VARIABLE KNOT, VARIABLE DEGREE SPLINE APPROXIMATION TO x^β

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We determine the asymptotic rate of approximation of the function x^β by splines with free knots and variable degree in terms of the total number of free parameters.

0. INTRODUCTION

There are several examples of functions, which can be approximated more efficiently by splines with variable knots than by splines with fixed knots. These functions generally have singularities and the knots are bunched near the singularities [4], [2]. Another possible way to improve the approximation of a function is by varying the degree of the polynomials which make up the approximating spline. An initial study of problems of this type was made in [3]. Here, we want to consider the very special problem of approximating the function x^β, $\beta > 0$ by piecewise polynomials with variable knots and degree. Our main result is to determine the asymptotic behavior of the degree of approximation to x^β in terms of the number of free parameters used in the approximation.

If $f \in C[a,b]$, let $E_n(f) = E_n(f,[a,b])$ denote the error in approximating f by polynomials of <u>order</u> n (degree $n-1$) i.e.

$$(0.1) \quad E_n(f) = \inf_{P \in \mathbb{P}_n} \|f-P\|_\infty [a,b]$$

where $\mathbb{P}_n$ is the space of polynomials of order n. Given $\underline{x} = (x_i)_0^m$, $0 = x_m < x_{m-1} < \cdots < x_o = 1$ and integers $n_0, \ldots, n_{m-1}$, let $S_{\underline{n},\underline{x}}$ denote the space of piecewise polynomials with knots $\underline{x}$ and order $\underline{n} = (n_0, \ldots, n_{m-1})$. If $S \in S_{\underline{n},\underline{x}}$ then S is a polynomial of order n_i on

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$[x_{i+1}, x_i]$, $i=0,\ldots,m-1$. If N is any positive integer, set

$$S_N = \bigcup_{\substack{\underline{n},\underline{x} \\ \sum n_i = N}} S_{\underline{n},\underline{x}} .$$

We are interested in giving asymptotic estimates for

$$E_N(f) = \inf_{S \in S_N} \|f - S\|_\infty [0,1]$$

for certain functions f. This involves finding asymptotically optimal knots $\underline{x}$ and order $\underline{n}$ so that $\sum n_i = N$ and

$$E_{\underline{n},\underline{x}}(f) \equiv \inf_{S \in S_{\underline{n},\underline{x}}} \|f - S\|_\infty [0,1] \sim E_N(f) .$$

Our main result is to show that for $f(x) = x^\beta$, $\beta > 0$, there are constants C_1 and $C_2 > 0$ depending only on β for which

$$(0.2) \quad C_2 N^{-2\beta-1} \delta^{2\sqrt{N\beta}} \le E_N(x^\beta) \le C_1 \delta^{2\sqrt{N\beta}}$$

with $\delta = \sqrt{2} - 1$. Thus, $E_N(x^\beta) \sim e^{-\gamma_\beta \sqrt{N} + 0(\log N)}$, $\gamma_\beta = -2\sqrt{\beta} \log \delta$.

1. POLYNOMIAL APPROXIMATION OF x^β

We shall need some rather fine estimates for $E_n(x^\beta;[a,b])$, especially how this quantity depends on $[a,b]$. We follow the ideas described in the classical book of Achieser [1] but take care to find the dependence of the constants on $[a,b]$.

LEMMA 1. *If* $0 < \beta < 1$, *there are constants* $C_1, C_2 > 0$ *depending only on* β, *such that for each* $n > 0$ *and each interval* $[a,b]$

$$(1.1) \quad C_1(a+b)^\beta n^{-2\beta-1} e^{-n\gamma} \le E_n(x^\beta,[a,b]) \le C_2(a+b)^\beta n^{-\beta-1} e^{-n\gamma}$$

with γ *defined by* $\cosh\gamma = \frac{b+a}{b-a}$.

Proof. By translation, we have for $\phi(t) = \left(t+\frac{b+a}{b-a}\right)^\beta$

$$E_n(x^\beta,[a,b]) = \left(\frac{b-a}{2}\right)^\beta E_n(\phi,[-1,1]) .$$

This gives via a comparison between L_2 and L_∞ norms that

$$(1.2) \quad \left(\frac{b-a}{2}\right)^\beta \left(\tfrac{1}{2} \sum_n^\infty |A_k|^2\right)^{\frac{1}{2}} \le E_n(x^\beta,[a,b]) \le \left(\frac{b-a}{2}\right)^\beta \sum_n^\infty |A_k|$$

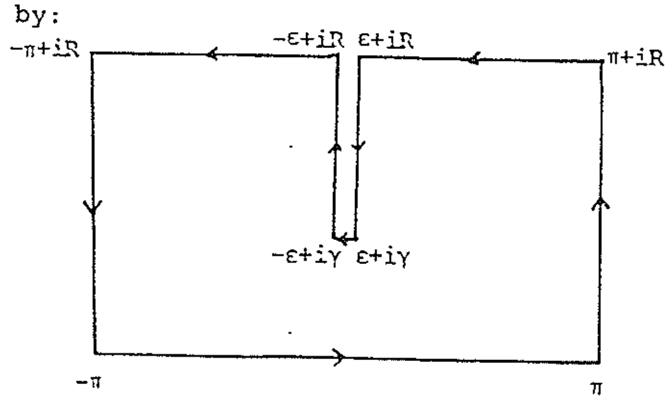
with the A_k the Fourier-Chebyskev coefficients of ϕ.
Making the substitution $t = \cos\theta$ and $\frac{b+a}{b-a} = \cosh\gamma$, we find

$$(1.3) \quad A_k = \frac{2}{\pi}\int_{-1}^{1} \phi(t)\, T_k(t) \frac{dt}{\sqrt{1-t^2}}$$

$$= \frac{2}{\pi}\int_0^\pi [\cos\theta + \cosh\gamma]^\beta \cos k\theta \, d\theta$$

$$= \frac{2(-1)^k}{\pi}\int_{-\pi}^{0} [\cosh\gamma - \cos\theta]^\beta \cos k\theta \, d\theta$$

This leads us to evaluate A_k by means of contour integration. Let $L(z) = \log|z| + i\,\mathrm{Arg}(z)$ be the branch of $\log(z)$ with branch cut the negative real axis $f(z) = e^{\beta L(\cosh\gamma - \cos z)}$. Then f is analytic inside the curve Γ described by:



Hence,

$$(1.4) \quad \int_\Gamma f(z) e^{ikz}\, dz = 0 .$$

Since $f(z)\cos kz$ has period 2π,

$$(1.5)\quad \int_{\pi}^{\pi+iR} + \int_{-\pi+iR}^{-\pi} f(z)e^{ikz}\,dz = 0 .$$

Also, for $k \geq 1$,

$$\sup_{-\pi \leq x \leq \pi} |f(x+iR)e^{ik(x+iR)}| \to 0 \qquad (R \to \infty)$$

so that

$$(1.6)\quad \int_{\pi+iR}^{\varepsilon+iR} + \int_{-\varepsilon+iR}^{-\pi+iR} f(z)e^{ikz}\,dz \to 0 \qquad \varepsilon \to 0 , R \to \infty .$$

Using (1.5) and (1.6) back in (1.4) and taking a limit as $\varepsilon \to 0$, $R \to \infty$, shows that

$$\begin{aligned}(1.7)\quad A_k &= \frac{(-1)^k}{\pi}\int_{-\pi}^{\pi} f(z)e^{ikz}\,dz \\ &= \frac{(-1)^k}{\pi}\int_{\gamma}^{\infty} e^{\pi i\beta}\,(\cosh y - \cosh\gamma)^{\beta}\, e^{-ky}\, i\,dy \\ &- \frac{(-1)^k}{\pi}\int_{\gamma}^{\infty} e^{-\pi i\beta}\,(\cosh y - \cosh\gamma)^{\beta}\, e^{-ky}\, i\,dy \\ &= \frac{2(-1)^{k+1}}{\pi}\sin\pi\beta\int_{\gamma}^{\infty}(\cosh y - \cosh\gamma)^{\beta}\, e^{-ky}\,dy \\ &= \frac{2(-1)^{k+1}}{\pi}\sin\pi\beta\, e^{-k\gamma}\int_{0}^{\infty}(\cosh(y+\gamma) - \cosh\gamma)^{\beta}\, e^{-ky}\,dy\end{aligned}$$

For $y \geq 1$, $\sinh\frac{y}{2}\sinh(\gamma+\frac{y}{2}) \leq \frac{1}{4}e^{\gamma+y}$ and for $y \leq 1$, $\sinh\frac{y}{2} \leq \frac{1}{2}ye^{y}$. Therefore, when $y > 0$

$$\cosh(y+\gamma) - \cosh\gamma = 2\sinh\frac{y}{2}\sinh(\gamma+\frac{y}{2}) \leq y\, e^{\frac{y}{2}}\, e^{\gamma+\frac{y}{2}} .$$

Using this in (1.7) gives the estimate for A_k

(1.8) $$|A_k| \le \frac{2}{\pi} e^{-k\gamma} . e^{\gamma\beta} \int_0^\infty y^\beta e^{-(k-\beta)y} dy$$

$$\le e^{\gamma\beta} e^{-k\gamma} (k-\beta)^{-\beta-1} \Gamma(\beta)$$

this gives the upper estimate (using (1.2))

$$E_n(x^\beta,[a,b]) \le C(b-a)^\beta e^{\gamma\beta} \sum_{k=n}^{\infty} (k-\beta)^{-\beta-1} e^{-k\gamma}$$

$$\le C(b-a)^\beta \left(\frac{b+a}{b-a}\right)^\beta (n-\beta)^{-\beta-1} e^{-n\gamma} \le C(a+b)^\beta n^{-\beta-1} e^{-n\gamma}$$

where we have used the definition of γ. This is the right-hand side of (1.1).

To prove the left-hand side of (1.1), we use the estimate

$$\cosh(y+\gamma) - \cosh\gamma = 2 \sinh\frac{y}{2} \sinh(\gamma+\frac{y}{2})$$

$$= 2 \sinh\frac{y}{2} [\sinh\gamma \cosh\frac{y}{2} + \cosh\gamma \sinh\frac{y}{2}]$$

$$\ge 2[\sinh\frac{y}{2}]^2 \cosh\gamma \ge y^2 \cosh\gamma$$

Hence, from (1.7),

$$|A_k| \ge \frac{2 \sin \pi\beta}{\pi} e^{-k\gamma}(\cosh\gamma)^\beta \int_0^\infty y^{2\beta} e^{-ky} dy$$

$$\ge Ce^{-k\beta} k^{-2\beta-1}(\cosh\gamma)^\beta$$

where C depends only on β. Using this in (1.2) gives

$$E_n(x^\beta,[a,b]) \ge (a+b)^\beta \left(\sum_n^\infty e^{-2k\gamma} k^{-4\beta-2}\right)^{\frac{1}{2}} \ge C(a+b)^\beta e^{-n\gamma} n^{-2\beta-1}$$

where again we used the definition of γ. This is the lower estimate in (1.1), so Theorem 1 is proved.

2. MAIN RESULTS

The estimates of the last section can be used to determine the asymptotic behavior of $E_N(x^\beta)$. For this purpose we will need the following interesting lemma.

LEMMA 2. If $\delta = \sqrt{2} - 1$, then for all $x > 0$

$$\left(\frac{1-\delta^x}{1+\delta^x}\right)^x \geq \frac{1-\delta}{1+\delta} = \delta .$$

Proof. Consider the function $f(x) = \left(\frac{1-e^{-x}}{1+e^{-x}}\right)^x$ and with $y = e^x$ the related function $g(y) = \left(\frac{1-1/y}{1+1/y}\right)^{\ln y} = \left(\frac{y-1}{y+1}\right)^{\ln y}$. The function f has a minimum on $(0,\infty)$ at x_0 if and only if g has a minimum on $(1,\infty)$ at $y_0 = e^{x_0}$. Let $G(y) = \ln(g(y)) = \ln y \ \ln\left(\frac{y-1}{y+1}\right)$. Then, G and g achieve their minima at the same points. To find the minima of G , we note that $G(y) < 0$ on $(1,\infty)$ and $\lim_{y \to 1} G(y) = \lim_{y \to \infty} G(y) = 0$. Thus G achieves its minima at points y for which

$$(2.1) \quad G'(y) = \left(\frac{2}{y^2-1}\right) \ln y + \frac{1}{y} \ln\left(\frac{y-1}{y+1}\right) = 0 .$$

There is symmetry in the function G which can be seen by using the transformation

$$(2.2) \quad z = \frac{y+1}{y-1} \quad \text{for} \quad y > 1 .$$

Then $y = \frac{z+1}{z-1}$ and $G'\left(\frac{y+1}{y-1}\right) = -\frac{(y-1)^2}{2} G'(y)$, so that $G'(y) = 0$ if and only if $G'\left(\frac{y+1}{y-1}\right) = 0$.

It is easy to check that the fixed point $y_0 = \sqrt{2} + 1$ of (2.2) satisfies $G'(y_0) = 0$. In view of the symmetry noted above we can show that y_0 is the unique minimum of G if we can show that $G'(y) \geq 0$ for $y \geq y_0$, or in view of (2.1), it is enough to show that

$$(2.6)\quad E_{n_i}(x^\beta,[x_{i+1},x_i]) \le C\delta^{2i\beta}(\phi(\delta^2))^{n_i} = C\delta^{2i\beta+n_i}$$

$$\le C\delta^{-1}\,\delta^{2m\beta} \le C\delta^{2m\beta}, \quad i < m .$$

For $i = m$, we have from Lemma 1,

$$E_1(x^\beta,[0,x_m]) \le C\delta^{2m\beta} .$$

Thus

$$(2.7)\quad E_{\underline{n},\underline{x}}(x^\beta) \le C\delta^{2m\beta} .$$

Now, we want to check how large $\sum n_i$ is. Let $s = [\frac{1}{\beta}]$. If $2(m-i)\beta < 2$ then $(m-i) < s$. Hence,

$$\sum_{i=1}^{m+1} n_i \le n_{m+1} + \sum_{\beta(m-i)<1} n_i + \sum_{\beta(m-i)\ge 1} n_i$$

$$\le 1 + s + 2\beta\sum_{s}^{m} j \le 1 + s + 2\beta\left[\frac{m(m+1)}{2} - \frac{s(s+1)}{2}\right]$$

$$\le (s+1)(1-\beta s) + \beta m(m+1) \le (s+1)(1-\frac{s}{s+1}) + \beta m(m+1)$$

$$\le 1 + \beta m(m+1) .$$

Thus if $N \ge \beta m(m+1) + 1$ then because of (2.6) and (2.7)

$$E_N(x^\beta) \le C\delta^{2m\beta} .$$

Consider any $N \ge \beta^{-1}$. Choose m as the largest integer with $\beta m(m+1) \le N - 1$. Then $m \ge \sqrt{\frac{N}{\beta}} - 2$, and so

$$E_N(x^\beta) \le C\delta^{2m\beta} \le C\delta^{2(\sqrt{\frac{N}{\beta}}-2)\beta} \le C\delta^{2\sqrt{N\beta}} .$$

This inequality is automatically satisfied when $N \le \beta^{-1}$, so the upper estimate in (2.4) is proved.

$$H(y) = 2y \ln y - (y^2-1)\ln\left(\frac{y+1}{y-1}\right) \geq 0 \qquad y \geq y_0 .$$

This is true when $y = y_0$ and so it is enough to check $H'(y)$. We find

$$(2.3) \quad H'(y) = 4 + 2 \ln y - 2y \ln \frac{y+1}{y-1} \geq 4 + 2 \ln y - \frac{2y}{y-1} ,$$

where we have used the fact that $\ln\left(\frac{y+1}{y-1}\right) = \ln(1+\frac{2}{y-1}) \leq \frac{2}{y-1}$, $y > 1$. Now the right-hand side of (2.3) is > 0 when $y = y_0$ and is increasing for $y > y_0$, so that $H'(y) > 0$ and in turn $H(y) \geq 0$, $y \geq y_0$, as desired.

We have shown that the function f achieves a unique minimum when $x = \ln(\sqrt{2}+1)$. Thus, the function $\left(\frac{1-\delta^x}{1+\delta^x}\right)^x = \left(\frac{1-e^{x \ln \delta}}{1+e^{x \ln \delta}}\right)^x$ achieves its unique minimum when $x \ln \delta = -\ln(\sqrt{2}+1) = \ln(\sqrt{2}-1) = \ln \delta$, that is when $x = 1$. This proves the lemma.

We can now state and prove our main result.

THEOREM 1. <u>If</u> $\delta = \sqrt{2}-1$, <u>then there exist constants</u> C_1 <u>and</u> $C_2 > 0$ <u>depending only on</u> β <u>for which</u>

$$(2.4) \quad C_1 N^{-2\beta-1} \delta^{2\sqrt{N\beta}} \leq E_N(x^\beta) \leq C_2 \delta^{2\sqrt{N\beta}} , \quad 0 < \beta < 1 , N \geq 1 .$$

<u>Proof</u>. We prove first the upper estimate. For any interval $I = [a,b]$, we have from Lemma 1 that

$$(2.5) \quad E_n(x^\beta, I) \leq Cb^\beta e^{-n\gamma} \leq Cb^\beta \left(\frac{b-a}{a+b+2\sqrt{ab}}\right)^n \leq Cb^\beta (\phi(\lambda))^n$$

where $\phi(\lambda) = \frac{1-\sqrt{\lambda}}{1+\sqrt{\lambda}}$ and $\lambda = \frac{a}{b}$.

With $\delta = \sqrt{2}-1$ as in Lemma 2, we have $\phi(\delta^2) = \delta$. For any integer $m > 0$, consider $\underline{x}$ with $x_i = \delta^{2i}$, $i = 0,1,\ldots,m$, $x_{m+1} = 0$ and $\underline{n}$ with $n_i = 1$ if $2(m-i)\beta < 2$ and $[2(m-i)\beta]$ otherwise. From (2.5), we have

To prove the lower estimate in (2.4), we will use the estimates (Lemma 1)

$$(2.8)\quad E_n(x^\beta,[a,b]) \geq Cn^{-2\beta-1}(a+b)^\beta\left(\frac{b-a}{b+a+2\sqrt{ab}}\right)^n \geq C_0 n^{-2\beta-1}\, b^\beta\, \phi(\lambda)\ ,$$

when $a > 0$

$$(2.9)\quad E_n(x^\beta,[0,b]) \geq C_0 n^{-2\beta-1}\, b^\beta$$

where ϕ and λ are as before and C_0 depends at most on β .

Suppose now that $\underline{x},\underline{n}$ are such that $N = \sum_0^k n_i$ and

$$(2.10)\quad E_{\underline{n},\underline{x}}(x^\beta) \leq C_0 N^{-2\beta-1}\, \delta^{2\beta m}$$

with $\delta = \sqrt{2}-1$ and C_0 the constant in (2.8) and (2.9). We will estimate N from below in terms of m . Write $x_i = \delta^{2s_i}$, $i = 0,1,\ldots,k$. From (2.8) and (2.10) we find

$$(2.11)\quad n_i^{-2\beta-1}\, \delta^{2s_i\beta}\, \phi(\delta^{2(s_{i+1}-s_i)})^{n_i} \leq N^{-2\beta-1}\, \delta^{2\beta m}\ ,$$

$i = 0,\ldots,k-1$.

Now each $n_i \leq N$ and

$$\phi\left(\delta^{2(s_{i+1}-s_i)}\right)^{n_i} = \left(\frac{1-\delta^{(s_{i+1}-s_i)}}{1+\delta^{(s_{i+1}-s_i)}}\right)^{n_i\left(\frac{s_{i+1}-s_i}{s_{i+1}-s_i}\right)}$$

$$\geq \delta^{\frac{n_i}{s_{i+1}-s_i}} \qquad i = 0,\ldots,k-1$$

because of Lemma 2. Hence, (2.11) implies that

$$2s_i\beta + \frac{n_i}{(s_{i+1}-s_i)} \geq 2\beta m\ , \qquad i = 0,\ldots,k-1\ .$$

That is,

$$(2.12)\quad n_i \geq 2\beta(m-s_i)_+(s_{i+1}-s_i)\ , \qquad i = 0,\ldots,k-1$$

with the usual notation $x_+ = \max(x,0)$. Because of (2.9), we also have $s_k \geq m$. Therefore, we can estimate $\sum_{i=1}^{k-1} n_i$ by an upper Riemann sum for $2\beta \int_0^{s_k} (m-t)_+ dt = \beta m^2$. This gives

$$N = \sum_{i=0}^{k-1} n_i \geq \beta m^2 ,$$

That is,

$$m \leq \sqrt{\frac{N}{\beta}} .$$

We have just shown that for any $\underline{x}, \underline{n}$ with $\sum n_i = N$, we have

$$(2.13) \quad E_{\underline{n},\underline{x}}(x^\beta) \leq C_0 N^{-2\beta-1} \delta^{2\beta m} \quad \text{implies} \quad m \leq \sqrt{\frac{N}{\beta}} .$$

Thus,

$$E_N(x^\beta) \geq C_0 N^{-2\beta-1} \delta^{2\beta} \delta^{2\sqrt{N\beta}} \geq C N^{-2\beta-1} \delta^{2\sqrt{N\beta}}$$

for otherwise $m = \sqrt{\frac{N}{\beta}} + 1$ would give an $\underline{n}$ and $\underline{x}$ which contradicts (2.13). This completes the proof of the lower estimate in (2.4) and the theorem.

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