MAXIMAL SINGULAR INTEGRALS ON L^∞

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The space $\mathrm{BMO}(\mathbf{R}^n)$ of functions of bounded mean oscillation consists of those locally integrable functions f on $\mathbf{R}^n$ for which

$$\|f\|_{\mathrm{BMO}(\mathbf{R}^n)} = \sup \frac{1}{|Q|} \int_Q |f(x) - f_Q|\, dx$$

is finite; here the supremum extends over all cubes Q in $\mathbf{R}^n$ (with sides parallel to the coordinate axes) and f_Q denotes the mean value of f over Q: $f_Q = |Q|^{-1} \int_Q f(x)\, dx$ (cf. [1, p. 141]).

Singular integral operators such as the Riesz transforms (the Hilbert transform when $n = 1$) do not preserve $L^\infty(\mathbf{R}^n)$ but instead carry $L^\infty(\mathbf{R}^n)$ boundedly into $\mathrm{BMO}(\mathbf{R}^n)$ (cf. [1, Theorem 1]). The purpose of the present note is to show that the corresponding maximal operators have the same property.

We consider Calderón – Zygmund kernels k, that is, kernels k on $\mathbf{R}^n$ satisfying the following conditions (cf. [2, p. 35]):

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(i) $\int_{|u| \geqslant 2|x|} |k(u - x) - k(u)|\, du \leqslant M_1 \qquad (x \neq 0);$

(ii) $|k(x)| \leqslant \dfrac{M_2}{|x|^n} \qquad (x \neq 0);$

(iii) $\int_{r < |x| < R} k(x)\, dx = 0 \qquad (0 < r < R < \infty),$

for some constants M_1 and M_2 depending only on k.

For each $\epsilon > 0$, let $k_\epsilon(x) = k(x)\chi_{(\epsilon,\infty)}(|x|)$. The singular integral operator K associated with k is defined on $L^\infty(\mathbf{R}^n)$ by the principal value integral

$$(Kf)(x) = \lim_{\epsilon \to 0} (k_\epsilon * f(x) - k_1 * f(0)),$$

and the corresponding maximal operator $\mathscr{K}$ is given by

$$\mathscr{K} f(x) = \sup_{\epsilon > 0} |k_\epsilon * f(x) - k_1 * f(0)|.$$

Theorem. *If f belongs to $L^\infty(\mathbf{R}^n)$, then $\mathscr{K}f$ belongs to* $\mathrm{BMO}(\mathbf{R}^n)$ *and*

$$\| \mathscr{K}f\|_{\mathrm{BMO}(\mathbf{R}^n)} \leqslant c \|f\|_{L^\infty(\mathbf{R}^n)}$$

where c is a constant depending only on n, M_1 and M_2.

Proof. It is enough to show that for each cube Q in $\mathbf{R}^n$ there exists α such that

(1) $$\int_Q |\mathscr{K}f - \alpha|\, dx \leqslant c \|f\|_{L^\infty(\mathbf{R}^n)} |Q|.$$

Fix Q and let $\widetilde{Q}$ be the cube concentric with Q but with diameter $5n$ times as large. Set $b = f\chi_{\widetilde{Q}}$ and $g = f - b$, and choose $\alpha = |Q|^{-1} \int_Q \mathscr{K} g\, dx$. A simple computation using the subadditivity of $\mathscr{K}$ gives

(2) $$\int_Q |\mathscr{K} f - \alpha|\, dx \leqslant \int_Q \mathscr{K} b\, dx + \int_Q |\mathscr{K} g - \alpha|\, dx.$$

We estimate separately the two terms on the right.

Since k is a Calderón – Zygmund kernel standard arguments show that $\mathscr{K}$ is bounded on $L^2(\mathbf{R}^n)$. Hence, using the Cauchy – Schwarz inequality, we have

$$\int_Q \mathscr{K}b\,dx \leqslant |Q|^{\frac{1}{2}} \|\mathscr{K}b\|_{L^2(\mathbf{R}^n)} \leqslant c\,|Q|^{\frac{1}{2}} \|b\|_{L^2(\tilde{Q})} \leqslant \tag{3}$$
$$\leqslant c\,|Q|^{\frac{1}{2}}\,|\tilde{Q}|^{\frac{1}{2}} \|b\|_{L^\infty(\mathbf{R}^n)} \leqslant c\,|Q|\,\|f\|_{L^\infty(\mathbf{R}^n)}.$$

To estimate the second term on the right of (2), note that

$$\int_Q |\mathscr{K}g - \alpha|\,dx = 2 \int_Q [\mathscr{K}g - \alpha]_+\,dx = 2 \int_\Omega (\mathscr{K}g - \alpha)\,dx,$$

where $\Omega = \{x \in Q\colon\ \mathscr{K}g(x) > \alpha\}$. Hence, in view of (2) and (3), in order to establish (1) it will suffice to show that

$$\mathscr{K}g - \alpha \leqslant c\,\|f\|_{L^\infty(\mathbf{R}^n)} \qquad (x \in \Omega). \tag{4}$$

To this end, let $G_\epsilon(z) = |k_\epsilon * g(z) - k_1 * g(0)|$. Fix x in Ω and choose any $\epsilon > 0$ for which $G_\epsilon(x) > \alpha$. Since α is the mean value of $\mathscr{K}g$ over Q, there is some point y in Q for which $G_\epsilon(y) < \alpha$. Hence

$$G_\epsilon(x) - \alpha \leqslant |G_\epsilon(x) - G_\epsilon(y)| \leqslant |k_\epsilon * g(x) - k_\epsilon * g(y)| \leqslant$$
$$\leqslant \|f\|_{L^\infty(\mathbf{R}^n)} \int_{(\tilde{Q})^c} |k_\epsilon(x-u) - k_\epsilon(y-u)|\,du.$$

But k_ϵ also satisfies conditions (i), (ii), and (iii) above and with constants independent of ϵ (cf. [2, p. 37]). Hence using (i) to estimate the preceding integral we obtain a uniform bound independent of x, y, and ϵ. This completes the proof.

REFERENCES

[1] C. Fefferman – E.M. Stein, H^p spaces of several variables, *Acta Math.*, 129 (1972), 137-193.

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