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THE N-WIDTH OF BV ∩ Lip α

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We determine asymptotically the n-width of the space of functions BV ∩ Lip α as a subset of C. Since the n-width behaves like $n^{-(\alpha+1)/2}$, it is seen that this class of functions has significantly smaller n-width than Lip α.

There are many theorems of analysis which show that the space $S_\alpha := BV \cap \text{Lip}\,\alpha$ is small when compared to Lip α. For example, we know that each $f \in S_\alpha$ has an absolutely convergent Fourier series if $\alpha > 0$ but this is not the case for Lip α unless $\alpha > \frac{1}{2}$. In this note, we shall measure the size of S_α by computing the n-width of the unit ball of S_α in the space $C_{2\pi}$ of 2π periodic continuous functions.

One way to measure the size of a compact set K of a Banach space X is through the notion of n-widths. If X_n is an n-dimensional subspace of X then

$$E(f,X_n) := \inf_{x_n \in X_n} \|f-x_n\|_X$$

is the error in approximating f by elements of X_n and

$$E(K,X_n) := \sup_{f \in K} E(f,X_n)$$

is the error in approximating the compact set K. The n-width of K looks for best n-dimensional subspaces:

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(1) $$d_n(K)_X : = \inf_{X_n} E(K,X_n) .$$

The asymptotic behavior of $d_n(K)_X$ gives some idea of the size of K in X. For example if $C_{2\pi}$ denotes the space of continuous 2π periodic functions, then the unit ball $U(\text{Lip}\ \alpha)$ of $\text{Lip}\ \alpha$ has n-width

(2) $$d_n(U(\text{Lip}\ \alpha))_{C_{2\pi}} \approx n^{-\alpha}, \quad n \to \infty .$$

Actually, there is a much larger set than $U(\text{Lip}\ \alpha)$ which also has n-width $n^{-\alpha}$. It was shown by B. S. Kashin [1] that the Sobolev space W_2^α satisfies:

$$d_n(U(W_2^\alpha))_{C_{2\pi}} \approx n^{-\alpha}, \quad \alpha > \tfrac{1}{2} .$$

Since $W_2^\alpha = B_2^{\alpha,2}$ (the Besov space), it follows from an interpolation of linear operators argument that

(3) $$d_n(U(B_2^{\alpha,r})) \approx n^{-\alpha}, \quad \alpha > \tfrac{1}{2}; \quad r > 0 .$$

Since $\text{Lip}\ \alpha \subsetneq B_2^{\alpha,r}$, (3) is a (substantial) improvement over (2).

The main result of this note is to determine the n-width of the set $K_\alpha : = \{f \in S_\alpha : \max (\|f\|_{BV}, \|f\|_{\text{Lip}\ \alpha}) \le 1\}$ in $C_{2\pi}$. The following result should be compared with (2).

Theorem 1. <u>For each</u> $0 < \alpha \le 1$, <u>there are constants</u> $C_1, C_2 > 0$ <u>such that</u>

(4) $$C_1\, n^{-(\alpha+1)/2} \le d_n(K_\alpha)_{C_{2\pi}} \le C_2\, n^{-(\alpha+1)/2} .$$

Proof. The upper estimate follows from an embedding of K_α into a Besov space. If $f \in K_\alpha$ and $h > 0$, we have

(5) $$\int_0^{2\pi} |f(x+h) - f(x)|^2 dx \le h^\alpha \int_0^{2\pi} |f(x+h) - f(x)|\, dx \le h^{\alpha+1}$$

The last inequality is apparent for absolutely continuous functions and follows for any f in K_α since such an f is the uniform limit of absolutely continuous functions from K_α.

It follows from (5) that $K_\alpha \subset U(B_2^{\beta,\infty})$, $\beta: = (\alpha+1)/2$ and so from (3),

$$d_n(K_\alpha) \leq d_n(U(B_2^{\beta,\infty})) \leq C\, n^{-\beta} = n^{-(\alpha+1)/2}$$

which is the right hand side of (4).

To prove the lower estimate in (4), we shall use an idea of W. Rudin [2] and construct sufficiently many functions in S_α which are mutually orthogonal. In part, the orthogonality will come from making supports disjoint and in part, by using Latin squares to guarantee orthogonality where supports are not disjoint.

It is enough to prove (4) when n is a 2 power. Let $N: = 2n$ and choose <u>m as the largest 2 power</u> such that

$$2m \leq (2N/\pi)^\alpha$$

It follows that

$$m \geq \tfrac{1}{4}(2N/\pi)^\alpha \tag{6}$$

Also $m \leq N$.

Now divide $[0,2\pi]$ into N/m intervals $I_\nu: = [x_\nu, x_{\nu+1}]$ $\nu = 0,1,\ldots,(N/m)-1$: $x_\nu: = 2\nu\pi\, m/N$. For each ν we choose points $x_{\mu\nu}: = x_\nu + (2\mu-1)\pi/N$, $\mu = 1,\ldots, m$ from I_ν.

We will also need a Latin square

$$A: = (a_{ij}) \qquad a_{ij} = \pm 1$$

of dimension $m \times m$. Let $v_1,\ldots,v_m$ denote the columns of A. Then, these vectors are orthogonal.

Here is how we construct our orthogonal functions. If $1 \leq i \leq N$, we write

$$i = km + \ell; \; 1 \le \ell \le m .$$

Define g_i as the piecewise linear function

$$g_i := \sum_{\mu=1}^{m} a_{\mu\ell} \, \psi_{\mu k}$$

with $\psi_{\mu k}$ the "hat function " which is supported on the interval $[x_{\mu k} - \pi/N, x_{\mu k} + \pi/N]$ has value zero at the end points of this interval and 1 at the point $x_{\mu k}$.

It is easy to see that the g_i are orthogonal. Indeed, suppose that $i_1 = k_1 m + \ell_1$ and $i_2 = k_2 m + \ell_2$. If $k_1 \neq k_2$ then the g_i are orthogonal because they have disjoint supports. If $k_1 = k_2$, then

$$\int_0^{2\pi} g_{i_1} g_{i_2} = \frac{\pi}{6N} v_{\ell_1} \cdot v_{\ell_2} = 0$$

because the columns of A are orthogonal.

The functions g_i have L_2 norm $(\frac{2\pi m}{3N})^{\frac{1}{2}}$. Therefore, the normalized functions

$$f_i := (\frac{3N}{2\pi m})^{\frac{1}{2}} g_i$$

form an orthonormal system.

Now suppose X_n is any n-dimensional subspace of $C_{2\pi}$ and let $\phi_1, \ldots, \phi_n$ be an orthonormal basis for X_n. We extend this orthonormal system by adjoining functions $\phi_{n+1}, \ldots$ to arrive at a complete orthonormal system. Now consider the matrix $B := (b_{ij})$ with $b_{ij} := (f_i, \phi_j)^2$ The sum of the entries in a given row of B is one and the sum of the entries in a given column is at most one. Hence

$$\sum_{i=1}^{N} E(f_i, X_n)^2 = \sum_{i=1}^{N} \sum_{j=n+1}^{\infty} (f_i, \phi_j)^2 = N - \sum_{i=1}^{N} \sum_{j=1}^{n} (f_i, \phi_j)^2$$

$$\geq N - n = n.$$

As a result

(7) $$\max_{1\leq i\leq N} E(f_i,X_n)^2 \geq \tfrac{1}{2}$$

It is easy to check that $\|g_i\|_{BV} = 2m$ and $\|g_i\|_{\text{Lip }\alpha} = (\frac{N}{\pi})^\alpha$. Hence the function $\tilde{f}_i := (\frac{\pi}{N})^\alpha g_i = (\frac{\pi}{N})^\alpha (\frac{2\pi m}{3N})^{\frac{1}{2}} f_i$ satisfy

(8) $$\|\tilde{f}_i\|_{BV} \leq \|\tilde{f}_i\|_{\text{Lip }\alpha} = 1$$

so that $\tilde{f}_i \in K_\alpha$

According to (7) and (8)

$$\sqrt{2\pi}\, E(K_\alpha,X_n)_{C_{2\pi}} \geq E(K_\alpha,X_n)_{L_2} \geq \max_{1\leq i\leq n} E(\tilde{f}_i,X_n)_{L_2} \geq (\tfrac{1}{\sqrt{2}})(\tfrac{\pi}{N})^\alpha(\tfrac{2\pi m}{3N})^{\frac{1}{2}}$$

$$\geq C\, N^{-\alpha-\frac{1}{2}}\, m^{\frac{1}{2}} \geq C\, N^{-(\alpha+1)/2} \geq C\, n^{-(\alpha+1)/2} .$$

Since X_n is arbitrary this gives the lower estimate in (4).

The approach above extends readily to the more general compact set

(9) $$K_{\alpha,\beta} := \{f: |f|_{B_p^{\alpha,r}} \leq 1 \ ; \ |f|_{B_q^{\beta,s}} \leq 1\} \ ; \ 1/p + 1/q = 1 \ ; \ p \leq q$$

with $\alpha, \beta, r, s > 0$ and $|\cdot|_{B_p^{\alpha,r}}$, the usual Besov space semi-norms. The space $B_p^{\alpha,r} \hookrightarrow B_q^{\beta,s}$ if $\alpha > \beta + 1/p - 1/q$ and $B_q^{\beta,s} \hookrightarrow B_p^{\alpha,r}$ if $\beta \geq \alpha$. Thus, the only interesting cases are when

$$\beta < \alpha \leq \beta + 1/p - 1/q .$$

Theorem 2. _If_ $K_{\alpha,\beta}$ _is defined as in_ (9) _with_ $1 \le p \le 2$, _and_ $\beta < \alpha \le \beta + 1/p - 1/q$, _then there are constants_ $C_1, C_2 > 0$ _such that_

$$C_1 n^{-(\alpha+\beta)/2} \le d_n(K_{\alpha,\beta})_{C_{2\pi}} \le C_2 n^{-(\alpha+\beta)/2} \tag{10}$$

Proof. The proof is essentially the same as Theorem 1 and so we only indicate the necessary changes. Let Δ_h^k denote the k-th difference with step size h. If $f \in K_{\alpha,\beta}$ and $k := [\alpha+\beta] + 1$, then

$$\int_0^{2\pi} |\Delta_h^k(f,x)|^2\, dx \le (\int_0^{2\pi} |\Delta_h^k(f,x)|^p\, dx)^{1/p} (\int_0^{2\pi} |\Delta_h^k(f,x)|^q dx)^{1/q}$$
$$\le C_0 h^\alpha ||f||_{B_p^{\alpha,r}} h^\beta ||f||_{B_q^{\beta,s}} \le C_0 h^{\alpha+\beta}$$

with C_0 depending only on k. Hence $K_{\alpha,\beta}$ is contained in a ball of radius $\sqrt{C_0}$ in $B_2^{\alpha+\beta,\infty}$ with C_0 depending only on k. It then follows from (3) that

$$d_n(K_{\alpha,\beta}))_{C_{2\pi}} \le \sqrt{C_0}\, d_n(U(B_2^{\alpha,\infty}))_{C_{2\pi}} \le C n^{-(\alpha+\beta)/2}$$

which is the upper estimate in (10)

For the lower estimate, we argue similarly to Theorem 1 except that we need smoother functions f_i. Let ϕ be a non-negative function in C^∞ which is supported on $[-\frac{1}{2},\frac{1}{2}]$ with

$$\int_{-\infty}^{\infty} \phi^2 = 1 \; ; \; ||\phi||_\infty \le 2$$

We suppose as before that n is a 2 power, $N := 2n$ and we define m as the largest 2 power such that $m \le N^{\gamma+1}$ with $\gamma := (\beta-\alpha)/(1/p-1/q)$. Note that $-1 \le \gamma < 0$. The points $x_{\mu\nu}$ are defined as in Theorem 1. Finally we define $h_0 := 2\pi/N$.

The function $\phi_{\mu\nu} := m^{-\frac{1}{2}} h_0^{-\frac{1}{2}} \phi(\frac{(x-x_{\mu\nu})}{h_0})$ is supported on $[x_{\mu\nu} - \frac{1}{2}h_0, x_{\mu\nu} + \frac{1}{2}h_0]$. If $1 \le i \le N$, we write $i = km + \ell$ with

$1 \le \ell \le m$ and define

$$f_i := \sum_{\mu=1}^{m} a_{\mu\ell}\, \phi_{\mu k}$$

where $a_{\mu\nu}$ are the entries of the Latin square A.

It follows that the f_i are an orthogonal system and therefore as in (7) for any n-dimensional space X_n

$$\max_{1\le i\le N} E(f_i, X_n)^2_{C_{2\pi}} \ge \tfrac{1}{2} \,. \tag{11}$$

We now check the norm of f_i in $B_p^{\alpha,r}$. Let $k > \alpha$. Now, $\|\phi_{\nu\mu}^{(k)}\|_\infty \le Cm^{-\frac{1}{2}} h_0^{-k-\frac{1}{2}}$ and since the $\phi_{\mu\nu}$ have disjoint supports and $|a_{\mu\nu}| = 1$,

$$|\Delta_h^k(f_i,x)| \le C \max\,(m^{-\frac{1}{2}} h_0^{-\frac{1}{2}} (h/h_0)^k, 1) \qquad \text{on support of } f_i.$$

Hence,

$$\|\Delta_h^k(f_i)\|_p \le C \max\,(m^{-\frac{1}{2}} h_0^{-\frac{1}{2}} (h/h_0)^k, 1)\,(m/N)^{1/p}$$

It follows that the modulus of smoothness $\omega_k(f_i)_p$ satisfies

$$\omega_k(f_i,h)_p \le C(m/N)^{1/p} \begin{cases} m^{-\frac{1}{2}} h_0^{-\frac{1}{2}} (h/h_0)^k, & h \le h_0 \\ 1 \quad , & h \ge h_0 \end{cases}$$

Hence,

$$|f_i|_{B_p^{\alpha,r}} = \left(\int_0^\infty (h^{-\alpha}\omega_k(f_i,h)_p^r \frac{dh}{h}\right)^{1/r} \le C\,(m/N)^{1/p}\,[m^{-\frac{1}{2}} h_0^{-\frac{1}{2}} h_0^{-\alpha} + h_0^{-\alpha}]$$

$$\le C\,(m/N)^{1/p}\, m^{-\frac{1}{2}}\, h_0^{-\frac{1}{2}}\, h_0^{-\alpha}$$

$$\le CN^{\gamma(1/p-\frac{1}{2})}\, N^{\alpha} \le CN^{(\alpha+\beta)/2}$$

The same computation shows that $|f_i|_{B_p^{\beta,s}} \leq C\, n^{(\alpha+\beta)/2}$.

The functions $g_i := C^{-1}\, n^{-(\alpha+\beta)/2}\, f_i$ are in $K_{\alpha,\beta}$ and according to (11) give the lower estimate in (10). □

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