# n-WIDTHS FOR $\mathscr{E}_p^\alpha$ SPACES

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The smoothness spaces $\mathscr{E}_p^\alpha$, introduced by DeVore and Sharpley in [5], coincide with the Sobolev spaces $W_p^\alpha$ for integer $\alpha$ and $p > 1$. Considering them as a natural extension of the Sobolev spaces for fractional $\alpha$ and values of $p > 0$, we compute the n-widths $d_n(U(\mathscr{E}_p^\alpha), L_q)$ for $\alpha > N/p - N/q$, $0 < p \leq \infty$, $1 \leq q \leq +\infty$.

## 1. Introduction

In the last several years smoothness spaces, defined using in some way the $L_p$ norm for $0 < p < 1$, have played an increasingly important role in approximation theory. For example, Brudnyi [2] described the functions that can be approximated in $L_q$ by rational functions of degree $n$ to the order $n^{-\alpha}$ in terms of Besov and Lipschitz spaces defined using $p = (\alpha + 1/q)^{-1}$. Likewise, spaces of generalized bounded variation $V_p^\lambda$, $0 < p \leq \infty$, arose naturally in the study of approximation by splines with free knots (see e.g. [1,2,3]). DeVore [4] used the Hardy-Littlewood maximal function as a mapping from $L_1$ to $L_p$, $1/2 < p < 1$, to give a short elementary proof of Popov's Theorem on rational approximation. This led DeVore to use other maximal functions, $f_{\alpha,p}^b$, and their associated spaces $\mathscr{E}_p^\alpha$ ($f_{\alpha,p}^b \in L_p$) to study rational functions. All of these spaces are in some sense a natural replacement for Sobolev spaces when $0 < p < 1$. An advantage of the $\mathscr{E}_p^\alpha$ spaces is that the maximal function $f_{\alpha,p}^b$ behaves like a (fractional) derivative and is relatively easy to use.

There has also been a resurgence of research in the theory of n-widths for smoothness spaces due largely to the deep results of Kashin [10-14] and Gluskin [6-7]. If the unit ball $U(X)$ of some quasi-normed space $X$ is compactly embedded in the Banach space $Y$, then the Kolmogorov n-width of

$U(X)$ is the quantity

$$d_n(U(X),Y) = \inf_{Y_n} \sup_{x \in U(X)} \inf_{y \in Y_n} \|x-y\|_Y , \tag{1.1}$$

where $Y_n$ ranges over all possible n-dimensional subspaces of $Y$. The problem is to determine $d_n(U(X),Y)$ as a function of $n$, at least up to constants.

When $X = \ell_p^m$, $1 \le p \le \infty$, and $Y = \ell_q^m$, $1 \le q \le \infty$, the results of Kashin and Gluskin complete the asymptotic determination of $d_n(U(\ell_p^m),\ell_q^m)$ as a function of $m,n$. By discretization techniques dating back to Maiorov [16], the finite dimensional results permit the determination of the n-widths $d_n(U(W_p^\alpha),L_q)$ and $d_n(U(B_p^{\alpha,r}),L_q)$ for Sobolev and Besov spaces on $\Omega = [0,1]^N$ (see [8-15]). The orders of $d_n(U(W_p^\alpha),L_q)$ and $d_n(U(B_p^{\alpha,r}),L_q)$ are the same as those for $d_n(U(\mathscr{C}_p^\alpha),L_q)$, $1 \le p \le \infty$, given in Theorem 1. (Some of the upper estimates on $d_n(U(W_p^\alpha),L_q)$ appearing in the literature have an additional power of $\log n$ that, in light of Gluskin's subsequent results [7], can be removed for all but two values of $\alpha$.) It is the purpose of the present note to extend these results to $0 < p < 1$ using $\mathscr{C}_p^\alpha$ spaces.

## 2. Spaces $\mathscr{C}_p^\alpha$ and their Widths

The spaces $\mathscr{C}_p^\alpha = \mathscr{C}_p^\alpha[0,1]^N$, $0 < p \le \infty$, $0 < \alpha$, are defined by means of certain maximal functions. For any cube $Q \subset \Omega = [0,1]^N$, and a function $f \in L_p(Q)$, let $\mathcal{P}_Q f$ be any best approximant to $f$ in the $L_p(Q)$ norm from the subspace $P_{(\alpha)}$ of polynomials with total degree $\le (\alpha)$, where $(\alpha)$ is the greatest integer $< \alpha$. Define

$$\begin{aligned} f^b_{\alpha,p}(x) &:= \sup_{\substack{Q \\ x \in Q \subset \Omega}} |Q|^{-\alpha/N}\Big(\frac{1}{|Q|}\int_Q |f-\mathcal{P}_Q f|^p\Big)^{1/p} \\ &= \sup_{\substack{Q \\ x \in Q \subset \Omega}} \inf_{\pi \in P_{(\alpha)}} |Q|^{-\alpha/N}\Big(\frac{1}{|Q|}\int_Q |f-\pi|^p\Big)^{1/p} . \end{aligned} \tag{2.1}$$

The space $\mathscr{C}_p^\alpha$ is the set of functions for which the quantity

(2.2) $$\|f\|_{\mathscr{C}_p^\alpha} := \|f\|_{L_p(\Omega)} + |f|_{\mathscr{C}_p^\alpha}, \quad |f|_{\mathscr{C}_p^\alpha} := \|f^b_{\alpha,p}\|_{L_p(\Omega)}$$

is finite. When $1 \le p \le \infty$, $\|f\|_{\mathscr{C}_p^\alpha}$ is a norm, but for $0 < p < 1$ it is only a quasi-norm.

The spaces $\mathscr{C}_p^\alpha$ are smoothness spaces in the sense that the statement "$f \in \mathscr{C}_p^\alpha$" implies smoothness or differentiability properties on $f$. Indeed, if $f \in L_p(\Omega)$ and $f^b_{\alpha,p}(x) < +\infty$, then the Peano derivative $D_\nu f$ exists at $x$ for any multi-index $\nu$ with $|\nu| < \alpha$. Moreover, when $p > 1$, the weak derivatives of order $\nu$, $|\nu| < \alpha$ exist, and we even have the relation

(2.3) $$W_p^\alpha = \mathscr{C}_p^\alpha \quad \text{if} \quad \alpha \in Z^+ \quad \text{and } 1 < p \le \infty.$$

When $p = 1$, the proper inclusion $\mathscr{C}_1^\alpha \subset W_1^\alpha$ holds for $\alpha \in Z^+$.

For non-integral $\alpha$, the spaces $\mathscr{C}_p^\alpha$ are related to other smoothness spaces of fractional order. We need the embedding with Besov spaces

(2.4) $$B_p^{\alpha,p} \subset \mathscr{C}_p^\alpha \subset B_p^{\alpha,\infty}, \quad \alpha \text{ not an integer, } 0 < p \le \infty.$$

These embeddings are proper and unimprovable within the scale of Besov spaces.

Various spaces $\mathscr{C}_p^\alpha$ are related by the embeddings

(2.5) $$\mathscr{C}_p^\alpha \subset \mathscr{C}_q^\beta, \quad 0 < p \le q \le \infty \quad \text{and} \quad 0 \le \beta \le \alpha - \frac{N}{p} + \frac{N}{q}.$$

(When $\alpha = 0$, $\mathscr{C}_p^\alpha := L_p$.) Of importance for n-widths is the compact embedding

(2.6) $$\mathscr{C}_p^\alpha \subset L_q, \quad \alpha > \frac{N}{p} - \frac{N}{q}, \quad 0 < p \le q \le \infty.$$

A detailed discussion of the spaces $\mathscr{C}_p^\alpha$, including the proofs of the above remarks, can be found in [5].

The above discussion indicates that the $\mathscr{C}_p^\alpha$ spaces form a natural framework for the extension of the n-width results for Sobolev spaces to the case $0 < p < 1$. To describe the results we divide the parameters $(p,q)$

into four regions (see figure 1):

$$\text{I: } 1 \le q \le p \le \infty; \qquad \text{II: } 0 < p \le q \le 2,\ 1 \le q$$
$$\text{III: } 2 \le p < q \le \infty; \qquad \text{IV: } 0 < p \le 2 \le q \le \infty.$$

THEOREM 1. *The asymptotic order of the n-widths* $d_n(U(\mathscr{E}_p^\alpha), L_q)$ *are as follows: For* $(p,q) \in$ I,

$$n^{-\alpha/N}, \text{ if } \alpha > \frac{N}{p} - \frac{N}{q}; \tag{2.7}$$

*for* $(p,q) \in$ II,

$$n^{-(\frac{\alpha}{N} - \frac{1}{p} + \frac{1}{q})}, \text{ if } \alpha > \frac{N}{p} - \frac{N}{q} \tag{2.8}$$

*for* $(p,q) \in$ III,

$$\begin{aligned} &n^{-\frac{\alpha}{N}}, \text{ if } \alpha > \left(\frac{N}{p} - \frac{N}{q}\right)\Big/\left(1 - \frac{2}{q}\right) \text{ or} \\ &n^{-\frac{q}{2}(\frac{\alpha}{N} - \frac{1}{p} + \frac{1}{q})}, \text{ if } \frac{N}{p} - \frac{N}{q} < \alpha < \left(\frac{N}{p} - \frac{N}{q}\right)\Big/\left(1 - \frac{2}{q}\right); \end{aligned} \tag{2.9}$$

*and for* $(p,q) \in$ IV,

$$\begin{aligned} &n^{-(\frac{\alpha}{N} - \frac{1}{p} + \frac{1}{2})}, \text{ if } \alpha > \frac{N}{p} \text{ or} \\ &n^{-\frac{q}{2}(\frac{\alpha}{N} - \frac{1}{p} + \frac{1}{q})}, \text{ if } \frac{N}{p} - \frac{N}{q} < \alpha < \frac{N}{p}. \end{aligned} \tag{2.10}$$

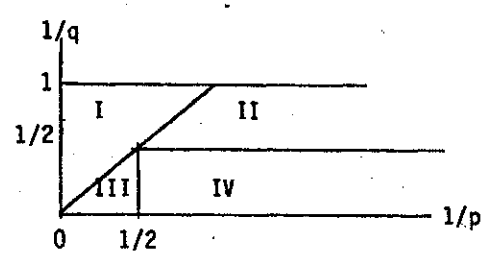


Figure 1

As mentioned in the introduction, the orders given in Theorem 1 hold for $d_n(U(B_p^{\alpha,r}),L_q)$ and $d_n(U(W_p^{\alpha}),L_q)$, when $1 \le p,q \le \infty$, $0 < r \le \infty$. When $\alpha = (\frac{N}{p} - \frac{N}{q})/(1 - \frac{2}{q})$ for $(p,q) \in$ III and $\alpha = \frac{N}{p}$ for $(p,q) \in$ IV, the lower estimate for the n-widths has the order given in the theorem but the upper estimate contains an additional power of log n. The closing of this gap is the remaining major problem.

When $1 \le p,q \le \infty$, Theorem 1 is an immediate consequence of the basic fact

$$(2.11) \qquad X \subset Z,\ W \subset Y \Rightarrow d_n(U(X),Y) \le O(1)d_n(U(Z),W)$$

together with the embeddings (2.3), (2.4) and the known orders for $d_n(U(W_p^{\alpha}),L_q)$, $d_n(U(B_p^{\alpha,r}),L_q)$. Throughout the remainder of the paper $O(1)$ will denote a generic constant, independent of $n$, which may be different at each occurance.

## 3. Upper Estiamtes for $0 < p < 1$

When $0 < p < 1$, we can successfully use the embeddings (2.5) with the fact (2.11). For $(p,q) \in$ IV, $0 < p < 1$, we can transfer to the case $p \ge 1$ by the embedding $\mathscr{E}_p^{\alpha} \subset \mathscr{E}_2^{\beta}$ where $\beta = \alpha - N/p + N/2$. Furthermore, $\beta > N/2$ iff $\alpha > N/p$, and $N/2 - N/q < \beta < N/2$ iff $N/p - N/q < \alpha < N/p$. Substituting $\beta = \alpha - N/p + N/2$ into the upper estimates for $d_n(U(\mathscr{E}_2^{\beta}),L_q)$ and using (2.11), we obtain the order (2.10) for the upper estimate of $d_n(U(\mathscr{E}_p^{\alpha}),L_q)$.

Similarly, the estimate in region II can be reduced to the case $p = q$. Indeed, for $(p,q) \in$ II, we have $\mathscr{E}_p^{\alpha} \subset \mathscr{E}_q^{\beta}$, $\beta = \alpha - N/p + N/q$, and the upper estimate follows from the estimate for $d_n(U(\mathscr{E}_q^{\beta}),L_q)$.

If the order of $d_n(U(\mathscr{E}_p^{\alpha}),L_q)$ were known, then from the embeddings (2.3), (2.4) together with the fact that the Besov spaces, $B_p^{\alpha,r}$ are interpolation spaces between pairs $\mathscr{E}_p^{\alpha_0}$, $\mathscr{E}_p^{\alpha_1}$ (see [5]), it would be possible to determine the orders of $d_n(U(W_p^{\alpha}),L_q)$ and $d_n(U(B_p^{\alpha,r}),L_q)$. A direct proof of the upper estimates in Theorem 1 uses the standard discretization techniques and Lemma 1 below.

Let $\{Q_j\}$ be a partition of $\Omega = [0,1]^N$ into $n$ equal cubes of volume 1/n. Let $PP_n$ denote the space of all functions $f$ such that $f|_{Q_j} \in P_{(\alpha)}$. Clearly, $\dim PP_n = O(1)n$.

LEMMA 1. <u>For</u> $f \in \mathscr{B}_p^\alpha$, <u>there is an</u> $S \in PP_n$ <u>such that for any</u> $q$ <u>with</u> $0 < p \le q \le \infty$ <u>and</u> $\alpha > N/p - N/q$, <u>we have</u>

$$\|f - S\|_{L_q(\Omega)} \le O(1)n^{-(\frac{\alpha}{N} - \frac{1}{p} + \frac{1}{q})}\|f\|_{\mathscr{B}_p^\alpha}. \tag{3.1}$$

We only sketch the proof. Select $S \in PP_n$ so that $S|_{Q_j} = P_{Q_j}f$, and use the local weak type inequality relating the decreasing rearrangement $[(f-S)\chi_{Q_j}]^*$ and $f^b_{\alpha,p}$ in Lemma 4.2 of [5]. From this inequality and Hardy's inequality, we obtain $\|f - S\|_{L_q(Q_j)} \le O(1)|Q_j|^{\frac{\alpha}{N} - \frac{1}{p} + \frac{1}{q}}\|f^b_{\alpha,p}\|_{L_p(Q_j)}$. To pass to the norms on $\Omega$, we use the fact that $\ell_p \subset \ell_q$ when $p \le q$.

## 4. Lower estimates

For the lower estimates, we follow the approach of Höllig [9]. The method requires the embedding of $\ell_p^m$ into $\mathscr{B}_p^\alpha$ through the coefficients of some spline expansion.

Let $M(x)$ be the tensor product of the one dimensional cardinal B-spline, normalized so that $\operatorname{supp} M(x) = [0,1]^N = \Omega$. The degree of the cardinal B-spline is chosen so large that $M \in \mathscr{B}_p^\alpha(R^N)$.

Let $\{Q_j\}$ be the decomposition of $[0,1]^N$ into cubes of volume $m^{-1}$. Choose $x_j \in Q_j$ so that $Q_j - x_j = m^{-1/N}\Omega$, and define $M_j(x) = M(m^{1/N}(x-x_j))$. Then $\operatorname{supp} M_j = Q_j$ and $\|M_j\|_{L_p(\Omega)} = m^{-1/p}\|M\|_{L_p(R^N)}$.

Let $S_m := \{S(x) = \sum c_j M_j(x) : \{c_j\} \in \ell_p^m\}$. Then $\dim S_m = m$ and $S_m \subset \mathscr{B}_p^\alpha$. We need the estimates

$$\|S\|_{L_r(\Omega)} \le O(1)m^{-1/r}\|\{c_j\}\|_{\ell_r^m}, \quad 0 < r \le \infty, \ S \in S_m \tag{4.1}$$

(4.2) $$\|\{c_j\}\|_{\ell_r^m} \leq O(1)m^{1/r}\|S\|_{L_r(\Omega)}, \quad 0 < r \leq \infty, \; S \in \mathcal{S}_m$$

and

(4.3) $$\|S\|_{\mathcal{C}_p^\alpha} \leq O(1)m^{\frac{\alpha}{N}-\frac{1}{p}}\|\{c_j\}\|_{\ell_p^m}, \quad 0 < p \leq \infty, \; \alpha > 0.$$

Inequalities (4.1) and (4.2) are straightforward (the second uses the fact that $M_j^r(x) \geq C^r > 0$ on a cube $Q_j^* \subset Q_j$ with $2|Q_j^*| = |Q_j|$). Before proving (4.3), we use it to derive the lower bounds.

Assume $0 < p < 1$ and $1 \leq q \leq \infty$. Let $P$ be a bounded projection from $L_q(\Omega)$ onto $\mathcal{S}_m \cap L_q(\Omega)$, for example, $Pf(x) = \sum_j a_j(\int_{Q_j} f)M_j(x)$, where $a_j = (\int_{Q_j} M_j(x)dx)^{-1}$. Then

(4.4) $$d_n(U(\mathcal{C}_p^\alpha) \cap \mathcal{S}_m, L_q \cap \mathcal{S}_m) \leq \|P\|d_n(U(\mathcal{C}_p^\alpha) \cap \mathcal{S}_m, L_q) \leq O(1)d_n(U(\mathcal{C}_p^\alpha), L_q).$$

Factoring the identity map $I: \ell_p^m \to \ell_q^m$ as

$$I: \ell_p^m \xrightarrow{T_1} \mathcal{C}_p^\alpha \cap \mathcal{S}_m \xrightarrow{J} L_q \cap \mathcal{S}_m \xrightarrow{T_2} \ell_q^m$$

where $J$ is the embedding operator, we obtain by (4.2)-(4.4)

(4.5) $$\begin{aligned} d_n(U(\ell_p^m), \ell_q^m) &\leq \|T_1\|\|T_2\|d_n(U(\mathcal{C}_p^\alpha) \cap \mathcal{S}_m, L_q \cap \mathcal{S}_m) \\ &\leq O(1)m^{\frac{\alpha}{N}-\frac{1}{p}+\frac{1}{q}}d_n(U(\mathcal{C}_p^\alpha), L_q). \end{aligned}$$

In [7], Gluskin gave the lower bounds,

(4.6) $$d_n(U(\ell_p^m), \ell_q^m) \geq O(1)\begin{cases} m^{1/q-1/p}, & (p,q) \in II \\ m^{1/q-1/2}(\frac{m}{n}-1)^{1/2}, & (p,q) \in IV \end{cases}$$

valid for $n^{q/2} \geq m \geq 2n$. We take $m = 2n$ in (4.5) and (4.6) to obtain

(2.8) and the first case in (2.10), and $m \sim n^{q/2}$ to obtain the second case in (2.10). It should be noted that Gluskin does not state (4.6) for $0 < p < 1$, but his proof does give this case.

It remains to prove (4.3). Let $0 < p < 1$. We have to estimate $|S|_{\mathscr{C}_p^\alpha}$, $S = \sum c_j M_j$, in terms of $\|\{c_j\}\|_{\ell_p^m}$. Let $x$ and $Q$ be given with $x \in Q \subset \Omega$. Set $Q_j^* := \{m^{1/N}(x-x_j): x \in Q\}$. Then $|Q_j^*| = m|Q_j|$. Choose $P_j^* \in P_{(\alpha)}$ to be a best approximation in $L_p(Q_j^*)$ to $M(x)$. Then $P_j(x) = P_j^*(m^{1/N}(x-x_j))$, $x \in Q$, is a best approximation in $L_p(Q)$ to $M_j(x)$. Let $P = \sum_j c_j P_j$. Then

$$\begin{aligned}\int_Q |S - P|^p &\le \sum_j |c_j|^p \int_Q |M_j - P_j|^p = m^{-1} \sum_j |c_j|^p \int_{Q_j^*} |M - P_j^*|^p \\ &\le m^{\alpha p/N} |Q|^{\alpha p/N+1} \sum_j |c_j|^p |Q_j^*|^{-\alpha p/N-1} \int_{Q_j^*} |M - P_j^*|^p \\ &\le m^{\alpha p/N} |Q|^{\alpha p/N+1} \sum_j |c_j|^p (\inf_{Q_j^*} M_{\alpha,p}^b)^p \\ &\le m^{\alpha p/N} |Q|^{\alpha p/N+1} \sum_j |c_j|^p (M_{\alpha,p}^b(m^{1/N}(x-x_j)))^p.\end{aligned}$$

Therefore,

$$S_{\alpha,p}^b(x) \le m^{\alpha/N} \{\sum |c_j|^p (M_{\alpha,p}^b(m^{1/N}(x-x_j)))^p\}^{1/p} , \tag{4.7}$$

and

$$\begin{aligned}|S|_{\mathscr{C}_p^\alpha} = \|S_{\alpha,p}^b\|_{L_p(\Omega)} &\le m^{\alpha/N} \{\sum_j |c_j|^p \|M_{\alpha,p}^b(m^{1/N}(\cdot - x_j))\|^p_{L_p(R^N)}\}^{1/p} \\ &\le O(1) m^{\alpha/N-1/p} \{\sum_j |c_j|^p\}^{1/p}.\end{aligned}$$

Since the proof of (4.3) for $1 \le p \le \infty$ is more technical and this case is not needed here, we give only a brief sketch of the argument. As above we arrive at (4.7) with $p = 1$; which is all that is required since $\|S_{\alpha,1}^b\|_{L_p(\Omega)} \sim \|S_{\alpha,p}^b\|_{L_p(\Omega)}$ (see [5]). However, since the supports of $M_{\alpha,1}^b(m^{1/N}(x-x_j))$ overlap significantly we cannot take the power and integration through the sum and pull out a common $\|M^b_{\alpha,1}\|_{L_p(R^N)}$ factor. The trick is to look at the integral over a fixed cube $Q_{j_0}$, and to separate the sum into the cubes touching $Q_{j_0}$ (good terms) and the rest (bad terms). For the small number of cubes touching $Q_{j_0}$ we can take the power and integration through the sum at the expense of a constant. For the remaining $Q_j$, we use the fact that the maximal function $M^b_{\alpha,1}(u)$ can be estimated in terms of the dist(u, supp M) to obtain $M^b_{\alpha,1}(m^{1/N}(x-x_j)) \le O(1)[m\ \mathrm{dist}(Q_j,Q_{j_0})]^{-\alpha-N}$ for $x \in Q_{j_0}$. After summing over $j_0$, the resulting estimate for the bad terms can be viewed as the convolution of $\{c_j\}$ with a $\ell_1$-kernel.

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