MAXIMAL SINGULAR INTEGRALS ON L^∞

C. BENNETT — R.A. DE VORE — R. SHARPLEY*

The space BMO$(\mathbf{R}^n)$ of functions of bounded mean oscillation consists of those locally integrable functions f on $\mathbf{R}^n$ for which

$$\|f\|_{\mathrm{BMO}(\mathbf{R}^n)} = \sup \frac{1}{|Q|} \int_Q |f(x) - f_Q|\, dx$$

is finite; here the supremum extends over all cubes Q in $\mathbf{R}^n$ (with sides parallel to the coordinate axes) and f_Q denotes the mean value of f over Q: $f_Q = |Q|^{-1} \int_Q f(x)\, dx$ (cf. [1, p. 141]).

Singular integral operators such as the Riesz transforms (the Hilbert transform when $n = 1$) do not preserve $L^\infty(\mathbf{R}^n)$ but instead carry $L^\infty(\mathbf{R}^n)$ boundedly into BMO$(\mathbf{R}^n)$ (cf. [1, Theorem 1]). The purpose of the present note is to show that the corresponding maximal operators have the same property.

We consider Calderón – Zygmund kernels k, that is, kernels k on $\mathbf{R}^n$ satisfying the following conditions (cf. [2, p. 35]):

*The research of all three authors was supported in part by National Science Foundation Grant No. MCS-80-01941.

(i) $\displaystyle\int_{|u|\geq 2|x|} |k(u-x)-k(u)|\,du \leq M_1 \qquad (x \neq 0);$

(ii) $\displaystyle |k(x)| \leq \frac{M_2}{|x|^n} \qquad (x \neq 0);$

(iii) $\displaystyle\int_{r<|x|<R} k(x)\,dx = 0 \qquad (0<r<R<\infty),$

for some constants M_1 and M_2 depending only on k.

For each $\epsilon > 0$, let $k_\epsilon(x) = k(x)\chi_{(\epsilon,\infty)}(|x|)$. The singular integral operator K associated with k is defined on $L^\infty(\mathbf{R}^n)$ by the principal value integral

$$(Kf)(x) = \lim_{\epsilon\to 0} (k_\epsilon * f(x) - k_1 * f(0)),$$

and the corresponding maximal operator $\mathscr{K}$ is given by

$$\mathscr{K} f(x) = \sup_{\epsilon>0} |k_\epsilon * f(x) - k_1 * f(0)|.$$

Theorem. *If f belongs to $L^\infty(\mathbf{R}^n)$, then $\mathscr{K} f$ belongs to* BMO$(\mathbf{R}^n)$ *and*

$$\| \mathscr{K} f\|_{\mathrm{BMO}(\mathbf{R}^n)} \leq c\,\|f\|_{L^\infty(\mathbf{R}^n)}$$

where c is a constant depending only on n, M_1 and M_2.

Proof. It is enough to show that for each cube Q in $\mathbf{R}^n$ there exists α such that

(1) $$\int_Q |\mathscr{K} f - \alpha|\,dx \leq c\,\|f\|_{L^\infty(\mathbf{R}^n)}|Q|.$$

Fix Q and let $\widetilde{Q}$ be the cube concentric with Q but with diameter $5n$ times as large. Set $b = f\chi_{\widetilde{Q}}$ and $g = f - b$, and choose $\alpha = |Q|^{-1}\int_Q \mathscr{K} g\,dx$. A simple computation using the subadditivity of $\mathscr{K}$ gives

(2) $$\int_Q |\mathscr{K} f - \alpha|\,dx \leq \int_Q \mathscr{K} b\,dx + \int_Q |\mathscr{K} g - \alpha|\,dx.$$

We estimate separately the two terms on the right.

Since k is a Calderón – Zygmund kernel standard arguments show that $\mathscr{K}$ is bounded on $L^2(\mathrm{R}^n)$. Hence, using the Cauchy – Schwarz inequality, we have

$$\tag{3} \begin{aligned} \int_Q \mathscr{K} b \, dx &\leqslant |Q|^{\frac{1}{2}} \| \mathscr{K} b \|_{L^2(\mathrm{R}^n)} \leqslant c |Q|^{\frac{1}{2}} \| b \|_{L^2(\tilde{Q})} \leqslant \\ &\leqslant c |Q|^{\frac{1}{2}} |\tilde{Q}|^{\frac{1}{2}} \| b \|_{L^\infty(\mathrm{R}^n)} \leqslant c |Q| \| f \|_{L^\infty(\mathrm{R}^n)}. \end{aligned}$$

To estimate the second term on the right of (2), note that

$$\int_Q | \mathscr{K} g - \alpha | \, dx = 2 \int_Q [\mathscr{K} g - \alpha]_+ \, dx = 2 \int_\Omega (\mathscr{K} g - \alpha) \, dx,$$

where $\Omega = \{x \in Q: \ \mathscr{K} g(x) > \alpha\}$. Hence, in view of (2) and (3), in order to establish (1) it will suffice to show that

$$\tag{4} \mathscr{K} g - \alpha \leqslant c \| f \|_{L^\infty(\mathrm{R}^n)} \qquad (x \in \Omega).$$

To this end, let $G_\epsilon(z) = | k_\epsilon * g(z) - k_1 * g(0) |$. Fix x in Ω and choose any $\epsilon > 0$ for which $G_\epsilon(x) > \alpha$. Since α is the mean value of $\mathscr{K} g$ over Q, there is some point y in Q for which $G_\epsilon(y) < \alpha$. Hence

$$\begin{aligned} G_\epsilon(x) - \alpha &\leqslant | G_\epsilon(x) - G_\epsilon(y) | \leqslant | k_\epsilon * g(x) - k_\epsilon * g(y) | \leqslant \\ &\leqslant \| f \|_{L^\infty(\mathrm{R}^n)} \int_{(\tilde{Q})^c} | k_\epsilon(x - u) - k_\epsilon(y - u) | \, du. \end{aligned}$$

But k_ϵ also satisfies conditions (i), (ii), and (iii) above and with constants independent of ϵ (cf. [2, p. 37]). Hence using (i) to estimate the preceding integral we obtain a uniform bound independent of x, y, and ϵ. This completes the proof.

REFERENCES

[1] C. Fefferman – E.M. Stein, H^p spaces of several variables, *Acta Math.*, 129 (1972), 137-193.

[2] E. M. Stein, *Singular integrals and differentiability properties of functions,* Princeton Univ. Press, 1970.

C. Bennett – R.A. DeVore – R. Sharpley
University of South Carolina, Columbia, South Carolina 29208, USA.