THE K FUNCTIONAL FOR (H_1, BMO)

R. DeVore[1]
Mathematics Research Center
University of Wisconsin-Madison
Madison, Wisconsin 53706, USA

1. Introduction. There are several theorems [1], [4], [5] which show that in some sense H_1 and BMO can serve as replacements for L_1 and L_∞ respectively for interpolation theory.[2] For example, it is known that the L_p spaces $1 < p < \infty$ are interpolation spaces between any of the pairs (X_1, X_∞) with X_1 either L_1 or H_1 and X_∞ either L_∞ or BMO. We are interested in the finer question of characterizing the K functionals for these pairs (X_1, X_∞). Actually the K functional is known or easily derived from known results in all but the one case (H_1, BMO). The characterization of the K functional for this latter pair is the main result of this paper.

Recall that for any pair of Banach spaces (X,Y), the Peetre K functional is defined for $f \in X + Y$ by

$$K(f,t,X,Y) := \inf_{f=h+g} (\|h\|_X + t\|g\|_Y), \quad t > 0.$$

Perhaps, it is useful to explain the interest in characterizing these K functionals. If T is a bounded operator on X_1 and X then T satisfies

$$(1.1) \qquad K(Tf,t) \leq cK(f,t) \quad \text{for all } f.\ X_1 + X_\infty$$

with $K(f,t) := K(f,t,X_1,X_\infty)$ the corresponding K-functional. The inequality (1.1) carries more information than any particular result on mapping of spaces. For example, $K(f,t,L_1,L_\infty) = tf^{**}(t)$ with f^* the decreasing rearrangement of f and $f^{**}(t) := \frac{1}{t}\int_0^t f^*(s)ds$; hence if T is bounded on L_1 and L_∞, then

$$(1.2) \qquad (Tf)^{**}(t) \leq cf^{**}(t).$$

---

1 This research is supported by NSF Grant 8101661.

2 All spaces are over $\mathbb{R}^n$ unless specifially stated otherwise.

It follows from (1.2) that T is bounded (for example) on $L \log L$; a result which is not included in the usual interpolation theorems for (L_1, L_∞) which give only that T is bounded on L_p, $1 < p < \infty$.

Another reason for studying K functionals is that they usually involve analytic quantities which are fundamental to the study of the particular pairs of spaces; f^{**} for (L_1, L_∞). Another example is (L_1, BMO) where C. Bennett and R. Sharpley [1] have shown

$$(1.3) \qquad K(f,t,L_1,\mathrm{BMO}) \approx t f^{\#*}(t) \qquad \text{for all } f \in L_1 + \mathrm{BMO} \text{ and } t > 0$$

with

$$f^{\#}(x) := \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f - f_Q| \ ; \ f_Q := \frac{1}{Q} \int_Q f \quad .$$

the Fefferman-Stein sharp function. We use the notation "$\approx$" to indicate that the quotient of the two expressions are bounded away from 0 and ∞ (independent of f and t in (1.3)). The fact that L_p spaces $1 < p < \infty$ are interpolation spaces between L_1 and BMO follows from (1.3) and the fact that for $1 < p < \infty$

$$(1.4) \qquad \|f^{\#}\|_{L_p} \approx \|f\|_{L_p} \qquad \text{for all } f \in L_p.$$

It is possible to characterize the K functional for (H_1, L_∞) from the work of C. Fefferman-N. Rivière and Y. Sagher [4]. They have shown that each (smooth) function f can be written as $f = b + g$ with

$$\|b\|_{H_1} + t\|g\|_{L_\infty} \le c \int_0^t (\mathcal{M}f)^*(s)ds = ct(\mathcal{M}f)^{**}(t)$$

with $\mathcal{M}$ the grand maximal function (see (2.11)). From this it follows that

$$K(f,t,H_1,L_\infty) \le ct(\mathcal{M}f)^{**}(t) \ .$$

On the other hand, $F \to (\mathcal{M}F)^{**}$ is subadditive and $\mathcal{M}$ maps H_1 into L_1 and L_∞ into L_∞; hence for $f = b + g$

$$t[\mathcal{M}f]^{**}(t) \le t(\mathcal{M}b)^{**}(t) + t(\mathcal{M}g)^{**}(t) \le c[\ \|\mathcal{M}b\|_{L_1} + t\|\mathcal{M}g\|_{L_\infty}]$$

$$\le c(\ \|b\|_{H_1} + t\|g\|_{L_\infty})$$

Taking an inf over all such decompositions gives

$$t(Mf)^{**}(t) \le cK(f,t,H_1,L_\infty)$$

or

(1.5) $$K(f,t,H_1,L_\infty) \approx t(Mf)^{**}(t), \quad f \in H_1 + L_\infty.$$

We shall characterize the K functional for the remaining pair (H_1, BMO). This characterization involves a generalization of the sharp function $f^\#$ to a new sharp function $f^\#_{H_1}$ which essentially replaces the role of L_1 by that of H_1 (see §2 for a precise definition). We then show that

(1.6) $$K(f,t,H_1,BMO) \approx tf^{\#*}_{H_1}(t)$$

As a consequence, any operator T which is bounded on H_1 and BMO satisfies

(1.7) $$(Tf)^{\#*}_{H_1} \le c\, f^{\#*}_{H_1}.$$

In the process of proving (1.6), we establish several results which compare $f^\#_{H_1}$ with other sharp functions. These in turn show that $\|f^\#_{H_1}\|_{L_p} \approx \|f\|_{L_p}$ (Corollary 3.4). Therefore the fact that L_p, $1 < p < \infty$ is an interpolation space between H_1 and BMO follows from (1.7) by applying L_p norms.

2. <u>Sharp Functions</u>. BMO is the space of all f satisfying

(2.1) $$\|f\|_{BMO} := \|f^\#\|_{L_\infty} < \infty.$$

It is a very useful fact that $f^\#$ can be replaced in (2.1) by (see [1, Cor. 4.7])

(2.2) $$f^\#_{0,p}(x) := \sup_{Q \ni x} \left(\frac{1}{|Q|}\int_Q |f-f_Q|^p\right)^{\frac{1}{p}}.$$

A word of explanation about notation; we are being consistent with [2] where maximal functions $f^\#_{\alpha,p}$, $\alpha,p > 0$ are introduced.

There is another important variant in the definition of $f^\#$. If we let $\mathbb{P}_N$

denote the space of polynomials of total degree N, then for any fixed N we have ([2, Lemma 4.4])

$$f^{\#}_{0,p} \approx F\,;\; F(x) := \sup_{Q\ni x}\ \inf_{\pi\in\mathbb{P}_N} \left(\frac{1}{|Q|}\int_Q |f-\pi|^p\right)^{\frac{1}{p}}. \tag{2.3}$$

We want now to introduce a sharp function which replaces the role of L_p in the definition of $f^{\#}_{0,p}$ by the space H_1. We must take some care since $(f-f_Q)\chi_Q$ is generally not in H_1. Let $A := 20\sqrt{n}$ (the precise value of A is not important; any A sufficiently large would do). For each cube Q with diameter d let Φ_Q denote the set of functions ϕ satisfying

(2.4) i) ϕ supported on AQ; $0 \le \phi \le 1$.

ii) $\phi \ge A^{-1}$ on $A^{-1}Q$.

iii) $\|D^\nu\phi\|_{L_\infty} \le A\, d^{-|\nu|}$, all $\nu \geq 0$.

We are using the notation λQ to denote the cube with the same center as Q and diameter λd with d the diameter of Q.

Given $\phi \in \Phi_Q$, consider the inner product

$$(f,g)_\phi := \frac{1}{|Q|}\int f\, g\, \phi \tag{2.5}$$

and denote by P_ϕ the orthogonal projection operator from $L_1(AQ)$ onto $\mathbb{P}_N$ with $N := n+1$, i.e. $P_\phi f$ is the unique polynomial in $\mathbb{P}_N$ which satisfies

$$(f-P_\phi f,\ \pi)_\phi = 0 \quad \text{for all } \pi \in \mathbb{P}_N. \tag{2.6}$$

If $f \in H_1$, define

$$f^{\#}_{H_1}(x) := \sup_{Q\ni x}\ \sup_{\phi\in\Phi_Q} \frac{1}{|Q|}\,\|(f-P_\phi f)\phi\|_{H_1}. \tag{2.7}$$

As we shall see in this and the next section, $f^{\#}_{H_1}$ is similar to $f^{\#}_{0,p}$. To begin with, we recall two equivalent norms for H_1 from the C. Fefferman-E. Stein [3] theory. If $k \in L_1$, let

$$\|k\|_{W_N} := \sum_{|\nu|\le N} \int (1+|x|)^N |D^\nu k(x)|\,dx \tag{2.8}$$

with $N := n+1$. <u>We fix a kernel</u> K <u>with the properties</u>:

(2.9) i) $K \geq 0$

ii) $\int K = 1$

iii) K supported on $|x| \leq 1$

iv) $\| D^{\nu} K \|_{L_\infty} \leq c$, for all $|\nu| \leq N$.

Then $\|K\|_{W_N} < \infty$. Let

$$f^{+}(x) := \sup_{\epsilon > 0} |f * K_\epsilon(x)|$$

Then,

(2.10) $$\|f^{+}\|_{L_1} \approx \|f\|_{H_1}$$

There is another important equivalent norm for H_1 given by the grand maximal function. Let $\alpha > 0$ be a fixed constant and

(2.11) $$\mathcal{M}f(x) := \sup_{\|k\|_{W_N} \leq 1} \sup_{|x_1 - x| < \alpha\epsilon} |f * k_\epsilon(x_1)|.$$

Then

(2.12) $$\|\mathcal{M}f\|_{L_1} \approx \|f\|_{H_1} .$$

Note in the definition (2.11) the kernels k are not required to have integral one.

We would like now to give estimates for $P_\phi f$ in terms of $\mathcal{M}f$. These are similar to those given in [4]. Let $\{\pi_i\}_{i=1}^{m}$, $m := \dim(\mathbb{P}_N)$ be an orthonormal basis for $\mathbb{P}_N$ with respect to the inner product $(\, , \,)_\phi$. Then

$$P_\phi f = \sum_{1}^{m} (f, \pi_i)_\phi \, \pi_i .$$

Using (2.4)ii, we have

$$\|\pi_i\|_{L_2(A^{-1}Q)} \leq (A \int \pi_i^2 \, \phi)^{\frac{1}{2}} = (A|Q|)^{\frac{1}{2}} .$$

It follows that for any $\lambda > 0$ (see [2, §3])

(2.13) $\|\pi_i\|_{L_\infty(\lambda Q)} \le c\,\|\pi_i\|_{L_\infty(A^{-1}Q)} \le c\,|Q|^{-\frac{1}{2}}\|\pi_i\|_{L_2(A^{-1}Q)} \le c$

with c depending only on λ, A and n.

Lemma 2.1. For any $\lambda > 0$, there is a constant $c > 0$ depending at most on λ, A and n such that

(2.14) $|(f,\pi_i)_\phi| \le c\,Mf(x)$, for all $x \in \lambda Q$

(2.15) $\|P_\phi f\|_{L_\infty(\lambda Q)} \le c\,Mf(x)$, for all $x \in \lambda Q$.

Proof. Clearly (2.15) follows from (2.14) and (2.13). To prove (2.14), we notice that (2.13) and Markov's inequality give that $\|D^\nu \pi_i\|_{L_\infty(\lambda Q)} \le c\,d^{-|\nu|}$, $|\nu| \le N$. Hence, using (2.4) we see that the kernel $k(u) := \pi_i(x-du)\phi(x-du)$ satisfies

$$\|D^\nu k\|_{L_\infty} \le c, \quad |\nu| \le N.$$

If $x \in \lambda Q$, then k is supported in $|u| \le A+\lambda$ and so $\|k\|_{W_N} \le c$. Therefore,

$$|(f,\pi_i)_\phi| = |\textstyle\int f\,\pi_i\phi| = |f * k_d(x)| \le c\,Mf(x). \qquad \square$$

Our next result estimates ψ^+ when $\psi := (f - P_\phi f)\phi$.

Lemma 2.2. If $\lambda \ge 2A$, there is a constant $c > 0$ depending only on λ and n such that for each cube Q with diameter d and center z and each $\phi \in \Phi_Q$, we have

(2.16) i) $\psi^+(x) \le c\,Mf(x)$, $x \in \lambda Q$

ii) $\psi^+(x) \le c\,|Q|\,d^{n+1}|x-z|^{-2n-1}\inf_{\lambda Q} f^\#$, $x \notin \lambda Q$.

Proof. For i), we consider two cases.

Case 1. $\epsilon \le d$. In this case, $k(u) := K(u)\phi(x-\epsilon u)$ satisfies $\|k\|_{W_N} \le c$ because of (2.4), and so

$$|(f\phi) * K_\epsilon(x)| = |f * k_\epsilon(x)| \le c\,Mf(x).$$

Also from (2.15) and (2.4)i),

$$|(\phi P_\phi f) * K_\epsilon(x)| \le \|\phi(P_\phi f)\|_{L_\infty} \le \|P_\phi f\|_{L_\infty(AQ)} \le c\, Mf(x).$$

Hence,

(2.17) $$|\psi * K_\varepsilon(x)| \le c\, Mf(x)$$

in this case.

Case 2. $\epsilon > d$. In this case, the kernel $k(u) := K(\frac{du}{\varepsilon})\phi(x-du)$ satisfies $\|k\|_{W_N} \le c$ and so

$$|(f\phi) * K_\varepsilon(x)| = (\frac{d}{\varepsilon})^n |f * k_d(x)| \le c\, Mf(x)$$

Also, from (2.15) and (2.4)i)

$$|(\phi P_\phi f) * K_\varepsilon(x)| \le \|\phi P_\phi f\|_{L_\infty} \le \|P_\phi f\|_{L_\infty(AQ)} \le c\, Mf(x).$$

Hence (2.17) holds in this case as well. Taking a sup over all $\varepsilon > 0$ in (2.17) gives i).

To prove ii), fix $x \notin \lambda Q$ and define $\delta := \mathrm{dist}(x, AQ)$. Then $\delta \ge c|x-z|$. If $\varepsilon < \delta$, then $\psi * K_\varepsilon(x) = 0$; hence we may assume $\varepsilon \ge \delta$. Now, there is a Taylor polynomial T of degree at most N such that

(2.18) $$|K_\varepsilon(u-x) - T(u)| \le c\, \varepsilon^{-2n-1} d^{n+1} \quad u \in \lambda Q$$

because derivatives of order N of $K_\varepsilon(\cdot - x)$ are less than $c\varepsilon^{-2n-1}$. Thus, using (2.18) and (2.6) gives

(2.19) $$|\psi * K_\varepsilon(x)| \le \int (f(u) - P_\phi f(u))\phi(u) K_\varepsilon(u-x)\,du|$$

$$= |\int (f(u) - P_u f(u)\phi(u) \, [K_\varepsilon(u-x) - T(u)]\,du|$$

$$\le c \frac{d^{n+1}}{\varepsilon^{2n+1}} \int_{AQ} |f - P_\phi f|$$

For any $\pi \in \mathbb{P}_N$, $P_\phi(\pi) = \pi$ and so

$$\int_{AQ} |f - P_\phi f| \le \int_{AQ} |f - \pi| + \int_{AQ} |P_\phi(f-\pi)|$$

$$\le c \int_{AQ} |f - \pi| \le c \int_{\lambda Q} |f - \pi|$$

because P_ϕ is a bounded operator on $L_1(AQ)$ (see (2.13)). Taking an inf over all π and using (2.3) gives

$$\int_{AQ} |f - P_\phi f| \le c|\lambda Q| \inf_{\lambda Q} f^* .$$

Using this in (2.19) completes the proof of ii). □

3. <u>Lower estimates for K functionals</u>. The main result of this paper is the following characterization of $K(f,t,H_1,BMO)$.

<u>Theorem 3.1</u>. There exist constants $c_1, c_2 > 0$ depending only on n such that for all $f \in H_1 + BMO$

(3.1) $$c_1 t f_{H_1}^{\#*}(t) \le K(f,t,H_1,BMO) \le c_2 t f_{H_1}^{\#*}(t)$$

In this section, we shall prove the lower estimate in (3.1); the upper estimate is proved in the next section.

The lower estimate in (3.1) rests on the behavior of $f_{H_1}^{\#*}(t)$ for t close to 0 and t close to ∞.

<u>Lemma 3.2</u>. For any $1 < p < \infty$; there are constants $c_1, c_2 > 0$ such that

(3.2) $$c_1 f^{\#} \le f_{H_1}^{\#} \le c_2 f_{0,p}^{\#}$$

in the sense that when one of these functions is finite the compared expression is also finite and smaller.

<u>Proof</u>. Suppose $x \in \mathbb{R}^n$ and Q is any cube containing x. There is a function $\phi \in \Phi_Q$ with $\phi \equiv 1$ on Q. Therefore,

(3.3) $$\frac{1}{|Q|} \int_Q |f-P_\phi f| \le \frac{1}{|Q|} \int |(f-P_\phi f)\phi| \le \frac{c}{|Q|} \|(f-P_\phi f)\phi\|_{H_1} \le c f_{H_1}^{\#}(x)$$

Taking a sup over all cubes Q containing x in (3.3) and using (2.3) with $p=1$ gives the lower estimate.

For the upper inequality, we will use the fact that $\|g^+\|_{L_p} \le c\,\|g\|_{L_p}$, for all $g \in L_p$ (see [3]). Let Q again be a cube which contains x and let $\phi \in \Phi_Q$. Then $\psi := (f - P_\phi f)\phi$, is supported on $\bar{Q} := 2AQ$. Using the estimates in (2.16)ii) we have

$$(3.4) \qquad \|(f - P_\phi f)\phi\|_{H_1} \le c \int \psi^+ \le c \int_{\bar{Q}} \psi^+ + c \int_{\bar{Q}^c} \psi^+$$

$$\le c\,[|Q|^{1-\frac{1}{p}} (\int_{\bar{Q}} \psi^{+})^p)^{\frac{1}{p}} + |Q|\; f^{\#}(x)]$$

$$\le c\,|Q|\;[(\frac{1}{|\bar{Q}|} \int_{\bar{Q}} |f - P_\phi f|^p)^{\frac{1}{p}} + f^{\#}(x)]$$

where the second to last inequality uses the fact that $\int_{\bar{Q}^c} |x-z|^{-2n-1} \le c d^{-n-1}$.

Since P_ϕ is a bounded (with norms depending only on n) projection from $L_p(\bar{Q})$ into $\mathbb{P}_N$, for any $\pi \in \mathbb{P}_N$, we can write $f - P_\phi f = f - \pi - P_\phi(f - \pi)$ and find

$$(\int_{\bar{Q}} |f - P_\phi f|^p)^{\frac{1}{p}} \le c\;(\int_{\bar{Q}} |f - \pi|^p)^{\frac{1}{p}}.$$

Taking an inf over $\pi \in \mathbb{P}_N$ and using this back in (3.4) gives

$$\frac{1}{|\phi|}\;\|(f - P_u f)\phi\|_{H_1} \le c\;[f^{\#}_{0,p}(x) + f^{\#}(x)].$$

Since $f^{\#} \le f^{\#}_{0,p}$ because of Hölder's inequality, we have the upper inequality in (3.2). □

As a corollary to this lemma, we have

<u>Corollary 3.3</u>. There are constants $c_1, c_2 > 0$ such that for each $f \in BMO$

$$(3.5) \qquad c_1\,\|f\|_{BMO} \le \|f^{\#}_{H_1}\|_{L_\infty} \le c_2\,\|f\|_{BMO}$$

<u>Proof</u>. This follows from (3.2) and the fact that $\|f\|_{BMO} \approx \|f^{\#}_{0,p}\|_{L^\infty}$, $1 \le p < \infty$.

<u>Corollary 3.4</u>. If $1 < q < \infty$, there are constants $c_1, c_2 > 0$ such that for each $f \in L_q$

$$\|f^{\#}_{H_1}\|_{L_q} \approx \|f\|_{L_q}$$

Proof. This follows from (3.2) by taking $1 < p < q$ in (3.2) applying L_q norms and using the fact (see [1]) that $\|f^{\#}_{0,p}\|_{L_q} \approx \|f\|_{L_q}$. □

We also need an estimate for $f^{\#}_{H_1}$ near H_1.

Lemma 3.5. There is a constant c such that for all $f \in H_1 + BMO$

$$f^{\#}_{H_1}(x) \le cM(\mathcal{M}f)(x) \tag{3.6}$$

with M the Hardy-Littlewood maximal operator.

Proof. Let Q be any cube in $\mathbb{R}^n$, $x \in Q$ and $\phi \in \Phi_Q$. With $\psi := (f - P_\phi f)\phi$ and $\bar{Q} := 2AQ$, we have from (2.16)

$$\|\psi\|_{H_1} \le c\|\psi^+\|_{L_1} \le c[\int_{\bar{Q}} \psi^+ + \int_{\bar{Q}^c} \psi^+] \le c[\int_{\bar{Q}} \mathcal{M}f + |Q|\, f^{\#}(x)]$$

$$\le c|Q|\,[M(\mathcal{M}f)(x) + f^{\#}(x)] \le c|Q|\, M(\mathcal{M}f)(x)$$

where we used the fact that $f^{\#} \le 2M(f) \le 2M(\mathcal{M}f)$. Dividing by $|Q|$ and taking a sup over all $\phi \in \Phi_Q$ and $Q \ni x$ gives (3.6). □

Corollary 3.5. There is a constant c such that for all $f \in H_1$ and $t > 0$

$$t\, f^{\#*}_{H_1}(t) \le c\,\|f\|_{H_1} \tag{3.7}$$

Proof. From (3.6), we have

$$t\, f^{\#*}_{H_1}(t) \le ct\, M(\mathcal{M}f)^*(t) \le c\,\|\mathcal{M}f\|_{L_1} \le c\,\|f\|_{H_1} \tag{3.8}$$

because M is of weak type (1,1). □

Proof of lower estimate in (3.1).

Suppose $f = b+g$ with $b \in H_1$ and $g \in BMO$. Since P_ϕ is a linear operator, it follows that $F \to F^{\#}_{H_1}$ is sub-linear. Using this and (3.5), we have

$$f^{\#}_{H_1} \le b^{\#}_{H_1} + g^{\#}_{H_1} \le b^{\#}_{H_1} + \|g^{\#}_{H_1}\|_{L_\infty}$$

$$\le b^{\#}_{H_1} + c\,\|g\|_{BMO}, \quad \text{a.e.,}$$

Hence from Corollary 3.5,

$$t\, f_{H_1}^{\#*}(t) \le t\, b_{H_1}^{\#*}(t) + ct\, \|g\|_{BMO} \le c[\, \|b\|_{H_1} + t\, \|g\|_{BMO}\,]$$

Taking now an inf over all such decompositions $f = b+g$ gives the lower inequality in (3.1). □

4. <u>The upper estimate.</u> To prove the upper estimate

$$K(f,t,H_1,BMO) \le ct\, f_{H_1}^{\#*}(t)$$

We need to decompose f as $f = b+g$ with b in H_1 and $g \in BMO$ satisfying

$$\|b\|_{H_1} + t\, \|g\|_{BMO} \le ct\, f_{H_1}^{\#*}(t)$$

The decomposition we give is similar to that given in [4] for H_1 and L_∞.

Fix $t > 0$ and let $E := \{x: f_{H_1}^{\#}(x) > f_{H_1}^{\#*}(t)\}$. Then E is an open set with $|E| \le t$. Let $\{Q_j\}_1^\infty$ be a Whitney decomposition of E into dyadic cubes with $d_j := \text{diam}\,(Q_j)$ and the usual properties [6, p. 167]:

(4.1) i) $\bigcup_1^\infty Q_j = E$

ii) $|Q_i \cap Q_j| = 0,\ i \ne j.$

iii) $d_j \le \text{dist}\,(Q_j,E^c) \le 4d_j,\ j=1,2,\ldots$

iv) If Q_i touches Q_j, then $\text{diam}\ Q_i \le 4\ \text{diam}\, Q_j$.

v) Any point $x \in E$ appears in at most N_0 of the cubes $\frac{5}{4} Q_j,\ j=1,2,\ldots$ with N_0 depending only on n.

If we let $Q_j^* := \frac{9}{8} Q_j$, then there is a partition of unity (ϕ_j) (denoted by (ϕ_j^*) in [5]) subordinate to (Q_j^*) with the properties

(4.2) i) $\sum_1^\infty \phi_j \equiv 1$ on E

ii) support of ϕ_j is contained in Q_j^*, $j=1,2,\ldots$

iii) $0 \le \phi_j \le 1$ and $\phi_j \equiv 1$ on $\frac{3}{4} Q_j$.

iv) $\| D^\nu \phi_j \|_{L_\infty} \le c\, d_j^{-|\nu|}$ for all ν and j.

With the abbreviated notation $P_j := P_{\phi_j}$, we define our decomposition for f by

$$(4.3)\qquad g := \sum_1^\infty (P_j f)\phi_j + f\chi_{E^c} \; ; \; b := \sum_1^\infty (f - P_j f)\phi_j$$

We begin by estimating the norm of b in H_1. This is particularly simple since if $\bar{Q}_j := 10\sqrt{n}\, Q_j$ then $\bar{Q}_j \cap E^c \neq \emptyset$ and therefore there is a point $x_j \in \bar{Q}_j \cap E^c$. Since $\phi_j \in \Phi_{\bar{Q}_j}$,

$$\|(f - P_j f)\phi_j\|_{H_1} \le |\bar{Q}_j|\, f^{\#}_{H_1}(x_j) \le c|Q_j|\, f^{\#*}_{H_1}(t)$$

Adding these estimates gives

$$(4.4)\qquad \|b\|_{H_1} \le c \sum_1^\infty |Q_j|\, f^{\#*}_{H_1}(t) \le c|E|\, f^{\#*}_{H_1}(t) \le ct\, f^{\#*}_{H_1}(t).$$

The estimate of the norm of g in BMO is somewhat more involved but similar to the Bennett-Sharpley argument [1,§6].

<u>Lemma 4.1</u>. We have $\|g\|_{BMO} \le c\, f^{\#*}_{H_1}(t)$ with c depending only on n.

<u>Proof</u>. Let Q be a cube in $\mathbb{R}^n$. According to (3.2), it is enough to show that there is a constant α such that

$$(4.5)\qquad \int_Q |g-\alpha| \le c|Q|\, f^{\#*}_{H_1}(t)$$

Let $\Lambda := \{i : Q_i^* \cap Q \neq \phi\}$. We shall consider three cases.

<u>Case 1</u>. $\Lambda = \phi$. Then $Q \subseteq E^c$ and $g=f$ on Q, so we may take $\alpha := f_Q$ and the lower estimate in (3.2) to find

$$(4.6)\qquad \int_Q |g-\alpha| = \int_Q |f-f_Q| \le |Q| \inf_Q f^{\#} \le c|Q| \inf_Q f^{\#}_{H_1} \le c|Q|\, f^{\#*}_{H_1}(t)$$

because $f^{\#}_{H_1} \le f^{\#*}_{H_1}(t)$ on E^c.

<u>Case 2</u>. There is an $i \in \Lambda$ with $\operatorname{diam}(Q) \le \frac{1}{64} \operatorname{diam}(Q_i)$ for some $i \in \Lambda$.

For the proof in this case we will use the fact that for each $h \in L_1(Q_i^*)$,

$$(4.7)\qquad \|P_i(h)\|_{L_\infty(Q_i^*)} \le \frac{c}{|Q_i^*|}\int_{Q_i^*}|P_i(h)| \le \frac{c}{|Q_i^*|}\int_{Q_i^*}|h|,\ i=1,2,\dots .$$

The first inequality in (4.7) is simply a comparison of polynomial norms (see e.g. [2,§3]) and the second is the boundedness of the projection P_i on $L_1(Q_i^*)$ which in turn follows from (2.13).

Now let Q_{j_0} be a largest cube among the Q_i with $i \in \Lambda$. Then

$$(4.8)\qquad Q \subset \frac{33}{32}Q_{j_0}^* \subset \frac{5}{4}Q_{j_0} \subset E.$$

If $i \in \Lambda$, then $\frac{9}{8}Q_i = Q_i^*$ intersects Q and hence intersects $\frac{5}{4}Q_{j_0}$. It follows from (4.1)iv) that Q_i and Q_{j_0} touch and therefore $Q_i^* \subset 4Q_{j_0}$. Define $\tilde{Q} := 10\sqrt{n}\,Q_{j_0}$ and $\alpha := f_{\tilde{Q}}$. According to (4.2)iii) $\tilde{Q} \cap E^c \ne \phi$ and therefore using (4.7) and (2.13) gives

$$(4.9)\qquad \int_Q|g-\alpha| = \int_Q\Big|\sum_{i\in\Lambda}P_i(f-\alpha)\phi_i\Big| \le \sum_{i\in\Lambda}\int_{Q_i^*\cap Q}|P_i(f-\alpha)|$$

$$\le c\sum_{i\in\Lambda}\frac{|Q_i^*\cap Q|}{|Q_i^*|}\int_{Q_i^*}|f-\alpha| \le c\frac{|Q|}{|Q_{j_0}|}\sum_{i\in\Lambda}\int_{Q_i^*}|f-\alpha|$$

$$\le c\frac{|Q|}{|\tilde{Q}|}\int_{\tilde{Q}}|f-f_{\tilde{Q}}| \le c|Q|\inf_Q f^{\#} \le c|Q|\, f^{\#*}_{H_1}(t)$$

where the third to last inequality uses the fact that $|Q_i| \ge \frac{1}{4^n}|Q_{j_0}|$ because Q_i touches Q_{j_0}; the second to last inequality uses the fact that $|\tilde{Q}| \le (10\sqrt{n})^n|Q_{j_0}|$; and the last inequality uses (3.2) and the fact that $\tilde{Q} \cap E^c \ne \phi$.

Case 3. $\Lambda \ne \phi$ and for all $i \in \Lambda$, $\operatorname{diam}(Q_i) \le 64\ \operatorname{diam}(Q)$. In this case $Q_i \subset Q_i^* \subset 129Q$ for all $i \in \Lambda$. Define $\tilde{Q} := 1290\sqrt{n}\,Q$. Since $10\sqrt{n}\,Q_i$ touches E^c and is contained in $\tilde{Q}$, we have $\tilde{Q} \cap E^c \ne \phi$. We let $\alpha := f_{\tilde{Q}}$. Using (4.7) and (4.1), we have

$$(4.10)\qquad \int_Q|g-\alpha| = \int_{Q\cap E}\Big|\sum_{i\in\Lambda}P_i(f-\alpha)\phi_i\Big| + \int_{Q\cap E^c}|f-\alpha|$$

$$\le \sum_{i\in\Lambda}\int_{Q_i^*}|P_i(f-\alpha| + \int_{Q\cap E^c}|f-\alpha|$$

$$\leq c \sum_{i\in\Lambda} \int_{Q_i^*} |f-\alpha| + \int_{Q\cap E^c} |f-\alpha|$$

$$\leq c \int_{\tilde{Q}} |f-\alpha| = \int_{\tilde{Q}} |f-f_{\tilde{Q}}| \leq c|\tilde{Q}| \inf_{\tilde{Q}} f^{\#}$$

$$\leq c\ |Q| \inf_{Q} f^{\#}_{H_1} \leq c\ |Q|\ f^{\#*}_{H_1}(t).$$

The three estimates (4.8–10) combine to prove (4.5) □

Lemma 4.1 and the estimate (4.4) shows that $f = b+g$ with

$$\|b\|_{H_1} + t\,\|g\|_{BMO} \leq ct\, f^{\#*}_{H_1}(t)$$

which establishes the upper estimate in (3.1) and completes the proof of Theorem 3.1.

5. Acknowledgement. The problem of characterizing the K functional for (H_1, BMO) was posed to my by my colleagues, C. Bennett and R. Sharpley. I thank them for this and various discussions concerning this work.

6. Postscript. I have recently heard that Björn Jawerth has also given a characterization of $K(f, t, H^1, BMO)$ using the area integral. His paper has the same title as this paper and will appear in the Proceedings of the American Math. Soc.

References.

1. Bennett, C., Sharpley, R.: Weak type inequalities for H_p and BMO. Proc. Symp. in Pure Math. 35(I), 201–229 (1979).
2. DeVore, R., Sharpley, R.: Maximal functions measuring smoothness. Mem. Amer. Math. Soc., in press.
3. Fefferman. C., Stein, E.: H^p spaces of several variables. Acta. Math. 129, 137–193 (1972).
4. Fefferman, C., Rivière, N.M., Sagher, Y.: Interpolation between H^p spaces: the real method. Trans. Amer. Math. Soc. 191, 75–81 (1974).
5. Hanks, R.: Interpolation by the real method between BMO, $L^\alpha (0<\alpha<\infty)$ and $H^\alpha (0<\alpha<\infty)$. Indiana Univ. Math. J. 26, 679–684 (1977).
6. Stein, E.M.: Singular integrals and differentiability properties of functions. Princeton: Princeton University Press 1970.