

PARTITIONS OF UNITY AND APPROXIMATION

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ABSTRACT. We show that for certain translation invariant spaces S , a necessary and sufficient condition for the eventual denseness of the corresponding scaled spaces S_h is that S contain a stable and locally supported partition of unity. These results have been motivated by recent work on approximation by multivariate piecewise polynomials on regular meshes.

1. Introduction. We are interested in the existence of partitions of unity in certain translation invariant spaces. We are motivated by recent investigations (listed in the references) of approximation by multivariate piecewise polynomial functions on regular partitions—a partition τ of \mathbb{R}^m is called regular if $j + \tau = \tau$ for all $j \in \mathbb{Z}^m$. These investigations study the degree of approximation from the spaces $\pi_k^\rho(h\tau)$, $h > 0$, which consist of pp (i.e., piecewise polynomial) functions in $C^\rho(\mathbb{R}^m)$ of total degree k on the dilated partition $h\tau$. It has been possible to determine the approximation order from these spaces only for very special partitions and certain values of ρ and k . In particular, there are no general results which characterize when $\bigcup_{h>0} \pi_k^\rho(h\tau)$ is dense in the space of continuous functions with compact support.

The main technique for proving denseness or giving lower estimates for the degree of approximation has been to construct a partition of unity in $\pi_k^\rho(\tau)$ consisting of certain multivariate B-splines known as box splines and to use this partition of unity to construct suitable approximation schemes called quasi-interpolants.

It has not been clear to us whether this represents the only viable approach to the construction of suitable approximants. For example, in [BDH] we have asked the following:

(1.1) Is $\bigcup_{h>0} \pi_k^\rho(h\tau)$ dense in $C_0(\mathbb{R}^s)$ if and only if $\pi_k^\rho(\tau)$ contains a good partition of unity?

By a *good* partition of unity we mean a (countable) collection Φ of functions with the properties that

- (1.2) (i) $\sup_{\phi \in \Phi} \text{diam supp } \phi < \infty$;
(ii) $\sum_{\phi \in \Phi} \phi = 1$;
(iii) $\sum_{\phi \in \Phi} |\phi| < \infty$.

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Actually, it is easy to see that the existence of a good partition of unity is sufficient for denseness. Namely, if $x_\phi \in \text{supp } \phi$, then $Lf := \sum_{\phi \in \Phi} f(x_\phi)\phi$ gives an operator L whose dilate $L_h := \sigma_{1/h}L\sigma_h$, with $(\sigma_h f)(x) := f(hx)$, provides an approximation $L_h f$ in $\pi_k^p(h\tau)$ which converges to f as $h \rightarrow 0$.

It turns out, but is much more difficult to prove, that the existence of a good partition of unity is necessary for denseness. Here we shall present a proof of this fact for the univariate case but for much more general spaces S than just pp ones, in an attempt to extract the essential features of pp function spaces on regular partitions.

The ideas of the proof seem to carry over to the multivariate case, but the constructions involved are much more involved. For this reason, we feel that it is beneficial to present here the basic ideas in their simplest setting and report elsewhere on their multivariate analogues.

2. The main results. We let S be a space of functions on \mathbf{R} with the following properties:

- (2.1) (i) S is translation invariant, i.e., $f \in S \Rightarrow f(\cdot \pm 1) \in S$;
 (ii) $\mu := \dim S|_{[0,1]} < \infty$;
 (iii) $\dim S = \infty$;
 (iv) S is closed under uniform convergence on compact sets.

We note for later reference that (i) and (iii) imply

$$(v) \dim S|_{[0,\infty)} = \infty.$$

These are the essential properties of pp spaces over regular meshes. However, it is easy to give examples of such spaces S which are not pp. For example, if Φ is a finite collection of compactly supported functions, then the closure under uniform convergence on compact sets of the span of the translates $\phi_j := \phi(\cdot - j)$, $j \in \mathbf{Z}$, $\phi \in \Phi$, has these properties. The results of this section will show that any space S satisfying (2.1) can be obtained this way modulo some finite-dimensional space.

Associated with S we have the dilated spaces $S_h := \sigma_{1/h}S$.

THEOREM 1. *If $\bigcup_{h>0} S_h$ is dense in $C_0(\mathbf{R})$, then S contains a good partition of unity.*

In preparation for the proof, we now construct certain compactly supported functions in S . Let S^- denote the subspace of S consisting of those functions which vanish on $(-\infty, 0]$. It will follow later that S^- is nontrivial. For now, we prove

LEMMA 2. *For each integer $n > 0$, there is an R_n such that any $\phi \in S$ vanishing on $[-R_n, 0]$ agrees on $[0, n]$ with some function from S^- .*

PROOF. For $r, j > 0$, we let $L_{r,j}$ denote the space of functions defined on $[0, j]$ which are restrictions of functions in S which vanish on $[-r, 0]$. Then $L_{r,j} \subseteq L_{r',j}$ whenever $r > r'$. Hence $(\dim L_{r,j})_{r=1}^\infty$ is a decreasing sequence of (nonnegative) integers and therefore is eventually constant. This implies that we can find an

increasing sequence (R_j) such that $L_{r,j} = L_j := L_{R_j,j}$ for all $r \geq R_j$, $j = 1, 2, \dots$. Now given $n > 0$ and $\phi \in L_n$, we define

$$n_i := n + i, \quad r_i := \max\{R_{n_i}, n_i\}, \quad i = 0, 1, \dots$$

Then, since $r_1 \geq R_{n_1} \geq R_n$, there is an $f_1 \in S$ which vanishes on $[-r_1, 0]$ and agrees with ϕ on $[0, n]$. Similarly, $r_2 \geq R_{n_2} \geq R_{n_1}$ and so there is an $f_2 \in S$ which vanishes on $[-r_2, 0]$ and agrees with f_1 on $[0, n_1]$. In this manner, we obtain a sequence (f_j) such that f_j vanishes on $[-r_j, 0]$ and agrees with f_i on $[0, n_i]$ for all $i < j$. Hence f_j converges uniformly on compact sets to some f , necessarily in S^- because of (2.1)(iv), with the desired properties. \square

Now let $R := R_1$ be the constant of Lemma 2. We improve Lemma 2 in the following way:

LEMMA 3. *If $f \in S$ vanishes on $[-R, 0]$, then $\phi := f\chi_{[0, \infty)}$ is in S^- .*

PROOF. From Lemma 2, we know an $f_0 \in S^-$ for which $f - f_0$ vanishes on $[-R, 1]$. Hence there is some f_1 with $f_1(\cdot + 1) \in S^-$, therefore also $f_1 \in S^-$, for which $f - f_0 - f_1$ vanishes on $[-R, 2]$. In this manner, we construct a sequence (f_j) in S^- for which $f_0 + \dots + f_j$ converges to ϕ . Since S^- is closed under uniform convergence on compact sets (by (2.1)(iv)), $\phi \in S^-$. \square

Next, we construct a basis of compactly supported functions for S^- . Recall the definition $\mu := \dim S|_{[0,1]}$ and set $M := \mu R$.

LEMMA 4. *If $f \in S^-$, then there is a function $\phi \in S$ which is supported on $[0, M]$ and agrees with f on $[0, 1]$.*

PROOF. Set $J := [M, M + R]$ and consider the functions $f_j := f(\cdot - j)$, $j = 0, \dots, M$. Since $\dim S|_J \leq M$, these functions are linearly dependent on J . Hence there are numbers c_j , not all zero, such that $\psi_0 := \sum c_j f_j$ vanishes on J . Let i be the smallest value of j for which $c_j \neq 0$. Then $\psi := \psi_0(\cdot - i)/c_i$ agrees with f on $[0, 1]$ and vanishes on $a + [-R, 0]$ with $a := M + R - i$. Therefore, by Lemma 3, $\psi_1 := \psi(\cdot + a)\chi_{[0, \infty)}$ is in S^- and so $\phi := \psi - \psi_1(\cdot - a)$ has the desired properties. \square

It follows from Lemma 4 that there is a finite collection B of functions supported on $[0, M]$ whose restrictions to $[0, 1]$ form a basis for $(S^-)|_{[0,1]}$. We use the abbreviation $b_j := b(\cdot - j)$, $b \in B$, $j \in \mathbf{Z}$. The peeling-off argument of Lemma 3 shows that the functions b_j , $b \in B$, $j = 0, 1, 2, \dots$, form a basis for S^- . But, in order to insure local linear independence of this basis, we need to choose B with some care. Precisely, we choose B so that $\text{length}(B) := \sum_{b \in B} \text{length}(b)$ is minimal. Here, we mean by $\text{length}(b)$ the smallest integer r for which $\text{supp } b \subseteq [i, i + r]$ for some integer i .

LEMMA 5. *For any interval $J := [r, r + R]$, with r an arbitrary integer and R as before, the collection*

$$\Phi_J := \{b_j : \text{supp } b_j \cap J \neq \emptyset\}$$

is linearly independent over J .

PROOF. Suppose that we have $f := \sum_{\phi \in \Phi_j} c_\phi \phi = 0$ on J . Then, by Lemma 3, we can find some $g \in S^-$ so that $f_1 := f - g(\cdot - (r + R))$ vanishes on $[r, \infty)$. If not all the coefficients c_ϕ were zero, then we could choose some $\phi := b_j$ with $c_\phi =: c_{b_j} \neq 0$ and so that j is as small as possible. Since B is linearly independent over $[0, 1]$, $(f_1)|_{[j, j+1]} \neq 0$, and hence we must have $j < r$. Since f_1 vanishes outside $[j, r]$, while $\text{supp } b_j$ intersects $[j, j + 1]$ as well as J , this would imply that

$$\text{length}(f_1) \leq r - j < \text{length}(b_j)$$

while $f_2 := f_1(\cdot + j) \in S^-$ and, on $[0, 1]$, is in the span of B with the coefficient of b_j not zero. But then $B' := (B \setminus \{b_j\}) \cup \{f_2\}$ would also supply a basis and $\text{length}(B') < \text{length}(B)$, contradicting our choice of a basis of minimal length. \square

Suppose now that $f = \sum c_\phi \phi$ with the sum over all $\phi = b_j$ with $j \in \mathbf{Z}$ and $b \in B$. Then the local linear independence just proved allows the conclusion that

$$(2.2) \quad |c_{b_j}| \leq \text{const} \|f|_{[j, j+R]}\|_\infty$$

for some constant independent of b and j .

Since $S|_{[-R, 0]}$ is finite dimensional, we can choose a finite-dimensional subspace T of S which is carried faithfully onto $S|_{[-R, 0]}$ by the restriction map $f \mapsto f|_{[-R, 0]}$. For $s \in S$, let s_T be the unique element of T which agrees with s on $[-R, 0]$. Then $(s - s_T)\chi_{[0, \infty)} \in S^-$ by Lemma 3, hence on $[0, \infty)$,

$$(2.3) \quad s = s_T + \sum_{j \geq 0} \sum_{b \in B} c_{b_j} b_j$$

for suitable coefficients c_{b_j} . Since $S|_{[0, \infty)}$ is infinite dimensional by (2.1)(v), this shows that $B \neq \emptyset$ and that S^- is also infinite dimensional. Further, since the linear space of all linear maps from T to $S|_{[-R, 0]}$ has dimension $(\dim T)^2$, the maps

$$T \rightarrow S|_{[-R, 0]}: f \mapsto f(\cdot - a)|_{[-R, 0]}, \quad a \in A,$$

must be linearly dependent whenever $A \subset \mathbf{Z}$ and $|A| > (\dim T)^2$. This together with Lemma 3 implies that, for any $A \subset \mathbf{Z}$ with more than $(\dim T)^2$ elements, there exist coefficients $c_a, a \in A$, so that

$$(2.4) \quad \max c_a = \max |c_a| = 1, \quad \chi_{[0, \infty)} \sum_{a \in A} c_a f(\cdot - a) \in S^- \quad \text{for all } f \in T.$$

PROOF OF THEOREM 1. Let \tilde{g} be the continuous piecewise linear function with integer breakpoints which has the value 1 at $x = 1, 2, 3$ and vanishes at all other integers. Since $\tilde{g} \in C_0(\mathbf{R})$, there is $\tilde{s}_h \in S_h$ such that \tilde{s}_h converges to \tilde{g} uniformly on \mathbf{R} . From now on, we consider only $h \in 1/\mathbf{N}$. We write $s_h := \sigma_h \tilde{s}_h$ and $g_h := \sigma_h \tilde{g}$. Then $\|g_h - s_h\| \rightarrow 0$ as $h \rightarrow 0$. Since $s_h \in S$, we have $s_h = t_h + s_h^-$ on $[0, \infty)$, with $t_h \in T$ and $s_h^- \in S^-$. Now choose $A \subset 8\mathbf{N}/h$ as above. Then the functions $g_h(\cdot - a), a \in A$, are supported in $[0, \infty)$ and have disjoint supports. We define

$$G_h := \sum_{a \in A} c_a g_h(\cdot - a),$$

with the c_a as given by (2.4). Then $f_h := \sum_{a \in A} c_a s_h(\cdot - a)$ is in S^- , and $\|f_h - G_h\| \rightarrow 0$ as $h \rightarrow 0$.

Now, because of (2.4), G_h is identically one on an interval of the form $[m_h - 1/h, m_h + 1/h]$. Hence, with $F_h := f_h(\cdot + m_h)$, we have $\|F_h - 1\|_{[-1/h, 1/h]} \rightarrow 0$ as $h \rightarrow 0$. Since $f_h \in S^-$, we can write $F_h = \sum_{j \in \mathbb{Z}} \sum_{b \in B} c_{b_j}(h) b_j$. In view of (2.2),

$$(2.5) \quad |c_{b_j}(h)| \leq \text{const} \quad \text{for all } b, j, h.$$

We can take therefore a subsequence h' of h so that, for all b and j , $c_{b_j}(h')$ converges as $h' \rightarrow 0$ to some c_{b_j} . It follows that $\sum_{j \in \mathbb{Z}} \sum_{b \in B} c_{b_j} b_j$ converges uniformly on compact sets to some function $F \in S$. Since we also have that F_h converges to F uniformly on compact sets, we must have $F = 1$. Thus $(c_{b_j} b_j)_{b \in B, j \in \mathbb{Z}}$ is a partition of unity in S . It is good because of (2.5). \square

REFERENCES

- [BD] C. de Boor and R. DeVore, *Approximation by smooth multivariate splines*, Trans. Amer. Math. Soc. **276** (1983), 775–785.
- [BDH] C. de Boor, R. DeVore and K. Höllig, *Approximation order from smooth pp functions*, Approximation Theory, IV (C. Chui, L. L. Schumaker and J. Ward, editors), Academic Press, New York, 1983, pp. 353–357.
- [BH₁] C. de Boor and K. Höllig, *Bivariate box splines and smooth pp functions on a three-direction mesh*, J. Comput. Appl. Math. **9** (1983), 13–28.
- [BH₂] ———, *Approximation order from bivariate C^1 -cubics: A counterexample*, Proc. Amer. Math. Soc. **87** (1983), 649–655.
- [DM] W. Dahmen and C. Micchelli, *On the optimal order approximation rates for criss-cross finite element spaces*, 1983.
- [J] R.-q. Jia, *Approximation by smooth bivariate splines on a three-direction mesh*, Approximation Theory, IV (C. Chui, L. L. Schumaker and J. Ward, editors), Academic Press, New York, 1983, pp. 539–545.

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