EMBEDDINGS OF BESOV SPACES INTO BMO

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In this note, we want to give an inequality which can be used to establish the known embeddings of Besov spaces into BMO. For R^n, $n \geq 1$, the space BMO is defined as the set of those functions f with

$$\sup_Q \frac{1}{|Q|} \int_Q |f-f_Q| \, du < \infty, \qquad f_Q = \frac{1}{|Q|} \int_Q f \, du$$

where the sup is taken over all cubes $Q \subseteq R^n$. It is better to describe BMO in terms of the sharp function $f^\#$ introduced by Fefferman and Stein

$$f^\#(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f-f_Q| \, du.$$

So, BMO is the set of those f for which $f^\#$ is in L_∞ and $\|f\|_{BMO} = \|f^\#\|_\infty$.

THEOREM 1. If $n = 1$, and $f \in L_p(R)$, then

$$(1) \qquad f^{\#*}(t) \leq C \sup_{s \geq t} s^{-1/p} \omega(f,s)_p, \qquad t > 0,$$

where $\omega(f,\cdot)_p$ is the L_p modulus of continuity of f and $f^{\#*}$ is the nonincreasing rearrangement of $f^{\#}$.

There are related inequalities for $n > 1$ but their statement and proof is more involved, so they will be reported on in a more comprehensive work with C. Bennett and R. Sharpley.

Recall that the Besov space $B_p^{\alpha,a}$ is defined as the set of those functions f for which

$$|f|_{B_p^{\alpha,a}} \equiv \left(\int_0^\infty \left(t^{-\alpha}\omega_k(f,t)_p\right)^a \frac{dt}{t}\right)^{1/a} < \infty, \quad \alpha < k$$

with the usual change to a sup when $a=\infty$. Thus the inequality (1) shows that $B_p^{1/p,\infty} \subset \mathrm{BMO}$.

Inequality (1) is typical of inequalities which give embeddings in that it compares the two analytic quantities $f^{\#}$ and $\omega(f,\cdot)_p$ which describe the corresponding spaces BMO and $B_p^{1/p,\infty}$. A similar inequality is

$$(2) \quad \omega_r(f,t)_q \leq C \int_0^t s^{-\theta}\omega_r(f,s)_p \frac{ds}{s}, \qquad \theta = n/p - n/q, \ t > 0.$$

This inequality gives the embeddings $B_p^{\alpha+\theta,a} \subset B_q^{\alpha,a}$ (see [3]). Another inequality of this type is

$$(3) \qquad f^{**}(t) \leq C\left(\|f\|_p + \int_t^\infty s^{-n/p}\omega_n(f,s)_p \frac{ds}{s}\right).$$

Here, $f^{**}(t) = t^{-1}\int_0^t f^*(s)\,ds$, the second rearrangement of f. This inequality gives the embeddings $B_p^{\theta,a} \subset L_{q,a}$ of Besov spaces into Lorentz spaces.

More insight into the nature of the inequalities (1), (2), and (3) can be gotten by viewing them in terms of the Peetre K functional. Indeed, all the quantities that appear in these inequalities are K functionals since

$$(4) \qquad \text{(i)} \quad K(f, t, L_1, L_\infty) = tf^{**}(t),$$

$$\text{(ii)} \quad K(f, t^k, L_p, W_p^k) \sim \omega_k(f, t)_p,$$

$$\text{(iii)} \quad K(f, t, L_1, BMO) \sim tf^{\#*}(t).$$

The identity (i) is a classical result of Peetre and Oklander (see [2]) while (ii) was recently proved for smooth domains $\Omega \subseteq R^n$ by Johnen and Scherer [4] and (iii) is due to Bennett and Sharpley [1]. The point of (4) is that it shows that each of the inequalities (1), (2), and (3) can be viewed as a comparison of two K functionals, a point of view which is expanded upon in [3].

Proof of Theorem 1. We shall use the results on K-functionals given in (4). Fix $t > 0$. From (4) (ii), there is a function φ such that

$$(5) \qquad \text{(i)} \quad \|f-\varphi\|_p \leq C\omega(f, t)_p ,$$

$$\text{(ii)} \quad \|\varphi'\|_p \leq Ct^{-1}\omega(f, t)_p .$$

It follows that

$$(6) \qquad \omega(\varphi, s)_p \leq C \begin{cases} st^{-1}\omega(f, t)_p, & s < t, \\ \omega(f, s)_p, & s \geq t. \end{cases}$$

Now $\|g\|_{BMO} \leq 2\|g\|_\infty$ for $g \in L_\infty$, and so

$$K(f-\varphi, t, L_1, BMO) \leq 2\,K(f-\varphi, t, L_1, L_\infty) = \int_0^t (f-\varphi)^*(s)\,ds$$

$$\leq t^{1/p'} \|(f-\varphi)^*\|_p \leq C\,t^{1/p'}\omega(f, t)_p$$

by (5)(i) with $1/p + 1/p' = 1$. Using this last inequality with (4) (iii) shows that

$$(7) \qquad (f-\varphi)^{**}(t) \leq C\, t^{-1} K(f-\varphi, t, L_1, BMO) \leq C\, t^{-1/p} \omega(f,t)_p.$$

We now need to estimate $\varphi^{**}(t)$. To do this let I be any interval, then

$$|\varphi(u) - \varphi_I(u)| \leq \frac{1}{|I|} \int_I |\varphi(v) - \varphi(u)|\, dv \leq \frac{1}{|I|} \int_{-|I|}^{|I|} |\varphi(u+s) - \varphi(u)|\, ds .$$

Integrating now with respect to u and applying Hölder's inequality gives

$$(8) \qquad \frac{1}{|I|} \int_I |\varphi(u) - \varphi_I(u)|\, du$$

$$\leq |I|^{-2} \int_{-|I|}^{|I|} \int_I |\varphi(u+s) - \varphi(u)|\, du\, ds$$

$$\leq |I|^{-2-1/p'} \int_{-|I|}^{|I|} \left(\int_I |\varphi(u+s) - \varphi(u)|^p du \right)^{1/p} ds$$

$$\leq |I|^{-1/p} \omega(\varphi, |I|)_p .$$

Now, if $|I| > t$, then (6) and (8) show that

$$(9) \qquad \frac{1}{|I|} \int_I |\varphi(u) - \varphi_I(u)|\, du \leq |I|^{-1/p} \omega(f, |I|)_p \leq \sup_{s \geq t} s^{-1/p} \omega(f, s)_p.$$

On the other hand, if $|I| < t$, then (6) and (8) show that

$$(10)\quad \frac{1}{|I|}\int_I |\varphi(u)-\varphi_I(u)|\,ds \leq C|I|^{1-1/p}\; t^{-1}\omega(f,t)_p \leq C\, t^{-1/p}\omega(f,t)_p.$$

These two inequalities show that

$$(11)\qquad \varphi^{\#*}(t) \leq \|\varphi^{\#}\|_{\infty} = \|\varphi\|_{BMO} \leq \sup_{s\geq t} s^{-1/p}\omega(f,s)_p .$$

For any functions g and h we have $(g+h)^{\#} \leq g^{\#}+h^{\#}$ and $(g+h)^{*}(t) \leq g^{*}(t/2) + h^{*}(t/2)$. So, (7) and (11) combine to show that

$$f^{\#*}(t) \leq \varphi^{\#*}(t/2) + (f-\varphi)^{\#*}(t/2) \leq C \sup_{s\geq t/2} s^{-1/p}\omega(f,s)_p$$

$$\leq C' \sup_{s\geq t} s^{-1/p}\omega(f,s)_p$$

because of the monotonicity of ω . This proves the theorem.

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