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# Weak- $L^\infty$ and BMO

By COLIN BENNETT<sup>1)</sup>, RONALD A. DEVORE<sup>2)</sup> and ROBERT SHARPLEY<sup>1)</sup>

Dedicated to Professor George G. Lorentz on the occasion  
of his seventieth birthday

## 1. Introduction

The Marcinkiewicz space weak- $L^p$  properly contains  $L^p$  when  $0 < p < \infty$  but it coincides with  $L^\infty$  when  $p = \infty$ . Consequently, the Marcinkiewicz interpolation theorem does not directly apply to operators that are unbounded on  $L^\infty$ . The main purpose of this paper is to construct a rearrangement-invariant space  $W$  that will play the role of “weak- $L^\infty$ ”, in the sense that it contains  $L^\infty$  and possesses the appropriate interpolation properties. The construction, which is motivated by elementary considerations in the Lions-Peetre real interpolation method, is valid for general measure spaces. However, if the underlying measure space is a cube in  $\mathbf{R}^n$ , then  $W$  has an alternative characterization in terms of the space BMO of functions of bounded mean oscillation.

The space  $W$  consists of those measurable functions  $f$  for which  $f^{**} - f^*$  is bounded (where  $f^*$  is the decreasing rearrangement of  $f$  and  $f^{**}(t) = t^{-1} \int_0^t f^*(s) ds$ ). Although no explicit use will be made of the fact, it is perhaps of some interest to note that the space  $W$  so-defined arises via the real interpolation method from the pair  $(L^\infty, L^1)$  in exactly the same way that the space weak- $L^1$  arises from the reversed pair  $(L^1, L^\infty)$ . This and other properties of  $W$  are developed in Section 2. In particular, a Marcinkiewicz-type interpolation theorem is established for  $W$  and it is shown that this result gives a direct proof of the  $L^p$ -boundedness of the Hilbert transform and related singular integral operators for all values of  $p$  with  $1 < p < \infty$ . With these properties, and the fact that  $W$  can be realized as a limit of the familiar spaces weak- $L^p$  as  $p \rightarrow \infty$ , the space  $W$  may justifiably be referred to as weak- $L^\infty$ .

The relationship between weak- $L^\infty$  and BMO is established in Section 3. A covering argument is used to relate the oscillation of a function  $f$  to that

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of its decreasing rearrangement  $f^*$ , and thereby to establish the main result that weak- $L^\infty(Q)$ , where  $Q$  is a cube in  $\mathbf{R}^n$ , is precisely the rearrangement-invariant hull of  $BMO(Q)$ .

In the final section the Hardy-Littlewood maximal operator is shown to be bounded from  $W$  into  $W$  and from  $BMO$  into  $BMO$ .

### 2. The space weak- $L^\infty$

The Peetre  $K$ -functional for the pair  $(L^1, L^\infty)$ , with respect to an arbitrary  $\sigma$ -finite measure space  $(X, \mu)$ , can be explicitly identified as follows:

$$K(f, t; L^1, L^\infty) = \int_0^t f^*(s) ds = t f^{**}(t) \quad (t > 0)$$

(cf. [2, p. 184]). The norm in the Marcinkiewicz space weak- $L^1$  is therefore given in terms of the  $K$ -functional by

$$(2.1) \quad \|f\|_{\text{weak-}L^1} \equiv \sup_{t>0} t f^*(t) = \sup_{t>0} t \frac{d}{dt} K(f, t; L^1, L^\infty).$$

If the roles of  $L^1$  and  $L^\infty$  are now reversed, then a simple computation, together with the identity  $K(f, t; L^\infty, L^1) = tK(f, t^{-1}; L^1, L^\infty)$ , shows that the functional corresponding to that on the right of (2.1) is simply  $\sup_{t>0} [f^{**}(t) - f^*(t)]$ .

*Definition 2.1.* Let  $W = W(X)$  denote the set of  $\mu$ -measurable functions  $f$  on  $X$  for which  $f^*(t)$  is finite for all  $t > 0$  and for which  $f^{**}(t) - f^*(t)$  is a bounded function of  $t$ . Let

$$(2.2) \quad \|f\|_W = \sup_{t>0} [f^{**}(t) - f^*(t)] \quad (f \in W).$$

It is clear that  $W$  contains  $L^\infty$ , and the containment is proper on the interval  $(0, 1)$  (or any nonatomic measure space) since  $\log(1/t)$ , for example, belongs to  $W(0, 1)$  but not to  $L^\infty(0, 1)$ . This logarithmic rate of growth for  $f^*$  at the origin is in fact the maximum attainable for any  $f$  in  $W$ . This follows at once from the elementary identity

$$(2.3) \quad f^{**}(t) - f^{**}(s) = \int_t^s [f^{**}(u) - f^*(u)] \frac{du}{u} \quad (0 < t \leq s < \infty)$$

by putting  $s = 1$  and using (2.2) to estimate the integrand. But such a growth condition does not characterize  $W$ , as easy examples show. The fact is that membership in  $W$  depends not on the growth of  $f^*$  or  $f^{**}$  but rather on the growth of the derivative of  $f^{**}$ . In fact, a simple computation gives

$$f^{**}(t) - f^*(t) = -t \frac{d}{dt} (f^{**}(t))$$

at each point of differentiability of  $f^{**}$ , that is, at each point of continuity of  $f^*$ . It should also be pointed out that  $W$  is not a linear space: there are in fact nonnegative functions in  $W$  whose sum is not in  $W$ . There are also functions  $f$  in  $W$  such that neither  $f_+$  nor  $f_-$  belongs to  $W$ .

When  $1 < p < \infty$ , it follows from (2.3) (with  $s = \infty$ ) that the functional

$$\left( \int_0^\infty \left( t^{1/p} [f^{**}(t) - f^*(t)] \right)^q \frac{dt}{t} \right)^{1/q} \quad (0 < q \leq \infty)$$

is finite if and only if  $f$  belongs to the Lorentz space  $L^{p,q}$ . With  $q = 1$ , this expression converges to  $\|f\|_{L^\infty}$  as  $p \rightarrow \infty$ . Thus  $L^\infty$  may be regarded in this way as the limit of the Lorentz spaces  $L^{p,1}$ . By the same token the space  $W$  is the limit as  $p \rightarrow \infty$  of the Lorentz spaces  $L^{p,\infty} = \text{weak-}L^p$ . This suggests the following definition.

Recall [10, p. 184] that a sublinear operator  $T$  is of *weak type*  $(1, 1)$  if it is a bounded map from  $L^1$  into  $\text{weak-}L^1$ :

$$(2.4) \quad \sup_{t>0} t(Tf)^*(t) \leq c \int_0^\infty f^*(t) dt \quad (f \in L^1).$$

By analogy,  $T$  will be said to be of *weak type*  $(\infty, \infty)$  if it is a bounded map from  $L^\infty$  into  $W$ :

$$(2.5) \quad \sup_{t>0} [(Tf)^{**}(t) - (Tf)^*(t)] \leq c \sup_{t>0} f^*(t) \quad (f \in L^\infty).$$

Our interpolation theorem will merely require that (2.4) and (2.5) hold for characteristic functions. Hence, in accordance with the Stein-Weiss terminology [10, p. 197], a sublinear operator  $T$  will be of *restricted weak type*  $(1, 1)$  (respectively, *restricted weak type*  $(\infty, \infty)$ ) if its domain contains all simple functions and if (2.4) (respectively, (2.5)) holds for all characteristic functions  $f = \chi_E$  of sets  $E$  of finite measure. The following interpolation theorem is best formulated in terms of the Calderón maximal operator  $S$  [3, p. 288]:

$$(Sf)(t) = \frac{1}{t} \int_0^t f(u) du + \int_t^\infty f(u) \frac{du}{u} \quad (t > 0).$$

**THEOREM 2.2.** *Let  $T$  be a sublinear operator of restricted weak types  $(1, 1)$  and  $(\infty, \infty)$ . Then, for all simple functions  $f$ ,*

$$(2.6) \quad (Tf)^{**}(t) \leq cS(f^{**})(t) \quad (t > 0)$$

and

$$(2.7) \quad \|Tf\|_{L^p} \leq c_p \|f\|_{L^p} \quad (1 < p < \infty),$$

where  $c$  depends only on  $T$ , and  $c_p$  only on  $p$  and  $T$ . In particular, if  $T$  is linear, then  $T$  has a unique extension to a bounded linear operator on  $L^p$  ( $1 < p < \infty$ ).

*Proof.* Let  $E$  be any  $\mu$ -measurable subset of  $X$  with  $0 < s = \mu(E) < \infty$ . Let  $\chi$  denote the characteristic function of  $E$  and let  $g = T\chi$ . Then the hypotheses on  $T$  (cf. (2.4) and (2.5)) give

$$(2.8) \quad tg^*(t) \leq cs \quad (t > 0)$$

and

$$(2.9) \quad g^{**}(t) - g^*(t) \leq c \quad (t > 0),$$

where  $c$  is a constant depending only on  $T$ . These estimates may be combined to give

$$(2.10) \quad g^*(t) \leq 2c \left\{ \left( \frac{s}{t} \wedge 1 \right) + \log^+ \left( \frac{s}{t} \right) \right\} \quad (t > 0).$$

This follows at once from (2.8) if  $t \geq s$ . In the remaining case where  $0 < t < s$ , the estimate (2.9) may be used to estimate the integrand in (2.3) (applied to  $g$ ) to give  $g^{**}(t) \leq g^{**}(s) + c \log(s/t)$ , and this yields (2.10) since successive applications of (2.9) and (2.8) show that  $g^{**}(s) \leq g^*(s) + c \leq 2c$ .

The right-hand side of (2.10) is precisely  $2cS(\chi^*)(t)$ , where  $S$  is the Calderón operator. Hence (2.10) may be written in the form

$$(T\chi)^*(t) \leq 2cS(\chi^*)(t) \quad (t > 0).$$

An integration of both sides and some further computation now yield the more desirable form

$$(2.11) \quad (T\chi)^{**}(t) \leq 2cS(\chi^{**})(t) \quad (t > 0),$$

the point being that the operation  $f \rightarrow f^{**}$  is subadditive whereas  $f \rightarrow f^*$  is not. This, together with the sublinearity of  $T$ , enables us, with standard arguments (cf. [3, pp. 286–287]), to pass from the estimate (2.11) for characteristic functions to the desired estimate (2.6) for all simple functions. The remaining assertions are routine consequences of this one.

The Hilbert transform  $H$  may be interpolated directly by the previous theorem. All that is needed is the Stein-Weiss estimate [10, p. 240]

$$(H\chi_E)^*(t) = \frac{1}{\pi} \sinh^{-1} \left( \frac{2|E|}{t} \right) \quad (t > 0),$$

valid for any subset  $E$  of  $(-\infty, \infty)$  with finite measure  $|E|$ . It follows at once from this identity that  $H$  is of restricted weak types  $(1, 1)$  and  $(\infty, \infty)$ , and hence that  $H$  may be interpolated by Theorem 2.2. The interpolation theorem applies also to the maximal Hilbert transform and, more generally, to the maximal operators associated with arbitrary Calderón-Zygmund singular integrals (cf. [9, p. 35]).

It is worth pointing out that Herz [5] has an interpolation theorem

which is somewhat loosely related to ours. The functional  $f^{**} - f^*$  is implicit in the proof and it plays a prominent role in some of Herz' applications to martingales. Our interpolation theorem may also be compared with a result of N. M. Rivi re [6], to the effect that if  $T$  is of weak type  $(1, 1)$  and maps  $L^\infty$  into BMO, then  $T$  is bounded on every  $L^p$  with  $1 < p < \infty$ . In view of Theorem 3.1 of the next section, this result is contained in ours, at least when the underlying measure space is a cube in  $\mathbf{R}^n$ .

### 3. Weak- $L^\infty$ and BMO

In this section the underlying measure space will be a fixed cube  $Q$  (with sides parallel to the coordinate axes) in  $\mathbf{R}^n$  with Lebesgue measure. For each integrable function  $f$  on  $Q$ , the *sharp function* of  $f$  relative to  $Q$  is defined by

$$(3.1) \quad f_Q^\sharp(x) = \sup_{Q \supset Q' \ni x} \frac{1}{|Q'|} \int_{Q'} |f(y) - f_{Q'}| dy \quad (x \in Q),$$

where  $f_{Q'} = 1/|Q'| \int_{Q'} f(y) dy$  and the supremum is taken over all cubes  $Q'$  that contain  $x$  and are contained in  $Q$ . If  $f_Q^\sharp$  is a bounded function of  $x$ , then  $f$  is said to belong to  $\text{BMO}(Q)$ . The norm is given by

$$(3.2) \quad \|f\|_{\text{BMO}(Q)} = \sup_{x \in Q} f_Q^\sharp(x).$$

It is well-known that BMO can serve as a useful substitute for  $L^\infty$  (cf. [4], [6], [7], [8], [11]). The next theorem shows that BMO for a cube  $Q$  is intimately connected with  $W(Q)$ .

**THEOREM 3.1.** (a) *If  $f$  belongs to  $L^1(Q)$ , then*

$$(3.3) \quad f^{**}(t) - f^*(t) \leq c(f_Q^\sharp)^*(t) \quad \left(0 < t < \frac{1}{6}|Q|\right),$$

where  $c$  is a constant depending only on  $n$ .

(b) *The space  $W(Q)$  is the rearrangement-invariant hull of  $\text{BMO}(Q)$  in the sense that an integrable function  $f$  belongs to  $W(Q)$  if and only if  $f$  is equimeasurable with some function  $g$  in  $\text{BMO}(Q)$ .*

The following covering lemma, which is a variant of Lemma 1.1 in [1], will be needed. The proof is similar so we omit it.

**LEMMA 3.2.** *Let  $\mathcal{O}$  be a relatively open subset of  $Q$  such that  $|\mathcal{O}| < (1/2)|Q|$ . Then there is a family of cubes  $Q_j$  ( $j = 1, 2, \dots$ ) with pairwise disjoint interiors such that*

- (i)  $|\mathcal{O} \cap Q_j| \leq 2^{-1}|Q_j| < |\mathcal{O} \cap Q_j|$  ( $j = 1, 2, \dots$ );
- (ii)  $\mathcal{O} \subset \bigcup_{j=1}^{\infty} Q_j \subset Q$ ;
- (iii)  $|\mathcal{O}| \leq \sum_{j=1}^{\infty} |Q_j| \leq 2^{n+1}|\mathcal{O}|$ .

*Proof of Theorem 3.1.* Since  $|f|_Q^* \leq 2f_Q^*$ , it is enough to establish (3.3) for nonnegative  $f$ . In that case, fix  $t$  with  $0 < t < (1/6)|Q|$  and let

$$E = \{x \in Q: f(x) > f^*(t)\}, \quad F = \{x \in Q: f_Q^*(x) > (f_Q^*)^*(t)\}.$$

Then  $|E \cup F| \leq 2t$  so there is a relatively open subset  $\mathcal{O}$  of  $Q$  with  $|\mathcal{O}| \leq 3t$  and  $E \cup F \subset \mathcal{O} \subset Q$ . In particular  $|\mathcal{O}| \leq (1/2)|Q|$  so by Lemma 3.2 there is a covering  $\{Q_j\}_{j=1}^\infty$  of  $\mathcal{O}$  satisfying conditions (i), (ii), and (iii) above. Now

$$\begin{aligned} t\{f^{**}(t) - f^*(t)\} &= \int_E \{f(x) - f^*(t)\}dx = \sum_{j=1}^\infty \int_{E \cap Q_j} \{f(x) - f^*(t)\}dx \\ &\leq \sum_j \int_{Q_j} |f(x) - f_{Q_j}|dx + \sum_j |E \cap Q_j| \{f_{Q_j} - f^*(t)\} \\ &= A + B, \text{ say.} \end{aligned}$$

If  $\Sigma'$  denotes the sum over those indices  $j$  for which  $f_{Q_j} > f^*(t)$ , then

$$B \leq \Sigma' |E \cap Q_j| \{f_{Q_j} - f^*(t)\} \leq \Sigma' |\mathcal{O} \cap Q_j| \{f_{Q_j} - f^*(t)\}.$$

Hence, by (i),

$$B \leq \Sigma' \int_{\mathcal{O}^c \cap Q_j} \{f_{Q_j} - f^*(t)\}dx \leq \Sigma' \int_{Q_j} |f_{Q_j} - f(x)|dx \leq A,$$

where the middle inequality holds because  $f(u) \leq f^*(t)$  on  $\mathcal{O}^c$ . This, together with the preceding estimate, gives

$$(3.4) \quad t\{f^{**}(t) - f^*(t)\} \leq 2A.$$

Now observe from (i) that each  $Q_j$  meets  $F^c$  in at least one point, say  $x_j$ . Then  $f_Q^*(x_j) \leq (f_Q^*)^*(t)$  because of the way  $F$  is defined, and so

$$A = \sum_j |Q_j| \left\{ \frac{1}{|Q_j|} \int_{Q_j} |f(x) - f_{Q_j}|dx \right\} \leq \sum_j |Q_j| f_Q^*(x_j) \leq \sum_j |Q_j| (f_Q^*)^*(t).$$

Hence, by (iii),

$$A \leq 2^{n+1} |\mathcal{O}| (f_Q^*)^*(t) \leq 2^{n+1} (3t) (f_Q^*)^*(t),$$

and this together with (3.4) establishes (3.3).

For part (b), note first that if  $t \geq (1/6)|Q|$ , then

$$f^{**}(t) - f^*(t) \leq f^{**}\left(\frac{1}{6}|Q|\right) \leq 6f^{**}(|Q|) = \frac{6}{|Q|} \int_Q |f(x)|dx.$$

The inequality (3.3) may be used to estimate  $f^{**} - f^*$  in the case  $t < (1/6)|Q|$ , so together these estimates give

$$(3.5) \quad \|f\|_{W(Q)} \leq c \left( \|f\|_{BMO(Q)} + \frac{1}{|Q|} \int_Q |f(x)|dx \right).$$

This shows that  $BMO(Q)$  is contained in  $W(Q)$  and hence, since  $W(Q)$  is rearrangement-invariant, that every function  $f$  equimeasurable to a

BMO( $Q$ )-function  $g$  must lie in  $W(Q)$ .

It will suffice to prove the converse for the unit cube  $Q = I^n$  (where  $I = [0, 1]$ ) since a linear change of variables reduces the general case to this one. But then if  $f \in W(I^n)$ , the function

$$g(x) = f^*(x_1) \quad (x = (x_1, x_2, \dots, x_n) \in I^n)$$

is equimeasurable with  $f$ , and for any subcube  $R = \prod_{i=1}^n [r_i, r_i + \alpha]$  of  $I^n$ ,

$$\begin{aligned} & \frac{1}{|R|} \int_R |g(x) - f^*(r_1 + \alpha)| dx_1 \cdots dx_n \\ &= \frac{1}{\alpha} \int_{r_1}^{r_1 + \alpha} [f^*(t) - f^*(r_1 + \alpha)] dt \\ &\leq \frac{1}{r_1 + \alpha} \int_0^{r_1 + \alpha} [f^*(t) - f^*(r_1 + \alpha)] dt \\ &= f^{**}(r_1 + \alpha) - f^*(r_1 + \alpha) \leq \|f\|_{W(Q)}. \end{aligned}$$

Hence  $g$  belongs to BMO( $Q$ ) and the proof is complete.

The preceding theorem fails when  $Q$  is replaced by all of  $\mathbf{R}^n$  since BMO( $\mathbf{R}^n$ ) contains functions (such as  $\log|x|$ ) which are unbounded at infinity and hence have decreasing rearrangements which are identically infinite. However, the theorem does contain "local" information pertinent to BMO( $\mathbf{R}^n$ ). For example, when  $f$  is in BMO( $\mathbf{R}^n$ ), the inequality (3.3) may be applied to the function  $(f - f_Q)\chi_Q$ . An integration of both sides produces the basic inequality (4.23) of [1] from which the John-Nirenberg lemma follows easily.

#### 4. Maximal operators

As in the previous section let  $Q$  be a fixed cube in  $\mathbf{R}^n$ . The Hardy-Littlewood maximal function  $M_Q f$  of an integrable function  $f$  on  $Q$  is given by

$$(M_Q f)(x) = \sup \frac{1}{|Q'|} \int_{Q'} |f(y)| dy \quad (x \in Q),$$

where the supremum is taken over all cubes  $Q'$  contained in  $Q$  and containing  $x$ . When  $Q$  is replaced by all of  $\mathbf{R}^n$ , the corresponding operator, defined for all locally integrable  $f$  on  $\mathbf{R}^n$ , will be denoted simply by  $M$ . The next result shows that such maximal operators are bounded on  $W$ .

**THEOREM 4.1.** (a) *If  $f$  belongs to  $W(Q)$ , then so does  $M_Q f$  and*

$$(4.1) \quad \|M_Q f\|_{W(Q)} \leq c \|f\|_{W(Q)},$$

where  $c$  depends only on the dimension  $n$ .



(b) *The same result holds if  $Q$  is replaced by  $\mathbf{R}^n$  and  $M_Q$  by  $M$ .*

*Proof.* (a) We may assume that  $f$  is nonnegative. Fix  $t < |Q|$  and let

$$b = \max(f - f^*(t), 0), \quad g = \min(f, f^*(t)),$$

so  $f = b + g$ . The weak  $(1, 1)$  and strong  $(\infty, \infty)$  properties of  $M_Q$  give

$$\begin{aligned} (M_Q f)^*(t) &\leq (M_Q b)^*(t-) + (M_Q g)^*(0+) \leq ct^{-1} \|b\|_{L^1} + \|g\|_{L^\infty} \\ &\leq ct^{-1} \int_0^t [f^*(s) - f^*(t)] ds + f^*(t). \end{aligned}$$

Hence  $(M_Q f)^*(t)$  is finite and

$$(4.2) \quad 0 \leq (M_Q f)^*(t) - f^*(t) \leq c \{f^{**}(t) - f^*(t)\} \quad (t > 0).$$

Now write

$$(M_Q f)^{**} - (M_Q f)^* = [(M_Q f)^{**} - f^{**}] + [f^{**} - f^*] + [f^* - (M_Q f)^*]$$

and

$$(M_Q f)^{**}(t) - f^{**}(t) = \frac{1}{t} \int_0^t [(M_Q f)^*(s) - f^*(s)] ds.$$

Then an application of (4.2) yields

$$(M_Q f)^{**}(t) - (M_Q f)^*(t) \leq c \sup_{0 < s \leq t} \{f^{**}(s) - f^*(s)\},$$

from which (4.1) follows. Exactly the same proof establishes part (b).

Next we show that  $M_Q$  is a bounded operator on  $BMO(Q)$ . Essentially the same result holds for  $\mathbf{R}^n$  except that functions  $f$  for which  $Mf$  is identically infinite must be ruled out ( $f(x) = \log|x|$  is an example).

**THEOREM 4.2.** (a) *If  $f$  belongs to  $BMO(Q)$ , then so does  $M_Q f$  and*

$$(4.3) \quad \|M_Q f\|_{BMO(Q)} \leq c \|f\|_{BMO(Q)},$$

where  $c$  depends only on the dimension  $n$ .

(b) *If  $f$  belongs to  $BMO(\mathbf{R}^n)$ , and if  $Mf$  is not identically infinite, then  $Mf$  belongs to  $BMO(\mathbf{R}^n)$  and*

$$(4.4) \quad \|Mf\|_{BMO(\mathbf{R}^n)} \leq c \|f\|_{BMO(\mathbf{R}^n)}$$

where  $c$  depends only on  $n$ .

*Proof.* (a) We may assume that  $f$  is nonnegative. Writing  $F$  for the maximal function  $M_Q f$  of  $f$ , we thus need to show

$$(4.5) \quad \frac{1}{|R|} \int_R |F(x) - F_R| dx \leq c \|f\|_{BMO(Q)}$$

for arbitrary subcubes  $R$  of  $Q$ .

Fix  $R$  and let  $3R$  denote the cube that is concentric with  $R$  and has three times the diameter. Let  $\tilde{R}$  be the smallest subcube of  $Q$  containing

$(3R) \cap Q$ , and for each  $x$  in  $R$  let

$$\begin{aligned} F_1(x) &= \sup\{f_{\bar{R}}: \bar{R} \subset \tilde{R} \text{ and } x \in \bar{R}\}, \\ F_2(x) &= \sup\{f_{\bar{R}}: \bar{R} \subset Q, x \in \bar{R}, \text{ and } \bar{R} \cap (Q \setminus \tilde{R}) \neq \emptyset\}. \end{aligned}$$

Clearly  $F = \max\{F_1, F_2\}$  on  $R$  so if

$$\Omega = \{x \in R: F(x) > F_R\}, \Omega_1 = \{x \in \Omega: F_1(x) \geq F_2(x)\} \text{ and } \Omega_2 = \Omega \setminus \Omega_1,$$

then

$$\frac{1}{|R|} \int_R |F(x) - F_R| dx = \frac{2}{|R|} \int_\Omega [F(x) - F_R] dx = \frac{2}{|R|} \sum_{i=1}^2 \int_{\Omega_i} [F_i(x) - F_R] dx.$$

Hence (4.5) will be established if we show that

$$(4.6) \quad \int_{\Omega_i} [F_i(x) - F_R] dx \leq c |R| \|f\|_{\text{BMO}(Q)} \quad (i = 1, 2).$$

Consider first the case  $i = 1$ . Since  $f_{\tilde{R}} \leq F(x)$  for all  $x$  in  $R$ , then certainly  $f_{\tilde{R}} \leq F_R$  so we may construct the Calderón-Zygmund decomposition [9, p. 17] for  $f$  and  $\tilde{R}$  with respect to the constant  $F_R$ . If the resulting sequence of pairwise disjoint cubes is denoted by  $\{R_k\}_{k=1}^\infty$ , and if  $\bar{R}_k$  denotes the "parent" cube of  $R_k$ , then the following properties hold:

- (i)  $\bigcup_k R_k \subset \tilde{R}$ ;
- (ii)  $f_{\bar{R}_k} \leq F_R < f_{R_k}$  ( $k = 1, 2, \dots$ );
- (iii)  $|\bar{R}_k| = 2^n |R_k|$  ( $k = 1, 2, \dots$ );
- (iv)  $f \leq F_R$  almost everywhere on  $E = \tilde{R} \setminus (\bigcup_k R_k)$ .

Define functions  $b$  and  $g$  on  $Q$  by

$$b = \sum_k (f - f_{\bar{R}_k}) \chi_{R_k}, \quad g = \sum_k f_{\bar{R}_k} \chi_{R_k} + f \chi_E$$

so  $f \chi_{\tilde{R}} = b + g$ . It follows from (ii) and (iv) that

$$(4.7) \quad \|g\|_{L^\infty(Q)} \leq F_R,$$

while on the other hand the John-Nirenberg lemma and (i) and (iii) give

$$(4.8) \quad \|b\|_{L^2(Q)} = \left\{ \sum_k \int_{R_k} |f - f_{\bar{R}_k}|^2 dx \right\}^{1/2} \leq \left\{ \sum_k |\bar{R}_k| \frac{1}{|\bar{R}_k|} \int_{\bar{R}_k} |f - f_{\bar{R}_k}|^2 dx \right\}^{1/2} \\ \leq c (\sum_k 2^n |R_k|)^{1/2} \|f\|_{\text{BMO}(Q)} \leq c |R|^{1/2} \|f\|_{\text{BMO}(Q)}.$$

Now it follows from the definition of  $F_1$  that

$$F_1 \leq M_Q(f \chi_{\tilde{R}}) = M_Q(b + g) \leq M_Q b + M_Q g,$$

so applying the Cauchy-Schwarz inequality we obtain

$$\begin{aligned} \int_{\Omega_1} F_1(x) dx &\leq |\Omega_1|^{1/2} \|M_Q b\|_{L^2(Q)} + |\Omega_1| \|M_Q g\|_{L^\infty(Q)} \\ &\leq c |R|^{1/2} \|b\|_{L^2(Q)} + |\Omega_1| \|g\|_{L^\infty(Q)}. \end{aligned}$$

Combining this with (4.7) and (4.8), and subtracting  $|\Omega_1| F_R$  from each side,

we obtain (4.6) for  $i = 1$ .

The remaining case  $i = 2$  will follow directly from the inequality

$$(4.9) \quad F_2(x) - F_R \leq c \|f\|_{\text{BMO}(Q)} \quad (x \in \Omega_2)$$

which we now prove. Fix  $x$  in  $\Omega_2$  and let  $P$  be any subcube of  $Q$  that contains  $x$  and has nonempty intersection with  $Q \setminus \tilde{R}$ . Clearly  $|P| \geq |R|$ . Let  $P'$  be the smallest subcube of  $Q$  containing both  $P$  and  $R$ . Then  $|P'| \leq 2^n |P|$ . Arguing as before, we note that  $f_{P'} \leq F_R$ . Hence

$$f_P - F_R \leq f_P - f_{P'} \leq \frac{1}{|P|} \int_P |f(y) - f_{P'}| dy \leq 2^n \|f\|_{\text{BMO}(Q)},$$

so taking the supremum over all such cubes  $P$  we obtain (4.9). This establishes part (a).

The maximal function  $F$  in the preceding proof is necessarily integrable over every cube  $R$  (contained in  $Q$ ) but this need not be the case when we extend to  $\mathbf{R}^n$ . However, if  $f$  belongs to  $\text{BMO}(\mathbf{R}^n)$  and  $R$  is any cube in  $\mathbf{R}^n$ , we can split the maximal function  $F = Mf$  into the two parts analogous to  $F_1$  and  $F_2$  in the proof above and estimate these separately. The function  $F_1$  is essentially a maximal function relative to a fixed cube and so may be estimated in terms of the BMO-norm of  $f$  exactly as in the proof above. The function  $F_2$  on the other hand is a supremum of averages of  $f$  over "large" cubes which, by means of a fixed dilation, may be taken to contain  $R$ . But then each of these averages is bounded above by the maximal function  $Mf$  evaluated at any point of  $R$ , so  $F_2$  is bounded by  $\inf_R Mf$ . Hence we arrive at the following estimate

$$\frac{1}{|R|} \int_R (Mf)(x) dx \leq c (\|f\|_{\text{BMO}(\mathbf{R}^n)} + \inf_{x \in R} (Mf)(x)).$$

Since  $R$  is arbitrary, it follows that the maximal function  $F = Mf$  of a function  $f$  in  $\text{BMO}(\mathbf{R}^n)$  is either identically infinite or else it is locally integrable (hence finite a.e. on  $\mathbf{R}^n$ ). In the latter case, having established that the mean  $F_R$  is finite, we may proceed exactly as in the proof of part (a) to show that  $F$  is in  $\text{BMO}(\mathbf{R}^n)$ . We omit the details.

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