MAXIMAL FUNCTIONS AND THEIR APPLICATION TO RATIONAL APPROXIMATION

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ABSTRACT. The Hardy Littlewood maximal functions as well as other maximal functions are used to give simple constructive proofs of the results of Popov and Brudnyi on rational approximation.

1. **INTRODUCTION.** Maximal functions are important tools in various areas of analysis, most notably in differentiation theory and the study of mapping properties of operators. As we shall see, such maximal functions can also be used in a fundamental way to derive results on the approximation by rational functions. Specifically, we shall show how to give simple, constructive proofs of results of V. Popov [8,9] and Yu. Brudnyi [1]. These results center on various smoothness conditions on a function f which guarantee that

$$r_n(f)_q = O(n^{-\alpha}) \tag{1.1}$$

where

$$r_n(f)_q := \inf_{\deg R=n} \|f-R\|_q . \tag{1.2}$$

Here $R = P/Q$ is a rational function and $\deg R = \max(\deg P, \deg Q)$: All norms in this paper are on $[0,1]$ unless otherwise indicated.

Popov gave the famous inequality

$$r_n(f)_\infty \le c\|f'\|_{BV}\, n^{-2} . \tag{1.3}$$

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This estimate should be contrasted with the approximation by polynomials where one can only prove the error is $O(n^{-1})$ with n the degree of the approximating polynomials (consider for example $f(x) = |x|$ on $[-1,+1]$ [6 p. 94]). It follows in a simple way from (1.3) (as was noted by G. Freud [5]) that for each $f \in \mathrm{Lip}\ 1$

$$(1.4) \qquad r_n(f)_\infty = o_f(n^{-1}).$$

This is a positive solution to the famous conjecture of D.J. Newman [7].

Maximal functions are used here to give local error estimates for the approximation by polynomials. For example, it follows from the remainder formula for interpolation by the Taylor polynomial $P_x f$ of degree $k-1$ that

$$(1.5) \qquad |f(y)-P_x f(y)| \le |y-x|^{k-1} \int_x^y |f^{(k)}| \le |y-x|^k \left(\frac{1}{|x-y|} \int_x^y |f^{(k)}|\right)$$
$$\le |y-x|^k \inf_{x \le u \le y} M(f^{(k)})(u)$$

where M is the Hardy Littlewood maximal function:

$$(1.6) \qquad Mg(x) := \sup_{I \ni x} \frac{1}{|I|} \int_I |g|$$

with the sup taken over all intervals $I \subset [0,1]$ which contain x.

One exploits (1.5) in rational approximations by choosing n intervals $I_1,\ldots,I_n$ where $\inf_{I_j} M f^{(k)}$ are approximately equal and taking $x = \xi_i$ with $\xi_i \in I_i$. The local polynomials P_{ξ_i} are then pieced together using a simple partition of unity with low degree rational functions to give a rational approximation to f. This technique is illustrated in its simplest form in §2 where we prove Popov's result (1.3) by using (1.5) with $k=2$.

The more general results of Brudnyi require extensions of (1.5) to estimate $\|f-P_x\|_{L_q(I)}$ in terms of the α-th order smoothness of f in L_p. There are two essential difficulties that prevents the use of simple maximal operators. First, α is not necessarily an integer and second and more importantly p is in general < 1. The usual notion of distributional derivative does not apply to $p < 1$. Fortunately there are maximal functions which will replace the role of $M(f^{(k)})$ in (1.5) for all $\alpha,p > 0$. These were introduced by A.P. Calderon [2] and studied extensively in [4]. With these maximal functions, one can show the existence of a polynomial $P_x f$ (now based on Peano derivatives of f) which generalizes (1.5) to all $\alpha, p > 0$.

In §4, we show that any function f of smoothness α in L_p (see §3 for the precise meaning of this) with $p > \sigma := \left(\alpha + \frac{1}{q}\right)^{-1}$ satisfies

$$r_n(f)_q = O(n^{-\alpha}). \tag{1.7}$$

The index σ is the smallest index which guarantees that functions with α order smoothness in L_p are in L_q. This same index occurs in optimal knot spline approximation (our techniques apply to this problem as well).

The estimate (1.7) is a slight variant of the results announced by Brudnyi [1]. He uses Besov spaces in his description of smoothness. We should note that we have not seen the proofs of Brudnyi's results, however he does state in [3, p. 320] that his proofs are not constructive.

2. POPOV'S THEOREM. Suppose $I_1,\dots,I_m$ are disjoint intervals in $[0,1]$ with $\bigcup_1^m I_j = [0,1]$ and $\xi_j \in I_j$, $j=1,\dots,m$. Let

$$\phi_j(y) := (1+|I_j|^{-2}(y-\xi_j)^2)^{-2}, \quad j=1,\dots,m. \tag{2.1}$$

Then,

$$\phi_j(y) \geq 2^{-2}, \quad y \in I_j; \; j=1,\dots,m. \tag{2.2}$$

Let $\phi := \sum_1^m \phi_j$, so that $\phi \geq 2^{-2}$ on $[0,1]$. The rational functions

$$R_j := \phi_j/\phi \tag{2.3}$$

satisfy

$$\text{(2.4)} \qquad \text{i)} \quad \sum_1^m R_j \equiv 1$$

$$\text{ii)} \quad R_j(y) \leq 4(1+|I_j|^{-2}(y-\xi_j)^2)^{-2} \qquad y \in [0,1].$$

THEOREM 2.1. If $f' \in BV$, there is a rational function R of degree $\leq n$ such that

$$\|f-R\|_\infty \leq c\|f'\|_{BV}\, n^{-2} \tag{2.5}$$

with c an absolute constant.

PROOF. Since any function f with $f' \in BV$ can be approximated uniformly by functions g with $\|g''\|_1 \le \|f'\|_{BV}$, it will be enough to prove (2.5) with $\|f'\|_{BV}$ replaced by $\|f''\|_1$. Also it is enough to construct R of degree $\le 16n$.

Now, the Hardy-Littlewood maximal function maps L_1 boundedly into L_r for all $r < 1$. This follows from the fact that M is of weak type (1,1) [10, p.5]. If we take $r = 3/4$ (actually any r strictly between 1/2 and 1 will do), then

$$\|M(f'')\|_r \le c_0 \|f''\|_1.$$

It will be enough to establish our result for f with $\|f''\|_1 = c_0^{-1}$. For such an f, choose intervals $I_1, \ldots, I_{2n}$ such that

(2.6) i) the I_j have pairwise disjoint interiors and $[0,1] = \bigcup_1^{2n} I_j$

ii) $|I_j| \le n^{-1}$, $j=1,\ldots,2n$

iii) $\int_{I_j} [M(f'')]^r \le n^{-1}$, $j=1,\ldots,2n$.

These intervals can be gotten by first finding n intervals which satisfy i) and iii) and then further subdividing them so as to guarantee ii).

Choose $\xi_j \in I_j$ so that $M(f'')(\xi_j) = \inf_{I_j} M(f'')$. The inf is attained since $M(f'')$ is lower semicontinuous. Let R_j be the partition of unity (2.3) for this choice of I_j and ξ_j. If $P_{\xi_j}(y) := f(\xi_j) + f'(\xi_j)(y-\xi_j)$, then (1.5) holds with $k=2$ and $x=\xi_j$, $j=1,\ldots,2n$. Set

$$R := \sum_1^{2n} P_{\xi_j} R_j.$$

Since $R_j = \phi_j/\phi$ with ϕ_j of degree ≤ 4 and ϕ of degree $\le 8n$, we have deg $R \le 16n$.

Now, we estimate $f-R$ using (1.5) and (2.4). First observe that from (1.5) and (2.6) iii), we have

$$(2.7) \qquad |f(y)-P_{\xi_j}(y)| \le (y-\xi_j)^2 M(f'')(\xi_j) = (y-\xi_j)^r \inf_{I_j} M(f'')$$
$$\le (y-\xi_j)^2 \Big(\frac{1}{|I_j|}\int_{I_j} M(f'')^r\Big)^{1/r} \le (y-\xi_j)^2 (n|I_j|)^{-1/r}.$$

Using this together with (2.4) ii), we have

$$(2.8)\qquad |f(y)-R(y)| \le \sum_1^{2n} |f(y)-P_{\xi_j}(y)||R_j(y)|$$

$$\le 4 \sum_1^{2n} (y-\xi_j)^2 (n|I_j|)^{-1/r}(1+|I_j|^{-2}(y-\xi_j)^2)^{-2}$$

$$\le 4 \sum_1^{\infty} S_\nu(y)$$

where $S_\nu(y)$ is the sum of those terms for intervals I_j which satisfy $2^{-\nu}n^{-1} < |I_j| \le 2^{-\nu+1}n^{-1}$ (recall $|I_j| \le n^{-1}$ because of (2.6) ii)).

Since $(y-\xi_j)^2 \le |I_j|^2(1+|I_j|^{-2}(y-\xi_j)^2)$, we have

$$(2.9)\qquad S_\nu(y) \le 4(2^\nu)^{1/r}2^{-2\nu}n^{-2} \sum_{I_j \in I_\nu} (1+|I_j|^{-2}(y-\xi_j)^2)^{-1}$$

with I_ν the set of those I_j which appear in S_ν. Each $I_j \in I_\nu$ has length $\ge 2^{-\nu}n^{-1}$ and the I_j are disjoint. Therefore for any integer $k \ge 0$ there are at most four ξ_j with $k\,2^{-\nu}n^{-1} \le |y-\xi_j| < (k+1)2^{-\nu}n^{-1}$. Using this in (2.9), we have

$$S_\nu(y) \le 16n^{-2}2^{-\nu(2-1/r)} \sum_0^{\infty} (1+(k/2)^2)^{-1} \le cn^{-2}2^{-\nu(2-1/r)}.$$

Thus, (2.8) gives

$$|f(y)-R(y)| \le cn^{-2} \sum_1^{\infty} 2^{-\nu(2-1/r)} \le cn^{-2}. \quad \square$$

3. MAXIMAL FUNCTIONS. Fix $\alpha,p > 0$. If I is an interval let $P_I f$ denote a best approximation to f from polynomials of degree (α) (greatest integer strictly less than α) in $L_p(I)$. We define

$$(3.1)\qquad f^b_{\alpha,p}(x) := \sup_{I \ni x} \frac{1}{|I|^\alpha}\left(\frac{1}{|I|}\int_I |f-P_I f|^p\right)^{1/p}$$

where the sup is taken over all intervals I which contain x.

The maximal function $f^b_{\alpha,p}$ measures the smoothness of f and is connected to many classical problems in analysis (see [4]). For example, for $k \ge (\alpha)+1$, we have the inequality,

$$(3.2)\qquad |\Delta_h^k(f,x)| \le ch^\alpha \sum_{\nu=0}^{k} f_{\alpha,p}^{b}(x+\nu h),\quad h>0,\ \text{a.e. in } x.$$

The proof of (3.2) is simple and illustrative, so we should at least sketch it. If $I \supset I^*$ are two intervals with $|I| \le 2|I^*|$ then

$$(3.3)\qquad \begin{aligned} \|P_I f - P_{I^*} f\|_{L_\infty(I^*)} &\le c|I|^{-1/p}\|P_I f - P_{I^*} f\|_{L_p(I^*)} \\ &\le c|I|^{-1/p}[\|f-P_I f\|_{L_p(I)} + \|f-P_{I^*}f\|_{L_p(I^*)}] \\ &\le c|I|^\alpha \inf_{I^*} f_{\alpha,p}^{b}. \end{aligned}$$

Here, the first equality is a comparison of polynomial norms (see [4, Lemma 3.1]). The inequality (3.3) holds without the restriction $|I| \le 2|I^*|$ since given any I and I^* we can choose $I_0 = I \supset I_1 \supset \cdots \supset I_n = I^*$ with $|I_j| = 2|I_{j+1}|$, $j=0,\ldots,n-2$ and $|I_{n-1}| \le 2|I_n|$. Then using (3.3), we find

$$(3.4)\qquad \begin{aligned} \|P_I f - P_{I^*} f\|_{L_\infty(I^*)} &\le \sum_{j=1}^{n} \|P_{I_{j-1}} f - P_{I_j} f\|_{L_\infty(I)} \le c\Big(\sum_{0}^{n} |I_j|^\alpha\Big)\inf_{I^*} f_{\alpha,p}^{b} \\ &\le c|I|^\alpha \inf_{I^*} f_{\alpha,p}^{b}. \end{aligned}$$

It follows from the Lebesgue differentiation theorem (see [4, Lemma 4.1]) that

$$(3.5)\qquad \lim_{I^*\downarrow\{x\}} P_{I^*} f(x) = f(x) \text{ a.e. }.$$

Hence for such x, taking a limit in (3.4) gives

$$(3.6)\qquad |P_I f(x) - f(x)| = \lim_{I^*\downarrow\{x\}} |P_I f(x) - P_{I^*} f(x)| \le c|I|^\alpha f_{\alpha,p}^{b}(x).$$

This gives (3.2) since given $x,\ldots,x+kh$, we choose I so that $|I| = kh$ and $x,\ldots,x+kh \in I$. Since $\deg P_I f < k$,

$$\begin{aligned} |\Delta_h^k(f,x)| = |\Delta_h^k(f-P_I f,x)| &\le c\sum_{0}^{k} |f(x+\nu h) - P_I f(x+\nu h)| \\ &\le ch^\alpha \sum_{0}^{k} f_{\alpha,q}^{b}(x+\nu h),\ \text{a.e. } x. \end{aligned}$$

We see from (3.2) that $f^b_{\alpha,p}$ behaves like a (fractional) derivative of f. In fact when α is an integer we have [4]

$$(3.7)\qquad c_1 f^{(\alpha)}(x) \le f^b_{\alpha,1}(x) \le c_2 M(f^{(\alpha)})(x) \quad \text{a.e.}$$

with c_1, c_2 depending only on α. Similar inequalities can be proved for other values of p.

Let us define the space of functions C^α_p as the set of functions f for which

$$\|f\|_{C^\alpha_p} := \|f\|_p + |f|_{C^\alpha_p}\ ; \quad |f|_{C^\alpha_p} := \|f^b_{\alpha,p}\|_p$$

are finite. These are smoothness spaces of order α in L_p. For example, it follows from (3.7) that

$$(3.8)\qquad C^k_p = W^k_p,\ 1 < p \le \infty;\ k=1,2,\dots\ .$$

Also, it is easy to show that $C^\alpha_\infty = \mathrm{Lip}\ \alpha$ for all $\alpha > 0$. Hence the C^α_p are generalizations of the Sobolev spaces to all $\alpha, p > 0$. For any p and α non-integral we have the embeddings

$$(3.9)\qquad B^{\alpha,p}_p \to C^\alpha_p \to B^{\alpha,\infty}_p$$

with $B^{\alpha,p}_p$ the Besov spaces (see [4]).

We now want to show how $f^b_{\alpha,p}$ can be used to extend inequality (1.5). We can no longer use ordinary derivatives in the definition of P_x and therefore we begin by introducing the notion of Peano derivatives.

The proof of (3.4) together with Markov's inequality shows that for $I \supset I^*$

$$(3.10)\qquad \|D^j(P_I f - P_{I^*} f)\|_{L_\infty(I^*)} \le c|I|^{\alpha-j} \inf_{I^*} f^b_{\alpha,p}\ .$$

Therefore, if $f^b_{\alpha,p}(x) < \infty$, then

$$(3.11)\qquad D_j f(x) := \lim_{I \downarrow \{x\}} D^j P_I f(x)$$

exists $j=0,\dots,(\alpha)$. The $D_j f$ are called the Peano derivatives of f. When these Peano derivatives exist, we define

$$P_x f(y) = \sum_{j=0}^{(\alpha)} D_j f(x) \frac{(y-x)^j}{j!} \tag{3.12}$$

The following lemma generalizes (1.5).

LEMMA 3.1. Suppose $1 \le q \le \infty$; $0 < p \le q$ and $\tau := \alpha + \frac{1}{q} - \frac{1}{p} > 0$. If $f \in C_p^\alpha$, then for almost all x and all intervals I containing x, we have

$$\|f - P_x f\|_{L_q(I)} \le c|I|^{\alpha+1/q} M_p(F)(x) \tag{3.13}$$

where $F := f_{\alpha,p}^b$ and $M_p(g) := (M(|g|^p))^{1/p}$.

PROOF. We first estimate $\|f-P_I f\|_{L_q(I)}$. Let $I_0 = \{I\}$ and in general the set of intervals I_j is gotten from I_{j-1} by halving the intervals in I_{j-1}. Define

$$S_j := \sum_{J \in I_j} P_J f \chi_J .$$

Then $S_0 = P_I f$ and $S_j \to f$ a.e. on I because of (3.5). It follows from the argument in (3.3) that whenever $J \in I_{j-1}$ and $J^* \in I_j$ with $J^* \subset J$, we have

$$\begin{aligned} \|P_J f - P_{J^*} f\|_{L_q(J^*)} &\le c|J|^{\alpha+1/q} \inf_{J^*} F \\ &\le c|J|^\tau \left(\int_{J^*} F^p\right)^{1/p}. \end{aligned}$$

Therefore,

$$\begin{aligned} \|S_j - S_{j-1}\|_{L_q(I)} &\le c|I|^\tau (2^{-j})^\tau \left\{\sum_{J \in I_j} \left(\int_J F^p\right)^{q/p}\right\}^{1/q} \\ &\le c|I|^\tau (2^{-j})^\tau \left(\int_I F^p\right)^{1/p} \end{aligned}$$

where in the last inequality we used the fact that an $\ell_{q/p}$ norm is smaller than an ℓ_1 norm $q/p \ge 1$. This last inequality gives

$$(3.14)\qquad \|f-P_If\|_{L_q(I)} = \lim_{j\to\infty} \|S_j-S_0\|_{L_q(I)} \le \sum_1^\infty \|S_j-S_{j-1}\|_{L_q(I)}$$

$$\le c|I|^{\alpha+1/q}\left(\frac{1}{|I|}\int_I F^p\right)^{1/p}$$

$$\le c|I|^{\alpha+1/q}M_p(F)(x).$$

In view of (3.10) and (3.11)

$$|D^j(P_If)(x)-D_jf(x)| \le c|I|^{\alpha-j}F(x)$$

and hence

$$(3.15)\qquad \|P_If-P_xf\|_{L_q(I)} \le |I|^{1/q}\|P_If-P_xf\|_{L_\infty(I)}$$

$$\le c|I|^{1/q}\sum_{j=0}^{(\alpha)} |D^j(P_If)(x)-D_jf(x)|\ \frac{|I|^j}{j!}$$

$$\le c|I|^{\alpha+1/q}F(x) \text{ a.e. } x.$$

Since $F \le M_p(F)$ a.e., (3.15) combines with (3.14) to give (3.3).□

4. **FURTHER RESULTS ON RATIONAL APPROXIMATION.** We need to modify slightly the partition of unity (2.3). Again, let I_j, $j=1,\ldots,m$ be intervals with disjoint interiors whose union is $[0,1]$ and let $\xi_j \in I_j$, $j=1,\ldots,m$. We define for $k:=(\alpha)+3$

$$\phi_j(y):=(1+|I_j|^{-2}(y-\xi_j)^2)^{-k};\quad \phi:=\sum_1^m \phi_j$$

and

$$(4.1)\qquad R_j:=\phi_j/\phi\ .$$

Then, the R_j satisfy

$$(4.2)\qquad \text{i)}\ \sum_1^m R_j \equiv 1 \text{ on } [0,1]$$

$$\text{ii)}\ |R_j(y)| \le 2^k(1+|I_j|^{-2}(y-\xi_j)^2)^{-k} \le c(1+|I_j|^{-1}|y-\xi_j|)^{-2k}\ .$$

Here the last inequality uses the fact that $(1+x^2)^{-k} \le c(1+x)^{-2k}$, $x \ge 0$.

THEOREM 4.1. Let $\alpha, p > 0$; $1 \le q \le \infty$ and $p > \sigma := (\alpha + \frac{1}{q})^{-1}$. If $f \in C_p^\alpha$ then

$$(4.3) \qquad r_n(f)_q \le c|f|_{C_p^\alpha} n^{-\alpha}$$

with c independent of n and f.

PROOF. The maximal function M_p boundedly maps L_p into L_r for all $r < p$. Choose $\sigma < r < p$. Then, there is a constant c_0 such that

$$\|M_p(F)\|_r \le c_0|f|_{C_p^\alpha}$$

with $F := f^b_{\alpha,p}$. It will be enough to prove (4.3) for functions f which satisfy $|f|_{C_p^\alpha} = c_0^{-1}$. In this case, $\|M_p(F)\|_r \le 1$ and hence we can choose intervals $I_1, \ldots, I_{2n}$ such that

(4.4) i) the I_j have disjoint interiors and $[0,1] = \bigcup_1^{2n} I_j$

ii) $|I_j| \le n^{-1}$, $j=1,\ldots,2n$

ii) $\int_{I_j} [M_p(F)]^r \le n^{-1}$.

For each $j=1,\ldots,2n$, choose a point $\xi_j \in I_j$ such that

$$M_p(F)(\xi_j) = \inf_{I_j} M_p(F).$$

Then,

$$(4.5) \qquad M_p(F)(\xi_j) \le \left(\frac{1}{|I_j|}\int_{I_j} M_p(F)^r\right)^{1/r} \le (n|I_j|)^{-1/r}.$$

Let P_{ξ_j} be the polynomial in Lemma 3.1 for $x = \xi_j$. The rational function

$$R := \sum_1^{2n} (P_{\xi_j} f)R_j$$

has degree $\leq 8kn$ and it will be enough to show that

$$\|f-R\|_q \leq cn^{-\alpha} \tag{4.6}$$

with c independent of f and n.

The proof of (4.6) is similar to the proof of Theorem 2.1. In fact, the case $q = \infty$ is identical and so we proceed only with the case $q < \infty$. Let I_ν be the set of those intervals I_j such that $2^{-\nu}n^{-1} < |I_j| \leq 2^{-\nu+1}n^{-1}$. Then, for any interval I_i, we have

$$\|f-R\|_{L_q(I_i)} \leq \sum_\nu \|S_\nu\|_{L_q(I_i)} \tag{4.7}$$

where $S_\nu := \sum_{I_j \in I_\nu} (f-P_{\xi_j})R_j$.

Suppose $I_i \in I_\mu$. Fix i for the moment. We want to estimate $\|S_\nu\|_{L_q(I_i)}$. It will be convenient to write $S_\nu = S_\nu^- + S_\nu^+$ where S_ν^- is the sum over all intervals in I_ν to the left of I_i and S_ν^+ is the sum over all intervals in I_ν to the right of I_i. We now estimate $\|S_\nu^-\|_{L_q(I_i)}$; the estimate for S_ν^+ is the same. If $\mu \geq \nu$, we let $J_0 := I_i$. If $\nu > \mu$, we take $m+1 := 2^{\nu-\mu}$ and let $J_0, \ldots, J_m$ be disjoint intervals of length $(m+1)^{-1}|I_i|$ whose union is I_i and which are ordered from left to right. It follows that for $I_j \in I_\nu$,

$$d_{ij} + s|I_j| \leq |y-\xi_j| \leq 2|I_j| + d_{ij} + s|I_j| \quad \text{when } y \in J_s$$

where $d_{ij} := \min\{|a-b| : a \in I_i, b \in I_j\}$. If $\bar{J}_s$ is the smallest interval which contains J_s and ξ_j, then $|\bar{J}_s| \leq d_{ij} + (s+2)|I_j|$. Using these facts with (3.13), (4.2) ii) and (4.5), we have, with
$a_{n,\nu} := (2^{-\nu})^{\alpha+1/q-1/r}\, n^{-(\alpha+1/q)}$,

$$\begin{aligned} \|S_\nu^-\|_{L_q(J_s)} &\leq \sum \|(f-P_{\xi_j})R_j\|_{L_q(J_s)} \\ &\leq ca_{n,\nu} \sum_{I_j \in I_\nu} (1+s+d_{ij}|I_j|^{-1})^{-2k+\alpha+1/q} \\ &\leq ca_{n,\nu} \sum_{\ell=1}^{\infty} (1+s+\ell)^{-2k+\alpha+1/q} \leq ca_{n,\nu}(1+s)^{-2k+\alpha+1/q+1}. \end{aligned} \tag{4.8}$$

Since $\tau := 2k - \alpha - \frac{1}{q} - 1 > 1$, we can use (4.8) and a similar estimate for S_ν^+ to find

$$(4.9) \qquad \|S_\nu\|_{L_q(I_i)} \leq ca_{n,\nu} \left(\sum_{s=1}^{\infty} (1+s)^{-\tau q} \right)^{1/q} = ca_{n,\nu} .$$

Using this back in (4.7) shows that $\|f-R\|_{L_q(I_i)} \leq cn^{-(\alpha+1/q)}$ and therefore

$$\|f-R\|_q = \left(\sum_1^{2n} \|f-R\|_{L_q(I_i)}^q \right)^{1/q} \leq cn^{-(\alpha+1/q)} \left(\sum_1^{2n} 1 \right)^{1/q} \leq cn^{-\alpha} . \quad \square$$

COROLLARY 4.2. If $1 \leq p,q \leq \infty$ and k is a non-negative integer ($k > 1$ if $p = 1$ and $q = \infty$) then for each $f \in W_p^k$ we have

$$r_n(f)_q \leq c\|f^{(k)}\|_p n^{-k} .$$

PROOF. If $p > 1$, this follows from Theorem 4.1 and (3.8). When $p = 1$, the same proof as Theorem 2.1 gives the result. $\square$

COROLLARY 4.3. If $\alpha, p > 0$; $p \leq q \leq \infty$ and $\alpha + \frac{1}{p} - \frac{1}{q} > 0$ then whenever α is not an integer and $f \in B_p^{\alpha,p}$, we have

$$r_n(f)_q \leq c\|f\|_{B_p^{\alpha,p}} n^{-\alpha} .$$

PROOF. This follows from Theorem 4.1 and (3.9). $\square$

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