APPROXIMATION ORDER FROM SMOOTH BIVARIATE PP FUNCTIONS

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The question of how smoothness requirements affect the approximation power of multivariate pp functions is explored, in the simple context of bivariate functions on a two- and three-direction mesh. The underlying emphasis is on the unresolved question: What is the precise relationship between the approximation power and the existence of a suitable local partition of unity in the approximating pp space?

This is a progress report on work [1] reported at the last Texas Approximation Theory conference by R. DeVore. It concerns approximation from

$$S := \pi^{\rho}_{k,\delta} := \pi_{k,\delta} \cap C^{\rho},$$

the space of pp (:= piecewise polynomial) functions of degree < k on some partition or subdivision δ and constrained to have continuous derivatives of all orders < ρ. For a smooth function f, one would expect to get

$$\text{dist}(f, S) \sim \text{const}_f\, |\delta|^m,$$

with $|\delta|$ the mesh size, and m an exponent which, for $\rho = -1$, is just $k+1$, but which may well decrease when we increase ρ. We are interested in the precise relationship between ρ and m.

For simplicity, we consider only the situation in which the various partitions are all obtained from a fixed partition, by scaling. Precisely, we consider the function $h \longmapsto \text{dist}(f, S_h)$, with $S_h := \sigma_h(S)$ and $\sigma_h f: x \longmapsto f(x/h)$. We say that m is the approximation order from the scale (S_h) in case

$(i)_m$ for all smooth functions, $\text{dist}(f,S_h) = O(h^m)$,

$(ii)_m$ for some smooth function, $\text{dist}(f,S_h) \neq o(h^m)$.

Because of the simple nature of such a scale, it is not hard to prove [2] that

(1) $$(i)_m \Longrightarrow \pi_{m-1} \subseteq S.$$

The converse clearly does not hold; take, e.g., $S = \pi_{m-1}$. What needs to be added to the right side of (1) to achieve equivalence with the left?

One would expect the approximation power of pp functions to come from their <u>local</u> flexibility. To make this precise, certain locally supported smooth pp functions called box splines were introduced in [1] and further studied in [2], [3]. These box splines are taylormade for S in case $\delta = \Sigma :=$ a square subdivision, i.e., a two-direction mesh, or $\delta = \Delta :=$ the triangulation obtained from Σ by drawing in all north-east diagonals, i.e., a three-direction mesh. In either case, denote by S_{loc} the span of the translates of the various box splines belonging to S. With this, one can show [3] that

(2) $$(i)_m \implies \pi_{m-1} \subseteq S_{loc} .$$

Since S_{loc} decreases when we increase the smoothness demand ρ, this makes the relationship between m and ρ somewhat more explicit. In particular, it allows the conclusion that the approximation order is 0 in case ρ is so large that there are no functions in S with compact support. According to [1], this happens unless

(3) $$\rho < \begin{cases} (k-2)/2 & , \text{ if } \delta = \Sigma \\ (2k-2)/3 & , \text{ if } \delta = \Delta \end{cases}$$

Further, even if ρ satisfies (3), the approximation order from (S_h) may be less than $k+1$ (the optimal one according to (1)), unless $\rho = -1$. Precisely [1],

$$m = k-\rho$$

for the two-direction mesh. For the three-direction mesh, the precise approximation order has not yet been found. According to [3], if ρ satisfies (3), i.e., $\rho < \rho(k) := \lfloor (2k-2)/3 \rfloor$, then

$$m \in [\rho(k)+2, m(k)] \quad \text{with} \quad m(k) := \min\{2(k-\rho), k+1\} .$$

In particular, $m = \rho(k)+2$ when $\rho = \rho(k)$ and $k \equiv 1(3)$. But the interval increases as ρ decreases. Yet, [3] expresses the hope that m

always stays within 1 of the upper bound $m(k)$. At this conference, Jia [6] reports that m stays always within 2 of that upper bound.

The conclusion of (2) is so much stronger than that of (1) that, for a while, we hoped that (2) could be reversed. In fact, [1] shows that

$$(i)_m \iff \pi_{m-1} \subseteq S_{loc}$$

in case of the two-direction mesh. This equivalence also holds for the three-direction mesh for all cases that can be settled by inspection, i.e., for $k = 0$, for $\rho < 1$, and for $\rho = 1$ and $k = 1, 2$. It breaks down, though, for the first nontrivial case, i.e., for $\rho = 1$ and $k = 3$: In this case, $m = 3$ even though $\pi_3 \subseteq S_{loc}$, as is shown in [4]. This means that yet something else has to be added to the conclusion in (2) to make it equivalent to $(i)_m$.

An alternative is to strengthen $(i)_m$. E.g., W. Dahmen at this conference reported on joint work [5] with C. A. Micchelli in which they make use of (2) to give the precise <u>controlled approximation order</u> from (S_h) for the case of maximum smoothness, i.e., $\rho = \rho(k)$ (and for the three-direction mesh). This is the approximation order achievable by a <u>quasi-interpolant</u>, i.e., an approximant of the form $\sigma_h Q \sigma_{1/h} f$, with

$$Qf \ := \ \Sigma_{j \in J} \lambda g(\cdot + j) M(\cdot - j) \quad ,$$

J the mesh points, λ a suitable compactly supported linear functional and M an element of S of compact support. It is entirely unclear, though, why, even for these simple meshes, the controlled approximation order should always be the same as the approximation order.

There is a simpler version of this problem of relating approximation power to smoothness, viz. the following question (related by (2) to $(i)_1$):

$$\text{for all } f \in C , \ \operatorname{dist}(f, S_h) = o(1) \ \overset{?}{\Longrightarrow} \ 1 \in S_{loc} \text{ stably}$$

I.e., is the eventual denseness of (S_h) in C sufficient for having a stable local partition of unity in S , i.e., some (M_j) with $\Sigma_j |M_j| < \infty$. sup. diam supp $M_j < \infty$, and $\Sigma M_j = 1$? It is certainly necessary.

References

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