing in (3.28). Define $Q_0 = Q$ and for each i, define $\bar{Q}_i$ to be the smallest rectangle containing Q_i and Q_{i-1}. Lemma 3.2 gives with f replaced by $f-P$, and observing that $\omega_\Lambda(P, \mathbf{\delta}, \bar{Q}_i \cap \Omega)_p = 0$,

$$\|f-P\|_p(\bar{Q}_i \cap \Omega) \leqq C(\|f-P\|_p(Q_{i-1} \cap \Omega) + \omega_\Lambda(f, \mathbf{\delta}, \bar{Q}_i \cap \Omega)_p),$$

where C depends only on $\boldsymbol{\eta}_i$ and Λ. A repeated application of this inequality gives, for any i,

$$\|f-P\|_p(\bar{Q}_i \cap \Omega) \leqq C(\|f-P\|_p(Q \cap \Omega) + \omega_\Lambda(f, \mathbf{\delta}, \Omega)_p) \leqq C\omega_\Lambda(f, \mathbf{\delta}, \Omega)_p.$$

Since $\tilde{Q} \cap \Omega \subseteq \bigcup (\bar{Q}_i \cap \Omega)$, it follows that

(3.32) $$\|f-P\|_p(\tilde{Q} \cap \Omega) \leqq C\omega_\Lambda(f, \mathbf{\delta}, \Omega)_p.$$

Now, if $\tilde{Q}_1, \cdots, \tilde{Q}_s$ are the rectangles covered by Q with $\Omega \subseteq \bigcup_{i=1}^s \tilde{Q}_i$, then

$$\|f-P\|_p(\Omega) \leqq C \sum_{i=1}^{s} \|f-P\|_p(\tilde{Q}_i \cap \Omega) \leqq C\omega_\Lambda(f, \mathbf{\delta}, \Omega)_p$$

because of (3.32).

Let us point out that the conditions on Ω can not be considerably weakened. Indeed, return to the example given after Lemma 3.1. If f_δ, $\delta > 0$ are the functions defined there and Ω is the cone defined there, then, as we have shown, $\omega_{(0,2)}(f_\delta, \mathbf{t}, \Omega)_\infty = 0$ and $\omega_{(2,0)}(f_\delta, \mathbf{t}, \Omega)_\infty \leqq C$ when $\mathbf{t} = (1, 1)$ with C independent of δ. Hence, if Theorem 3.2 were valid for Ω, there would be polynomials $P_\delta \in \mathbb{P}_\Lambda$ with

$$\|f_\delta - P_\delta\|_\infty \leqq C$$

for each $\delta > 0$, which is easily seen to be impossible.

4. Approximation by Λ-splines. In this section, we give direct estimates for approximation by piecewise Λ-polynomials and splines. Our first results are for piecewise Λ-polynomials, i.e., no global smoothness requirements. These estimates can be given for general gridding; however, we restrict ourselves to the case of rectangular decompositions.

Let $\Delta=\{Q_\nu: \nu\in A\}$ be a collection of pairwise disjoint rectangles with A an index set from Z_+^n and $\Omega\subseteq \bigcup_{\nu\in A} Q_\nu$. In the following, δ_ν will always denote the side length vector of Q_ν, whereas in general, δ_R will be the side length vector of some rectangle R. We will be able to prove direct estimates for piecewise Λ-polynomial approximation for domains with the following property:

(4.1) *Ω is coordinatewise convex* (3.22) *and there is a $K>0$ such that for each $Q_\nu\in\Delta$ with $Q_\nu\cap\Omega\neq\varnothing$ there is a rectangle $R_\nu\subset\Omega$ with $\delta_{R_\nu}\leqq K\delta_\nu$ and R_ν covers Q_ν in a distance K (cf.* (3.28)). *Note that R_ν need not be in Δ.*

We can assume as we will that the constant $K\geqq 1$ in (4.1). Furthermore, denote by A_Ω the set of all ν for which $Q_\nu\cap\Omega\neq\varnothing$ and define $\delta:=\max_{\nu\in A_\Omega}\delta_\nu$, where the maximum is taken coordinatewise.

THEOREM 4.1. *If Ω satisfies* (4.1) *and $f\in W_p^\Lambda(\Omega)$, then there is a piecewise polynomial S such that $S|_{Q_\nu}\in\mathbb{P}_\Lambda$ for each ν, and*

$$\|f-S\|_p(\Omega)\leqq C\sum_{\alpha\in\partial\Lambda}\delta^\alpha\|D^\alpha f\|_p(\Omega). \tag{4.2}$$

If in addition Ω satisfies (3.23), *then for each $f\in L_p(\Omega)$ there is an S with $S|_{Q_\nu}\in\mathbb{P}_\Lambda$ for each ν, and*

$$\|f-S\|_p(\Omega)\leqq C\omega_\Lambda(f,\delta,\Omega)_p. \tag{4.3}$$

In these two estimates C depends at most on p, Λ, Ω and the constant K.

Proof. Consider first (4.2). Let $f\in W_p^\Lambda(\Omega)$. If $Q_\nu\cap\Omega\neq\varnothing$, then let R_ν be the rectangle guaranteed by (4.1). Since $R_\nu\subseteq\Omega$, we can use Corollary 2.1 to find a polynomial $P_\nu\in\mathbb{P}_\Lambda$ with

$$\|f-P_\nu\|_p(R_\nu)\leqq C\sum_{\alpha\in\partial\Lambda}\delta_\nu^\alpha\|D^\alpha f\|_p(R_\nu).$$

Now R_ν covers Q_ν in a distance K, and so using the argument given in the proof of Theorem 3.2 shows that

$$\begin{aligned}\|f-P_\nu\|_p(Q_\nu\cap\Omega)&<C(\|f-P_\nu\|_p(R_\nu)+\omega_\Lambda(f,\delta_\nu,\tilde Q_\nu\cap\Omega)_p)\\ &\leqq C\sum_{\alpha\in\partial\Lambda}\delta_\nu^\alpha\|D^\alpha f\|_p(\tilde Q_\nu\cap\Omega),\end{aligned} \tag{4.4}$$

where $\tilde Q_\nu$ is the rectangle with the same center as Q_ν but side length vector $(2K^2+1)\delta_\nu$. Now, let S be defined to be P_ν on the rectangle Q_ν for each ν. We want to add up the estimates (4.4). For $p=\infty$, (4.4) already gives (4.2). For $1\leqq p<\infty$, we note that if $\alpha\in\partial\Lambda$,

$$\sum_{\mathbf{x}}\delta_\nu^{p\alpha}\leqq C\delta^{p\alpha},$$

where $\sum_{\mathbf{x}}$ indicates that the sum is taken over all ν such that $\mathbf{x}\in\tilde Q_\nu$. Using this back in (4.4) gives

$$\begin{aligned}\|f-S\|_p^p(\Omega)&=\sum_\nu\|f-P_\nu\|_p^p(Q_\nu\cap\Omega)\leqq C\sum_{\alpha\in\partial\Lambda}\sum_\nu\delta_\nu^{p\alpha}\|D^\alpha f\|_p^p(\tilde Q_\nu\cap\Omega)\\ &\leqq C\sum_{\alpha\in\partial\Lambda}\delta^{p\alpha}\|D^\alpha f\|_p^p(\Omega)\leqq C\Big(\sum_{\alpha\in\partial\Lambda}\delta^\alpha\|D^\alpha f\|_p(\Omega)\Big)^p.\end{aligned}$$

This is the desired estimate (4.2).

To prove (4.3), we observe that when (3.23) holds then the functional K_Λ is equivalent to the modulus of smoothness ω_Λ (cf. Theorem 3.1). Hence, given $f \in L_p(\Omega)$, choose $g \in W_p^\Lambda(\Omega)$ so that

$$\|f-g\|_p(\Omega)+\sum_{\alpha\in\partial\Lambda}\delta^\alpha\|D^\alpha g\|_p(\Omega)\leqq 2K_\Lambda(f,\delta,\Omega)_p\leqq C\omega_\Lambda(f,\delta,\Omega)_p.$$

Now let S be a piecewise Λ-polynomial chosen to satisfy (4.2) for g. Then,

$$\|f-S\|_p(\Omega)\leqq\|f-g\|_p(\Omega)+\|g-S\|_p(\Omega)$$
$$\leqq C(\|f-g\|_p(\Omega)+\sum_{\alpha\in\partial\Lambda}\delta^\alpha\|D^\alpha g\|_p(\Omega))\leqq C\omega_\Lambda(f,\delta,\Omega)_p$$

as desired.

We now want to establish direct estimates for smooth spline approximation. We can only do this when Δ is a tensor product grid. Therefore, for the remainder of this section we assume that $\mathbf{\Delta}=\Delta_1\otimes\cdots\otimes\Delta_n$ with $\Delta_i: 0=x_i^{(0)}<x_i^{(1)}<\cdots<x_i^{(m_i)}=1$, $Q_\nu=\prod_1^n[x_i^{(\nu_i)},x_i^{(\nu_i+1)}]$, $\boldsymbol{\delta}_\nu$ the side length vector of Q_ν and $\boldsymbol{\delta}=\max_\nu\boldsymbol{\delta}_\nu$ with the maximum taken coordinatewise. Furthermore, we will concentrate on the case of coordinate degree $<\mathbf{r}$, and so $\Lambda_\mathbf{r}=\{\boldsymbol{\alpha}:\boldsymbol{\alpha}<\mathbf{r}\}$ and $\mathbb{P}_\mathbf{r}=\mathbb{P}_{\Lambda_\mathbf{r}}$. Let $\mathcal{S}_\mathbf{r}(\mathbf{\Delta})=\{S: S|_{Q_\nu}\in\mathbb{P}_\mathbf{r}$ for each ν, and $D^\alpha S\in C(I^n)$ for each $\boldsymbol{\alpha}\leqq\mathbf{r}-2\mathbf{e}\}$. Here, $I=[0,1]$ and $\mathbf{e}=(1,\cdots,1)$. Thus $\mathcal{S}_\mathbf{r}(\mathbf{\Delta})=\mathcal{S}_{r_1}(\Delta_1)\otimes\cdots\otimes\mathcal{S}_{r_n}(\Delta_n)$, where $\mathcal{S}_{r_i}(\Delta_i)$ is the set of one variable splines of degree $<r_i$, smoothness r_i-2 and knot sequence Δ_i. It is well-known that a basis for the space $\mathcal{S}_{r_i}(\Delta_i)$ is given by the B-splines N_{j,i,r_i} and any $S\in\mathcal{S}_{r_i}(\Delta_i)$ can be written as

$$S=\sum_{-r_i<j<m_i}\lambda_{j,i}(S)N_{j,i,r_i},$$

where the $\lambda_{j,i}$ are biorthogonal functionals. A basis for $\mathcal{S}_\mathbf{r}(\mathbf{\Delta})$ is then given by the splines $N_\nu(\mathbf{x})=N_{\nu_1,1,r_1}(x_i)\cdots N_{\nu_n,n,r_n}(x_n)$, and the corresponding linear functionals λ_ν can be calculated by

$$\lambda_\nu(S)=\lambda_{\nu_n,n}(\cdots\lambda_{\nu_2,2}(\lambda_{\nu_1,1}(S))\cdots),$$

where at each stage $\lambda_{\nu_k,k}$ is applied to $\lambda_{\nu_{k-1},k-1}(\cdots)$ as a function of x_k with the other variables held fixed.

The approximation scheme of de Boor–Fix [2] is described by the operators

$$L_\Delta(f)=\sum_\nu\tilde{\lambda}_\nu(f)N_\nu, \tag{4.5}$$

where $\tilde{\lambda}_\nu$ is a suitable norm preserving extension of λ_ν and the sum is taken over all ν for which supp N_ν intersects Ω. If we want L_Δ to be a projector onto $\mathcal{S}_\mathbf{r}(\mathbf{\Delta})$, then the set supp $\lambda_\nu\cap$ supp $N_\nu\cap\Omega$ must be nonempty and the norm of $\tilde{\lambda}_\nu$ on L_p is controlled by the size of this latter set. However, in general domains, supp $N_\nu\cap\Omega$ may be very small so that the norm of $\tilde{\lambda}_\nu$ will be large, making the operator L_Δ ineffective for approximation. For this reason, we will only require that $\tilde{\lambda}_\nu$ agree with λ_ν on $\mathbb{P}_\mathbf{r}$. In this way, we can move the support of $\tilde{\lambda}_\nu$ outside of supp N_ν.

Let $A_\Omega=\{\nu: Q_\nu\cap\Omega\neq\varnothing\}$ and $\tilde{A}_\Omega=\{\boldsymbol{\mu}:\boldsymbol{\nu}-\mathbf{r}\leqq\boldsymbol{\mu}\leqq\boldsymbol{\nu}+\mathbf{r}$ with $\nu\in A_\Omega\}$. Given ν, we set $\bar{Q}_\nu=$ supp $N_\nu=\bigcup\{Q_\mu:\boldsymbol{\nu}\leqq\boldsymbol{\mu}\leqq\boldsymbol{\nu}+\mathbf{r}\}$. Given a rectangle Q and a constant $K>0$, let $Q(K)$ denote the rectangle with the same center as Q but a side length vector $K\boldsymbol{\delta}_Q$, where $\boldsymbol{\delta}_Q$ is the side length vector of Q. We will assume the following property of Ω and $\mathbf{\Delta}$ for the remainder of this section.

(4.6) *Ω is coordinatewise convex (3.22), and there are constants $K_1, K_2 > 1$ such that for each $\nu \in \tilde{A}_\Omega$ there are rectangles $R_\nu \subseteq Q_\nu(K_1) \cap \Omega$, $T_\nu \subseteq \Omega$ with*
(i) $\delta_{R_\nu} \geqq K_1^{-1} \delta_\nu$,
(ii) *T_ν covers the rectangle $\bigcup_{\nu - \mathbf{r} \leqq \mu \leqq \nu} Q_\mu(|\mathbf{r}| K_1) \equiv S_\nu$ in at most length K_2 (cf. (3.28)) and $\delta_{T_\nu} \leqq K_1^{-1} \delta_{S_\nu}$.*

Note that the rectangles R_ν and T_ν need not be in Δ. Among other things, this condition will be satisfied by any coordinatewise convex domain which can be decomposed into subdomains each of which satisfy a cone property for a cone which has vertex at the origin and contains a coordinate axis, provided that the rectangles in Δ are small enough.

LEMMA 4.1. *Suppose Ω and Δ satisfy (4.6) and ν is any index with $\bar{Q}_\nu \cap \Omega \neq \varnothing$ where $\bar{Q}_\nu = \operatorname{supp} N_\nu$. Then, there is a functional $\tilde{\lambda}_\nu \in L_p^*(\Omega)$ supported on the rectangle $\bar{Q}_\nu(K_1)$ with K_1 as in (4.6), and*

$$\text{(4.7)} \qquad \begin{aligned} &\text{(i)} \quad \tilde{\lambda}_\nu(P) = \lambda_\nu(P) \quad \text{for all } P \in \mathbb{P}_{\mathbf{r}}, \\ &\text{(ii)} \quad |\tilde{\lambda}_\nu(f)| \leqq C |\bar{Q}_\nu|^{-1/p} \|f\|_p(\Omega \cap \bar{Q}_\nu(K_1)) \quad \text{for all } f \in L_p(\Omega). \end{aligned}$$

with C depending only on $\mathbf{r}$ and K_1.

Proof. Given ν with $\bar{Q}_\nu \cap \Omega \neq \varnothing$, choose Q_μ as the largest rectangle with $\nu \leqq \mu \leqq \nu + \mathbf{r}$. From [2], it follows that λ_ν can be represented as

$$\text{(4.8)} \qquad \lambda_\nu(S) = \sum_{0 \leqq \alpha \leqq r} c_\alpha D^\alpha(S)(\mathbf{a}), \qquad S \in \mathcal{S}_{\mathbf{r}}(\Delta),$$

where $\mathbf{a}$ is the center of Q_μ and c_α are constants which satisfy

$$c_\alpha \leqq C \delta_\mu^\alpha, \qquad \mathbf{0} \leqq \alpha \leqq \mathbf{r}$$

with C depending only on $\mathbf{r}$. Now, $\mu \in \tilde{A}_\Omega$, hence, according to (4.6), there is a rectangle $R_\mu \subseteq Q_\mu(K_1) \cap \Omega$ with $\delta_{R_\mu} \geqq K_1^{-1} \delta_\mu$. Let $\mathbf{a}'$ be the center of R_μ so that $\mathbf{a} - K_1 \delta_\mu \leqq \mathbf{a}' \leqq \mathbf{a} + K_1 \delta_\mu$. For any $P \in \mathbb{P}_{\mathbf{r}}$,

$$D^\alpha P(\mathbf{a}) = \sum_{0 \leqq \beta \leqq \mathbf{r} - \alpha} D^{\alpha+\beta} P(\mathbf{a}') \frac{(\mathbf{a} - \mathbf{a}')^\beta}{\beta!}.$$

Using this with (4.8), we get that, for any $P \in \mathbb{P}_{\mathbf{r}}$,

$$\text{(4.9)} \qquad \lambda_\nu(P) = \sum_{0 \leqq \beta \leqq \mathbf{r}} c'_\beta D^\beta P(\mathbf{a}')$$

and

$$\text{(4.10)} \qquad c'_\beta \leqq C \delta_\mu^\beta, \qquad \mathbf{0} \leqq \beta \leqq \mathbf{r}.$$

Now, $D^\beta P(\mathbf{a}')$ can be estimated by Markov's inequality to give

$$|D^\beta P(\mathbf{a}')| \leqq C \delta_\mu^{-\beta - (1/p)\mathbf{e}} \|P\|_p(R_\mu),$$

where $\mathbf{e} = (1, 1, \cdots, 1)$. The rectangle R_μ is contained in $\bar{Q}_\nu(K_1) \cap \Omega$ and, because of (4.6)(i), is comparable in size to $\bar{Q}_\nu(K_1)$ (recall that Q_μ was chosen as the largest rectangle with $\nu \leqq \mu \leqq \nu + \mathbf{r}$). Hence, using (4.10) and our estimates for derivatives in (4.9) gives

$$|\lambda_\nu(P)| \leqq C |\bar{Q}_\nu|^{-1/p} \|P\|_p(R_\mu) < C |\bar{Q}_\nu|^{-1/p} \|P\|_p(\Omega \cap \bar{Q}_\nu(K_1))$$

with C depending only on $\mathbf{r}$ and K_1. The Hahn–Banach theorem now guarantees the existence of an extension $\tilde{\lambda}_\nu$ of λ_ν from $\mathbb{P}_r$ to all of $L_p(\Omega)$ which will satisfy (4.7)(i) and (ii). Hence, the lemma is proved.

The approximation operator L_Δ is now defined by

$$L_\Delta(f) = \sum \tilde{\lambda}_\nu(f) N_\nu, \tag{4.11}$$

where the $\tilde{\lambda}_\nu$ are defined by Lemma 4.1, and the sum is taken over all those ν for which $(\operatorname{supp} N_\nu) \cap \Omega \neq \varnothing$.

THEOREM 4.2. *If Ω and Δ satisfy* (4.6) *and L_Δ is defined by* (4.11), *then for each $f \in W_p^{\mathbf{r}}(\Omega)$,*

$$\|f - L_\Delta(f)\|_p(\Omega) \leq C \sum_{i=1}^n \delta_i^{r_i} \|D^{r_i e_i} f\|_p(\Omega), \tag{4.12}$$

where $\boldsymbol{\delta} = \max \boldsymbol{\delta}_\nu$, the maximum being taken coordinatewise over all ν for which $(\operatorname{supp} N_\nu) \cap \Omega \neq \varnothing$. If in addition Ω satisfies (3.23) *then for each $f \in L_p(\Omega)$,*

$$\|f - L_\Delta(f)\|_p \leq C\omega_{\mathbf{r}}(f, \boldsymbol{\delta}, \Omega)_p. \tag{4.13}$$

Here, C depends only on $\mathbf{r}$, p, Ω, *and the constants of* (4.6).

Proof. Consider first (4.12). If $Q_\nu \cap \Omega \neq \varnothing$, then let T_ν be the rectangle guaranteed by (4.6). From Theorem 2.1, we know that there is a polynomial $P \in \mathbb{P}_r$ with

$$\|f - P\|_p(T_\nu) \leq C \sum_{i=1}^n \delta_{T_\nu}^{r_i e_i} \|D^{r_i e_i} f\|_p(T_\nu).$$

Now, T_ν covers $S_\nu = \bigcup_{\nu - \mathbf{r} \leq \mu \leq \nu} Q_\mu(|\mathbf{r}| K_1)$ and so the argument used before (cf. Theorem 3.2 and (4.4)) gives that

$$\|f - P\|_p(\Omega \cap S_\nu) \leq C \sum_{i=1}^n \delta_{\tilde{S}_\nu}^{r_i e_i} \|D^{r_i e_i} f\|_p(\Omega \cap \tilde{S}_\nu) \tag{4.14}$$

where $\tilde{S}_\nu = \bigcup_{\nu - r < \mu < \nu} Q_\mu(|\mathbf{r}| K_1(2K_2^2 + 1))$. When $x \in Q_\nu \cap \Omega$, we can write

$$\begin{aligned} f(\mathbf{x}) - L_\Delta(f, \mathbf{x}) &= f(\mathbf{x}) - P(\mathbf{x}) - L_\Delta(f - P, \mathbf{x}) \\ &= f(\mathbf{x}) - P(\mathbf{x}) - \sum_{\nu - \mathbf{r} \leq \mu \leq \nu} \lambda_\mu(f - P) N_\mu(\mathbf{x}) \end{aligned} \tag{4.15}$$

because of the support of the N_μ. From Lemma 4.1, we know that for any $\nu - \mathbf{r} \leq \mu \leq \nu$ we have

$$\begin{aligned} |\lambda_\mu(f - P)| &\leq C |\bar{Q}_\mu|^{-1/p} \|f - P\|_p(\Omega \cap \bar{Q}_\mu(K_1)) \\ &\leq C |\bar{Q}_\mu|^{-1/p} \|f - P\|_p(\Omega \cap S_\nu), \end{aligned} \tag{4.16}$$

where we used the fact that $\bar{Q}_\mu(K_1) \subseteq S_\nu$. If we use (4.16) in (4.15), we find

$$\begin{aligned} \|f - L_\Delta(f)\|_p(\Omega \cap Q_\nu) &\leq \|f - P\|_p(\Omega \cap Q_\nu) + C\|f - P\|_p(\Omega \cap S_\nu) \sum_{\nu - \mathbf{r} \leq \mu \leq \nu} |\bar{Q}_\mu|^{-1/p} \left(\int_{Q_\mu} N_\mu^p \right)^1 \\ &\leq C \|f - P\|_p(\Omega \cap S_\nu), \end{aligned}$$

where we used the facts that $Q_\nu \subseteq S_\nu$ and $N_\mu(\mathbf{x}) \leq 1$.

Using now (4.14), we find

$$\|f - L_\Delta(f)\|_p^p(\Omega \cap Q_\nu) \leq C \sum_{i=1}^n \|D^{r_i e_i} f\|_p^p(\Omega \cap \tilde{S}_\nu) \delta_{\tilde{S}_\nu}^{p r_i e_i}. \tag{4.17}$$

We now want to add up the estimates (4.17) over ν. To do this, observe that if Q_μ is the largest rectangle with $\nu-\mathbf{r}\leqq\mu\leqq\nu+\mathbf{r}$, then it is easy to check that $S_\nu\subseteq\tilde{Q}_\mu=Q_\mu(2|\mathbf{r}|^2K_1(2K_2^2+1))$ and $\delta_{S_\nu}\leqq C\delta_\mu$. Using this in (4.17) and summing over ν gives

$$\begin{aligned}\|f-L_\Delta(f)\|_p^p(\Omega)&\leqq C\sum_\nu\|f-L_\Delta(f)\|_p^p(\Omega\cap Q_\nu)\\&\leqq C\sum_{i=1}^n\sum_\mu\|D^{r_ie_i}f\|_p^p(\Omega\cap\tilde{Q}_\mu)\delta^{pr_ie_i}\\&\leqq C\sum_{i=1}^n\|D^{r_ie_i}f\|_p^p(\Omega)\delta^{pr_ie_i}.\end{aligned}$$

In the last inequality, we used the fact that $\sum_{\mathbf{x}}\delta_\mu^{pr_ie_i}\leqq C\delta^{pr_ie_i}$, where $\sum_{\mathbf{x}}$ denotes that the sum is taken over all those μ for which $\mathbf{x}\in\tilde{Q}_\mu$. Taking a pth root of this last inequality establishes (4.12).

The proof of (4.13) is easy. If Ω satisfies (3.23), then according to (3.23), there is a function g for which

$$\|f-g\|_p(\Omega)+\sum_{i=1}^n\delta_i^{r_i}\|D^{r_ie_i}g\|_p(\Omega)\leqq 2K_{\mathbf{r}}(f,\delta,\Omega)_p\leqq C\omega_{\mathbf{r}}(f,\delta,\Omega)_p$$

with C depending only on $\mathbf{r}$ and Ω. Thus, using (4.12), we find

$$\begin{aligned}\|f-L_\Delta(f)\|_p(\Omega)&\leqq(1+\|L_\Delta\|)\|f-g\|_p(\Omega)+\|g-L_\Delta(g)\|_p(\Omega)\\&\leqq C(\|f-g\|_p(\Omega)+\sum_{i=1}^n\delta_i^{r_i}\|D^{r_ie_i}g\|_p(\Omega))\\&\leqq C\omega_{\mathbf{r}}(f,\delta,\Omega)_p.\end{aligned}$$

Here, we have used the fact that L_Δ is a bounded operator from $L_p(\Omega)$ into $L_p(\Omega)$ (this follows easily from the estimates (4.7)(i)). This establishes (4.13).

5. Inverse theorems. We now want to examine to what extent the estimates given in the last section are sharp. There is a myriad of results which can be proven depending on the domain Ω, the set Λ, the smoothness of the splines and, most importantly, the properties of the sequence of the partitions (Δ_k). We choose to focus on two types of results, one which applies to approximation by piecewise polynomials from $\mathbb{P}_\Lambda$ and the second which applies to approximation by splines of coordinate degree $\mathbf{r}$ and fixed smoothness $\boldsymbol{\rho}$. These then complement the direct theorems of § 4.

Let us begin with the case of approximation by piecewise Λ-polynomials. Suppose (Δ_k) is a sequence of rectangular partitions (not necessarily tensor product), with δ_k the coordinatewise maximum of the sidelengths of the rectangles in Δ_k. We denote by $\mathcal{S}_\Lambda(\Delta_k)$ the space of piecewise Λ-polynomials on Δ_k. We have shown in § 4 that under certain conditions on Ω, for each $f\in L_p(\Omega)$, there is an $S_k\in\mathcal{S}_\Lambda(\Delta_k)$ with

$$\|f-S_k\|_p(\Omega)\leqq C\omega_\Lambda(f,\delta_k,\Omega)_p. \tag{5.1}$$

For example, if Ω satisfies (3.23) and if in addition (4.1) holds for each k with a constant K which is not dependent on k, then (5.1) will be valid with a constant C independent of f and k. We can show that under certain conditions on (Δ_k), a converse estimate to (5.1) is valid. That is, we will be able to estimate $\omega_\Lambda(f,\delta_k,\Omega)_p$ in terms of the errors $\|f-S_j\|_p(\Omega)$, $j\geqq k$.

Since the approximating splines satisfy no smoothness condition, it is not possible to prove any meaningful inverse theorems without some sort of a "mixing condition" on the sequence of partitions $(\mathbf{\Delta}_k)$ (see e.g., [8]). These conditions prevent a given hyperplane from appearing too often in the partitions $\mathbf{\Delta}_k$. The simplest mixing conditions arise in the case $p=\infty$. We say that $(\mathbf{\Delta}_k)$ satisfies the ∞ mixing condition (compare [8]) if

(5.2) *there is a constant $K_1>0$ such that for each k and each $\mathbf{x}\in\Omega$, there is a partition $\mathbf{\Delta}_j$, $j=j(\mathbf{x})\geqq k$, with $\mathbf{x}$ in a rectangle R of $\mathbf{\Delta}_j$ and* dist $(\mathbf{x}, \mathscr{C}R)\geqq K_1\delta_k$, *where the distance is coordinatewise distance.*

For example, if $\mathbf{\Delta}_k=\Delta_k\otimes\Delta_k\otimes\cdots\otimes\Delta_k$ with $\Delta_k=\{i/k\}_0^k$, then (5.2) is satisfied. Note also that if $(\mathbf{\Delta}_k)$ satisfies (5.2) for a rectangle then it satisfies (5.2) for any subdomain of this rectangle.

The following generalizes some results of Ju. Brudnyi [6]. Brudnyi worked with uniform partitions and total degree.

LEMMA 5.1. *Suppose that $(\mathbf{\Delta}_k)$ satisfies* (5.2) *and $f\in C(\Omega)$ with $S_k\in\mathscr{S}_\Lambda(\mathbf{\Delta}_k)$ satisfying*

$$\|f-S_k\|_\infty(\Omega)=\varepsilon_k,\qquad k=1,2,\cdots.$$

There is a constant $C>0$, independent of f or k, such that for each $k\geqq 1$,

$$\omega_\Lambda(f,\delta_k,\Omega)_\infty\leqq C\sup_{j\geqq k}\varepsilon_j.$$

Proof. If $\boldsymbol{\alpha}\in\partial\Lambda$ and $\mathbf{x}\in\Omega, \mathbf{t}\leqq|\boldsymbol{\alpha}|^{-1}K_1\delta_k$, then choose j and the rectangle R according to (5.2). We then have that S_j is in $\mathbb{P}_\Lambda$ on R, so

$$|\Delta_{\mathbf{t}}^{\alpha}(f,\mathbf{x})|=|\Delta_{\mathbf{t}}^{\alpha}(f-S_j,\mathbf{x})|\leqq C\varepsilon_j\leqq C\sup_{i\geqq k}\varepsilon_i,$$

where we have used the fact that all the points which appear in the computation of $\Delta_{\mathbf{t}}^{\alpha}(f,\mathbf{x})$ are in R, because dist $(\mathbf{x},R)\geqq K_1\delta_k$. Taking a supremum over such $\mathbf{t}$ and $\mathbf{x}\in\Omega$,

$$\omega_\alpha(f,\delta_k,\Omega)_\infty\leqq C\omega_\alpha(f,|\boldsymbol{\alpha}|^{-1}K_1\delta_k,\Omega)_\infty\leqq C\sup_{i\geqq k}\varepsilon_i,$$

and this proves the lemma because of the definition of ω_Λ.

To see what sort of information Lemma 5.1 gives, suppose that $\delta_{k+1}\leqq\delta_k$, $k=1,\cdots$ (for simplicity) and consider the case of coordinate order $\mathbf{r}$, $\mathbb{P}_\Lambda=\mathbb{P}_{\mathbf{r}}$. If $f\in C(\Omega)$ and there are splines $S_k\in\mathscr{S}_{\mathbf{r}}(\mathbf{\Delta}_k)$ with

$$\|f-S_k\|_\infty(\Omega)=o(\min_{1\leqq i\leqq n}\delta_k^{r_i e_i}),\qquad k\to\infty,$$

then $f\in\mathbb{P}_{\mathbf{r}}$, because $\omega_{r_i e_i}(f,\delta_k,\Omega)=o(\delta_k^{r_i e_i})$ implies $D^{r_i e_i}f=0$. This is a saturation result. More generally, if $s_i>0$, $1\leqq i\leqq n$ and

$$\|f-S_k\|_\infty(\Omega)=O(\min_{1\leqq i\leqq n}\delta_k^{s_i e_i}),\qquad k\to\infty,$$

then for each i, $|\Delta_{t_i}^{r_i}(f,\mathbf{x})|=O(t^{s_i})$. This last statement uses the fact that $\delta_{k+1}\geqq C\delta_k$ which follows from the condition (5.2).

The condition (5.2) is too weak to apply to L_p approximation when $p<\infty$. A sufficient condition to obtain similar results to that of Lemma 5.1 for $p<\infty$ is

(5.3) *there is a constant $K_1>0$ and an $N>0$ such that for each k, there exist integers $j_1,\cdots,j_N\geqq k$, such that for each $\mathbf{x}\in\Omega$, we can find j_i and a rectangle R from the partition $\mathbf{\Delta}_{j_i}$ with* dist $(\mathbf{x},\mathscr{C}R)\geqq K_1\delta_k$.

However, this condition excludes some interesting cases, such as the discrete equally spaced partitions $\mathbf{\Delta}_k = \Delta_k \otimes \cdots \otimes \Delta_k$ with $\Delta_k = \{i/k\}_0^k$. The equally spaced partitions dependent on a continuous parameter would satisfy an analogous condition to (5.3), however.

For $1 \leqq p < \infty$, it is possible to formulate mixing conditions dependent on p, under which inverse theorems hold. These are quite difficult to describe for general partitions and so we consider only the case of tensor product partitions, $\mathbf{\Delta}_k = \Delta_{1,k} \otimes \cdots \otimes \Delta_{n,k}$, $\mathbf{\delta}_k = (\delta_{1,k}, \cdots, \delta_{n,k})$, where $\Delta_{i,k} = \{x_{i,k}^{(j)}\}$. We say that $(\mathbf{\Delta}_k)$ satisfies the p-mixing condition, if

(5.4) *there exists $K_1 > 0$, so that for each k, we can find $a_{i,j}$ (depending on k), $i = 1, \cdots, n$, $j > k$, with $\sum_{j=k}^{\infty} a_{i,j} = 1$ and for each $\mathbf{x} \in \Omega$,*

$$\sum_{j=k} a_{i,j} \operatorname{dist}(\mathbf{x}, \mathbf{\Delta}_{i,j})^{r_i p+1} \geqq \delta_{i,k}^{r_i p+1}, \qquad 1 \leqq i \leqq n,$$

where $r_i \mathbf{e}_i \in \partial\Lambda$, $i = 1, \cdots, n$.

For example, when $\mathbf{\Delta}_k$ are the discretized equally spaced partitions as given above, then (5.4) is satisfied with $a_{i,j} = 1/k$, $i = 1, \cdots, n$, $j = k, \cdots, 2k-1$.

THEOREM 5.1. *Let $1 \leqq p \leqq \infty$ and let $(\mathbf{\Delta}_k)$ satisfy the p-mixing condition. There is a $C > 0$ such that for each $f \in L_p(\Omega)$ for which there are $S_k \in \mathscr{S}_\Lambda(\mathbf{\Delta}_k)$ with*

$$|f - S_k|_p(\Omega) = \varepsilon_k, \qquad k = 1, 2, \cdots,$$

$$\omega_\Lambda(f, \mathbf{\delta}_k, \Omega)_p \leqq C \sup_{j \geqq k} \varepsilon_j.$$

Proof. We have shown the case of $p = \infty$ in Lemma 5.1. The case of $1 \leqq p < \infty$ is proved in the same way that the one variable results for equally spaced partitions were established in [7]. We do not repeat the details.

The results given above required some sort of a mixing condition and as such excluded the interesting case of nested partitions. We want now to describe an approach to inverse theorems that supposes nothing about the sequence of partitions but instead assumes something about the approximating splines. Such a program was carried out for the one variable case in [9] and we will develop here only the details that are substantially different. We should mention that this approach could also be used to give the mixing condition results.

To describe the general feature, it is enough to consider the case of two variables. We begin with the case $Q = I^2$, $I = [0, 1]$. Let $\mathbf{\Delta}_k = \{Q_v^{(k)}\}$ be partitions of Q (not necessarily tensor product). We denote by $\mathscr{S}_\Lambda(\mathbf{\Delta}_k, \mathbf{\rho})$ the space of piecewise Λ-polynomials with respect to $\mathbf{\Delta}_k$ such that $D^\alpha S \in C(\Omega)$ whenever $S \in \mathscr{S}_\Lambda(\mathbf{\Delta}_k, \mathbf{\rho})$ and $\mathbf{\alpha} \leqq \mathbf{\rho}$. Here, $\mathbf{\rho} = -(1, 1)$ means that no smoothness is required and we continue to use the notation $\mathscr{S}_\Lambda(\mathbf{\Delta}_k) = \mathscr{S}_\Lambda(\mathbf{\Delta}_k, -(1, 1))$ in this case.

Suppose now that $f \in L_p(Q)$ and $S_k \in \mathscr{S}_\Lambda(\mathbf{\Delta}_k, \mathbf{\rho})$ with

$$\|f - S_k\|_p(Q) \leqq \varepsilon_k, \qquad k = 1, 2, \cdots.$$

We are interested in estimating the modulus of smoothness $\omega_\Lambda(f, \cdot, Q)_p$ in terms of the sequence (ε_k). We begin by observing simply that

$$\begin{aligned} \omega_\Lambda(f, \mathbf{\delta}, Q)_p &\leqq \omega_\Lambda(f - S_k, \mathbf{\delta}, Q)_p + \omega_\Lambda(S_k, \mathbf{\delta}, Q)_p \\ &\leqq C\varepsilon_k + \omega_\Lambda(S_k, \mathbf{\delta}, Q)_p. \end{aligned} \tag{5.5}$$

Hence, we are left with the task of estimating $\omega_\Lambda(S_k, \cdot, Q)_p$. We first estimate this modulus of smoothness for general S in terms of the jumps of S and its derivatives.

For a fixed partition Δ, and $S \in \mathcal{S}_\Lambda(\Delta)$, we define

$$(5.6)\qquad \begin{aligned}[S]_i(y) &= S(x_i+, y) - S(x_i-, y),\\ J_{p,1}(S, y) &= \left(\sum_{i=1}^{n(y)} |[S]_i(y)|^p\right)^{1/p},\end{aligned}$$

where $\{(x_i, y)\}_{i=1}^{n(y)}$ are the points of this form which are on the edge of some rectangle from Δ. We define $J_{p,2}(S, x)$ in a similar way. Define

$$(5.7)\qquad H_{p,1}(S) = \left(\int_0^1 |J_{p,1}(S, y)|^p \, dy\right)^{1/p}$$

with a similar definition for $H_{p,2}$.

In [9], we have shown how to estimate the smoothness of a one variable spline with knots $\{t_i\}$ in terms of the jumps in S and its derivatives. For example, for such S,

$$\begin{aligned}\Delta_h(S, x) = S(x+h) - S(x) &= \sum_{x,h} [S]_i + \int_0^h S'(x+u)\, du\\ &= \sum_{x,h} [S]_i + I_h(S', x),\end{aligned}$$

where $I_h(g, x) = \int_0^h g(x+u)\, du$ and $[S]_i = S(t_i+) - S(t_i-)$. Here $\sum_{x,h}$ means that the sum is taken over those i with $x < t_i < x+h$. A repeated application of this identity leads to the inequality

$$(5.8)\qquad |\Delta_h^r(S, x)| \leqq C \sum_{j=0}^{r-1} h^j \sum_{x,rh} |[S^{(j)}]_i| + |I_h^r(S^{(r)}, x)|.$$

We can use this inequality in our multidimensional estimates.

For our original partition Δ, let d_1 denote the smallest side length in the x direction among the rectangles of Δ. Define d_2 in a similar way.

LEMMA 5.2. *For $Q = I^2$, $S \in \mathcal{S}_\Lambda(\Delta)$ and $\boldsymbol{\alpha} \in \partial\Lambda$, we have*

$$\begin{aligned}\omega_{\boldsymbol\alpha}(S, \mathbf{t}, Q)_p \leqq C(1+t_1/d_1)^{1/p'} t_1^{1/p} &\sum_{\nu<\alpha_1} \mathbf{t}^{(\nu,0)} H_{p,1}(S^{(\nu,0)})\\ + (1+t_2/d_2)^{1/p'} t_2^{1/p} &\sum_{\nu<\alpha_2} \mathbf{t}^{(\alpha_1,\nu)} H_{p,2}(S^{(\alpha_1,\nu)}).\end{aligned}$$

Proof. Let $\boldsymbol\alpha \in \partial\Lambda$ with $\alpha_1\alpha_2 \neq 0$. The other cases are handled similarly, in fact with somewhat simpler arguments. Set $T(x, y) = \Delta_{h_2}^2(S, (x, y))$. If we apply (5.8) to T as a function of x with y fixed and $r = \alpha_1$, we find

$$(5.9)\qquad |\Delta_{\mathbf h}^{\boldsymbol\alpha}(S, (x, y))| \leqq C \sum_{\nu=0}^{\alpha_1-1} h_1^\nu \sum_{x,\alpha_1 h_1} |[T^{(\nu,0)}]_i(y)| + |I_{h_1,1}^{\alpha_1}(T^{(\alpha_1,0)}, (x, y)|$$

with the obvious interpretation of $I_{h_1,1}^{\alpha_1}$. Now $\sum_{x,\alpha_1 h_1}$ involves at most $\alpha_1(1+h_1/d_1)$ values of i for a given ν and each jump $[T^{(\nu,0)}]_i$ can be estimated by $C\sum_{j=0}^{\alpha_2} |[S^{(\nu,0)}]_i(y+jh)|$. Hence, applying L_p norms gives (see [9] for the minor details)

$$(5.10)\qquad \begin{aligned}\|\Delta_{\mathbf h}^{\boldsymbol\alpha}(S)\|_p(Q_{\boldsymbol\alpha,\mathbf h}) \leqq C(1+h_1/d_1)^{1/p'} h_1^{1/p} &\sum_{\nu=0}^{\alpha_1-1} h_1^\nu H_{p,1}(S^{(\nu,0)})\\ + Ch_1^{\alpha_1} &\|\Delta_{h_2,2}^{\alpha_2}(S^{(\alpha_1,0)})\|_p(Q_{\boldsymbol\alpha,\mathbf h})\end{aligned}$$

with $Q_{\alpha,\mathbf{h}}$ denoting the set of points $(x, y) \in Q$ for which $(x+\alpha_1 h_1, y+\alpha_2 h_2) \in Q$. In (5.10), the term $h_1^{1/p}$ appears since each jump $[S^{(\nu,0)}]_i(y)$ appears for at most an interval of length Ch_1.

If we argue in a similar way to estimate $\Delta^2_{h_2,2}(S^{(\alpha_1,0)})$ in terms of jumps in its derivatives, we find from (5.8) with x fixed,

$$|\Delta^{\alpha_2}_{h_2,2}(S^{(\alpha_1,0)}, (x, y))| \leqq C \sum_{\nu=0}^{\alpha_2-1} h_2^\nu \sum_{y,\alpha_2 h_2} |[S^{(\alpha_1,\nu)}]_j(x)|,$$

where the term involving $I^2_{h_2,2}$ does not appear because $S^{(\alpha_1,\alpha_2)}=0$. Estimating as in (5.10) gives

$$\|\Delta^{\alpha_2}_{h_2,2}(S^{(\alpha_1,0)})\|_p(Q_{\alpha,\mathbf{h}}) \leqq C(1+h_2/d_2)^{1/p'} h_2^{1/p} \sum_{\nu=0}^{\alpha_2-1} h_2^\nu H_{p,2}(S^{(\alpha_1,\nu)}).$$

When this estimate is put back into (5.10) and we take a supremum over $\mathbf{h} \leqq \mathbf{t}$, we get the desired estimate for $\omega_\alpha(f, \mathbf{t}, Q)_p$.

From this point on, we can carry out our program in a similar way to that given in [9] for the one variable case. Suppose that (Δ_k) is a sequence of partitions and $d_{1,k}$, $d_{2,k}$ are the minimum side lengths of the rectangles in Δ_k and $\mathbf{d}_k = (d_{1,k}, d_{2,k})$.

LEMMA 5.3. *If $S_k \in \mathscr{S}_\Lambda(\Delta_k)$, $k = 1, 2, \cdots$, and $\alpha \in \Lambda$, then*

$$\begin{aligned} &d_{1,k}^{1/p} \sum_{\nu<\alpha_1} \mathbf{d}_k^{(\nu,0)} H_{p,1}(S_k^{(\nu,0)}) + d_{2,k}^{1/p} \sum_{\nu<\alpha_2} \mathbf{d}_k^{(\alpha_1,\nu)} H_{p,2}(S_k^{(\alpha_1,\nu)}) \\ &\leqq \|S_k - S_{k-1}\|_p(Q) + d_{1,k}^{1/p} \sum_{\nu<\alpha_1} \mathbf{d}_k^{(\nu,0)} H_{p,1}(S_{k-1}^{(\nu,0)}) + d_{2,k}^{1/p} \sum_{\nu<\alpha_2} \mathbf{d}_k^{(\alpha_1,\nu)} H_{p,2}(S_{k-1}^{(\alpha_1,\nu)}). \end{aligned} \tag{5.11}$$

The proof of this lemma is essentially a coordinatewise version of the proof of Lemma 3 in [9] and we do not give the somewhat lengthy and technical details.

Let us now illustrate how Lemmas 5.2 and 5.3 can be used to derive inverse theorems. We restrict ourselves from here on to the case of coordinate degree $\mathbf{r} = (r_1, r_2)$. Suppose that $S_k \in \mathscr{S}_\Lambda(\Delta_k, \boldsymbol{\rho})$ are such that

$$|f - S_k|_p(Q) \leqq \varepsilon_k.$$

Take $\alpha = (r_1, 0)$ and let $\sum_k$ denote the left-hand side of (5.11) for this α. Then, (5.11) says that

$$\textstyle\sum_k \leqq (\varepsilon_k + \varepsilon_{k-1}) + (d_{1,k}/d_{1,k-1})^{\rho_1+1+1/p} \sum_{k-1}, \tag{5.12}$$

because $\|S_k - S_{k-1}\|_p(Q) \leqq \|f - S_k\|_p(Q) + \|f - S_{k-1}\|_p(Q)$ and any term $H_{p,1}(S^{(\nu,0)})$, $\nu \leqq \rho_1$ is 0. Hence, we can estimate

$$\begin{aligned} \omega_{(r_1,0)}(f, \mathbf{d}_m, Q)_p &\leqq C\varepsilon_m + \omega_{(r_1,0)}(S_m, \mathbf{d}_m, Q)_p \\ &\leqq C\varepsilon_m + \textstyle\sum_m \leqq C \displaystyle\sum_{k=1}^m d_{1,m}^{\theta_1} d_{1,k}^{-\theta_1} (\varepsilon_k + \varepsilon_{k-1}) \end{aligned} \tag{5.13}$$

with $\theta_1 = \rho_1 + 1 + 1/p$. Here, the first inequality is (5.5), the second inequality is Lemma 5.2 and the last inequality is a repeated application of (5.12) and the fact that $\sum_0 = 0 (S_0 \in \mathscr{S}_\Lambda)$. Of course, a similar estimate holds for $\alpha = (0, r_2)$, namely,

$$\omega_{(0,r_2)}(f, \mathbf{d}_m, Q)_p \leqq C \sum_{k=1}^m d_{2,m}^{\theta_2} d_{2,k}^{-\theta_2} (\varepsilon_k + \varepsilon_{k-1}) \tag{5.14}$$

with $\theta_2 = \rho_2 + 1 + 1/p$.

Suppose now further that $\mathbf{d}_k \geqq C\boldsymbol{\delta}_k$, for each k. Then, (5.13) and (5.14) show that if $\mathbf{s}<\boldsymbol{\rho}$ and $\varepsilon_k = O\,(\min \boldsymbol{\delta}_k^{s_i/e_i})$, $i = 1, 2$, then

$$\omega_{r_i e_i}(f, \mathbf{t}, Q)_p \leqq C t_i^{s_i}, \qquad i = 1, 2$$

for each $\mathbf{t} > \mathbf{0}$. We summarize these results in the following theorem which is stated for the general dimension n. As above, $\mathbf{d}_k$ is the coordinatewise minimum of the side length vectors of the rectangles from $\boldsymbol{\Delta}_k$, and $\boldsymbol{\delta}_k$ is the coordinatewise maximum of these side length vectors.

THEOREM 5.2. *Let* $(\boldsymbol{\Delta}_k)$ *be a sequence of partitions for which* $\mathbf{d}_k > C\boldsymbol{\delta}_k$ *and* $\boldsymbol{\delta}_{k+1} \geqq C\boldsymbol{\delta}_k$, $k = 1, 2, \cdots$. *Suppose that* $\mathbf{r} > \mathbf{0}$, $\mathbf{0} < \boldsymbol{\rho} < \mathbf{r} - \mathbf{e}$, $\mathbf{s} < \boldsymbol{\theta} = \boldsymbol{\rho} + (1 + 1/p)\mathbf{e}$ *and* $f \in L_p(Q)$. *If*

(i) *there exist* $S_k \in (\boldsymbol{\Delta}_k, \boldsymbol{\rho})$ *with* $\|f - S_k\|_p(Q) = O(\boldsymbol{\delta}_k^{s_i/e_i})$, $1 \leqq i \leqq n$, *then*

(ii) $\omega_{r_i e_i}(f, \mathbf{t}, Q)_p = O(\mathbf{t}^{s_i e_i})$, $\qquad 1 \leqq i \leqq n$.

If in addition $\boldsymbol{\delta}_k^{s_i/e_i} = O(\boldsymbol{\delta}_k^{s_j/e_j})$ *for all* $1 \leqq i, j \leqq n$, *then* (ii) *implies* (i).

Proof. We have shown above how (i) implies (ii). That (ii) implies (i) under the additional hypothesis is contained in Theorem 4.2 because the splines $L_{\boldsymbol{\Delta}_k}(f)$ are in $\mathscr{S}_\Lambda(\boldsymbol{\Delta}_k, \boldsymbol{\rho})$ for each $\boldsymbol{\rho} < \mathbf{r} - \mathbf{e}$.

Let us reiterate that Theorem 5.2 makes no requirements about the sequence $(\boldsymbol{\Delta}_k)$ so that it applies for example to nested sequences. If $(\boldsymbol{\Delta}_k)$ satisfies a mixing condition then, as was shown in Theorem 5.1, the inverse theorems hold with no smoothness requirement on the splines. Theorem 5.2 can not be improved in the sense that $\mathbf{s}$ can not equal $\boldsymbol{\theta}$ (see [9]).

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