EMBEDDINGS OF BESOV SPACES INTO BMO

R. DeVore

Department of Mathematics, Computer Science and Statistics, University of South Carolina, Columbia, SC, USA

In this note, we want to give an inequality which can be used to establish the known embeddings of Besov spaces into BMO. For R^n, $n \geqslant 1$, the space BMO is defined as the set of those functions f with

$$\sup_Q \frac{1}{|Q|} \int_Q |f-f_Q| \, du < \infty, \qquad f_Q = \frac{1}{|Q|} \int_Q f \, du$$

where the sup is taken over all cubes $Q \subseteq R^n$. It is better to describe BMO in terms of the sharp function $f^{\#}$ introduced by Fefferman and Stein

$$f^{\#}(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f-f_Q| \, du.$$

So, BMO is the set of those f for which $f^{\#}$ is in L_∞ and $\|f\|_{BMO} = \|f^{\#}\|_\infty$.

THEOREM 1. If $n = 1$, and $f \in L_p(R)$, then

$$(1) \qquad f^{\#*}(t) \leqslant C \sup_{s \geqslant t} s^{-1/p} \omega(f,s)_p, \qquad t > 0,$$

where $\omega(f,.)_p$ is the L_p modulus of continuity of f and $f^{\#*}$ is the nonincreasing rearrangement of $f^{\#}$.

There are related inequalities for $n > 1$ but their statement and proof is more involved, so they will be reported on in a more comprehensive work with C. Bennett and R. Sharpley.

Recall that the Besov space $B_p^{\alpha,a}$ is defined as the set of those functions f for which

$$|f|_{B_p^{\alpha,a}} \equiv \left(\int_0^\infty \left(t^{-\alpha}\omega_k(f,t)_p\right)^a \frac{dt}{t} \right)^{1/a} < \infty, \quad \alpha < k$$

with the usual change to a sup when $a = \infty$. Thus the inequality (1) shows that $B_p^{1/p,\infty} \subset \mathrm{BMO}$.

Inequality (1) is typical of inequalities which give embeddings in that it compares the two analytic quantities $f^{\#}$ and $\omega(f,.)_p$ which describe the corresponding spaces BMO and $B_p^{1/p,\infty}$. A similar inequality is

$$(2) \quad \omega_r(f,t)_q \leq C \int_0^t s^{-\theta}\omega_r(f,s)_p \frac{ds}{s}, \qquad \theta = n/p - n/q,\ t > 0.$$

This inequality gives the embeddings $B_p^{\alpha+\theta,a} \subset B_q^{\alpha,a}$ (see [3]). Another inequality of this type is

$$(3) \qquad f^{**}(t) \leq C\left(\|f\|_p + \int_t^\infty s^{-n/p}\omega_n(f,s)_p \frac{ds}{s}\right).$$

Here, $f^{**}(t) = t^{-1} \int_0^t f^*(s)\,ds$, the second rearrangement of f. This inequality gives the embeddings $B_p^{\theta,a} \subset L_{q,a}$ of Besov spaces into Lorentz spaces.

More insight into the nature of the inequalities (1), (2), and (3) can be gotten by viewing them in terms of the Peetre K functional. Indeed, all the quantities that appear in these inequalities are K functionals since

(4) $\quad$ (i) $K(f,t,L_1,L_\infty) = tf^{**}(t)$,

(ii) $K(f,t^k,L_p,W_p^k) \sim \omega_k(f,t)_p$,

(iii) $K(f,t,L_1,BMO) \sim tf^{**}(t)$.

The identity (i) is a classical result of Peetre and Oklander (see [2]) while (ii) was recently proved for smooth domains $\Omega \subseteq R^n$ by Johnen and Scherer [4] and (iii) is due to Bennett and Sharpley [1]. The point of (4) is that it shows that each of the inequalities (1), (2), and (3) can be viewed as a comparison of two K functionals, a point of view which is expanded upon in [3].

Proof of Theorem 1. We shall use the results on K-functionals given in (4). Fix $t>0$. From (4) (ii), there is a function φ such that

(5) $\quad$ (i) $\|f-\varphi\|_p \leqslant C\omega(f,t)_p$.

(ii) $\|\varphi'\|_p \leqslant Ct^{-1}\omega(f,t)_p$.

It follows that

$$(6) \qquad \omega(\varphi,s)_p \leqslant C\begin{cases} st^{-1}\omega(f,t)_p, & s<t, \\ \omega(f,s)_p, & s\geqslant t. \end{cases}$$

Now $\|g\|_{BMO} \leqslant 2\|g\|_\infty$ for $g\in L_\infty$, and so

$$K(f-\varphi,t,L_1,BMO) \leqslant 2\,K(f-\varphi,t,L_1,L_\infty) = \int_0^t (f-\varphi)^*(s)\,ds$$
$$\leqslant t^{1/p'}\|(f-\varphi)^*\|_p \leqslant C\,t^{1/p'}\omega(f,t)_p$$

by (5)(i) with $1/p + 1/p' = 1$. Using this last inequality with (4) (iii) shows that

$$(7) \qquad (f-\varphi)^{**}(t) \leq C\, t^{-1} K(f-\varphi, t, L_1, BMO) \leq C\, t^{-1/p} \omega(f,t)_p.$$

We now need to estimate $\varphi^{**}(t)$. To do this let I be any interval, then

$$|\varphi(u) - \varphi_I(u)| \leq \frac{1}{|I|} \int_I |\varphi(v) - \varphi(u)|\, dv \leq \frac{1}{|I|} \int_{-|I|}^{|I|} |\varphi(u+s) - \varphi(u)|\, ds.$$

Integrating now with respect to u and applying Hölder's inequality gives

$$(8) \qquad \frac{1}{|I|} \int_I |\varphi(u) - \varphi_I(u)|\, du$$

$$\leq |I|^{-2} \int_{-|I|}^{|I|} \int_I |\varphi(u+s) - \varphi(u)|\, du\, ds$$

$$\leq |I|^{-2-1\, p'} \int_{-|I|}^{|I|} \left(\int_I |\varphi(u+s) - \varphi(u)|^p du \right)^{1/p} ds$$

$$\leq |I|^{-1/p} \omega(\varphi, |I|)_p.$$

Now, if $|I| > t$, then (6) and (8) show that

$$(9) \qquad \frac{1}{|I|} \int_I |\varphi(u) - \varphi_I(u)|\, du \leq |I|^{-1/p} \omega(f, |I|)_p \leq \sup_{s \geq t} s^{-1/p} \omega(f, s)_p.$$

On the other hand, if $|I| < t$, then (6) and (8) show that

(10) $$\frac{1}{|I|}\int_I |\varphi(u)-\varphi_I(u)|\,ds \leqslant C|I|^{1-1/p}\ t^{-1}\omega(f,t)_p \leqslant C\ t^{-1/p}\omega(f,t)_p.$$

These two inequalities show that

(11) $$\varphi^{**}(t) \le \|\varphi^{\#}\|_{\infty} \approx \|\varphi\|_{BMO} \leqslant \sup_{s \geqslant t} s^{-1/p}\omega(f,s)_p .$$

For any functions g and h we have $(g+h)^{\#} \leqslant g^{\#}+h^{\#}$ and $(g+h)^{*}(t) \leqslant g^{*}(t/2) + h^{*}(t/2)$. So, (7) and (11) combine to show that

$$f^{**}(t) \leqslant \varphi^{**}(t/2) + (f-\varphi)^{**}(t/2) \leqslant C \sup_{s \geqslant t/2} s^{-1/p}\omega(f,s)_p$$

$$\leqslant C' \sup_{s \geqslant t} s^{-1/p}\omega(f,s)_p$$

because of the monotonicity of ω . This proves the theorem.

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