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APPROXIMATION BY RATIONAL FUNCTIONS

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ABSTRACT. Making use of the Hardy-Littlewood maximal function, we give a new proof of the following theorem of Pekarski: If f' is in $L \log L$ on a finite interval, then f can be approximated in the uniform norm by rational functions of degree n to an error $O(1/n)$ on that interval.

It is well known that approximation by rational functions of degree n can produce a dramatically smaller error than that for polynomials of degree n. The best example of this is Newman's theorem [3] which shows that the function $f(x) = |x|$ can be approximated on $[-1, 1]$ by rational functions of degree n to an error $O(\exp(-c\sqrt{n}))$, whereas for polynomials of degree n the error is known to be larger than c/n. Other authors have shown that such improvement also occurs for certain classes of functions. For example, V. Popov [5] showed that if $f' \in L_p[0, 1]$, with $p > 1$, then $r_n(f) = O(n^{-1})$ where $r_n(f)$ is the error in approximating f by rational functions R of degree at most n in the *uniform norm*:

$$r_n(f) := \inf_{\deg(R)=n} \|f - R\|_\infty[0, 1].$$

To obtain this order of approximation for polynomials requires roughly speaking that $f' \in L_\infty$. A striking limiting version of Popov's result was given by A. A. Pekarski [4], who showwed that the same conclusion holds when $f' \in L \log L$, i.e. if $|f'| \log(1 + |f'|)$ is integrable.

The Popov and Pekarski proofs of these theorems are quite technical, and it was the purpose of [2] to introduce an elementary technique using maximal functions and partitions of unity for rational functions in order to give a simpler proof of Popov's results. The point of this note is to show that a modification of the technique in [2], albeit a little tricky, will also prove Pekarski's theorem.

The idea in [2] is to partition $[0, 1]$ into a set $\mathcal{I}$ of disjoint intervals I and construct associated rational functions ψ_I which form a partition of unity: $\sum_{I \in \mathcal{I}} \psi_I \equiv 1$. Our rational approximation R is then given by

$$R(x) := \sum_{I \in \mathcal{I}} f(x_I)\psi_I(x) \tag{1}$$

with x_I the center of I. Of course, the intervals I depend on f.

The rational functions ψ_I are constructed using a standard method for partitions of unity. Namely, $\psi_I := \phi_I/\Phi$ with $\Phi := \sum \phi_I$. In the case of Popov's theorem, the ϕ_I depend only on the interval I and all can be taken of degree 4. The intervals

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I are determined by using the Hardy-Littlewood maximal functions M which is defined for $g \in L_1$ by

$$Mg(x) := \sup_{J \ni x} \frac{1}{|J|} \int_J |g|,$$

where the sup is taken over all intervals $J \subset [0,1]$ which contain x.

To prove the Pekarski theorem, we will need to let the degree of ϕ_I depend on f. The desired properties of ϕ_I are given in the following lemma.

LEMMA 1. *For each even integer $m \geq 8$, and each interval I there is a nonnegative rational function ϕ_I of degree at most $6m$ with the following properties:*
(i) $\phi_I(x) \geq 1$, $x \in I$,
(ii) $\phi_I(x) \leq 8 \cdot 2^{-\sqrt{\lambda}/4}$, *if* $2^{-m/\lambda}|I| \leq \text{dist}(x, I) \leq 1/2$ *and* $0 < \lambda < m$,
(iii) $\phi_I(x) \leq 4(a^2+1)^{-m}$, *if* $\text{dist}(x, I) \geq a|I|$ *and* $a > 0$.

We postpone the proof of this lemma until the end of the paper. We now use this result to prove the following.

THEOREM. *There is an absolute constant $c > 0$ such that for $n = 1, 2, \ldots$*

$$r_n(f) \leq c\|M(f')\|_1 n^{-1}, \qquad n = 1, 2, \ldots,$$

whenever $M(f')$ is in $L_1[0,1]$.

REMARK: It is well known (see e.g. [**1**]) that $g \in L \log L$ is equivalent to $M(g) \in L_1$ and therefore this theorem is equivalent to Pekarski's.

PROOF. It is enough to consider functions f with $\|M(f')\|_1 = 1$. It follows that $\|f'\|_1 \leq 1$ and hence there is a collection $\mathcal{I}$ of at most n intervals I which are a disjoint partition of $[0,1]$ and satisfy

$$\frac{1}{n} \leq \int_I |f'| \leq \frac{2}{n}, \qquad I \in \mathcal{I}. \tag{2}$$

For each $I \in \mathcal{I}$, we let m_I be the smallest integer which is both larger than 7 and also larger than $4n \int_I M(f')$. If ϕ_I is the function of Lemma 1 for the interval I and for $m = m_I$, we let $\Phi := \sum_{I \in \mathcal{I}} \phi_I$. By Lemma 1, $\Phi \geq 1$, on $[0,1]$ and hence the functions ψ_I satisfy

$$\psi_I(x) \leq \phi_I(x), \qquad 0 \leq x \leq 1. \tag{3}$$

We now take R as in (1) with x_I the center of I. Since $\sum m_I \leq 16n$, R has degree $\leq 96n$. To estimate $|f(x) - R(x)|$, we let I_0 denote the interval of $\mathcal{I}$ which contains x; I_1 the interval of $\mathcal{I}$ immediately to the right of I_0; I_{-1} the interval immediately to the left of I_0; and so on. We have

$$f(x) - R(x) = \sum_{I \in \mathcal{I}} (f(x) - f(x_I))\psi_I(x) =: \textstyle\sum_{-1} + \sum_0 + \sum_1 \tag{4}$$

Where $\sum_{-1}$ denotes the sum over those $I = I_k$ with $k < -1$, $\sum_1$ the sum over those $I = I_k$ with $k > 1$ and $\sum_0$ the sum of the terms $k = -1, 0, 1$. Clearly, $|f(x) - f(x_{I_k})| \leq 2(|k|+1)/n$. Since the ψ_I are nonnegative and add up to one, we have

$$\textstyle\sum_0 \leq 12/n. \tag{5}$$

The estimates for $\sum_{-1}$ and $\sum_1$ are the same and therefore we estimate only $\sum_1$. For this, we fix $k > 1$ and estimate the term in $\sum_1$ corresponding to $I = I_k$. We have

$$e_k := |f(x) - f(x_I)|\psi_I(x) \le \frac{2(k+1)}{n}\psi_I(x) \le \frac{4k}{n}\phi_I(x). \tag{6}$$

We write $\operatorname{dist}(x, I) =: a|I|$, with $a \ge 0$, and we consider three cases.

Case $a \ge \sqrt{k}$. Then since $m \ge 8$, by (iii) of Lemma 1, we have $\psi_I(x) \le \phi_I(x) \le 4k^{-4}$ and consequently

$$e_k \le 16k^{-3}n^{-1}. \tag{7}$$

Case $1/2 \le a \le \sqrt{k}$. The smallest interval J which contains x and I has length $(a+1)|I|$ and on I,

$$M(f') \ge \frac{1}{|J|}\int_J |f'| \ge \frac{k}{n(a+1)|I|}$$

and therefore $m \ge 4n\int_I M(f') \ge 4k/(a+1) \ge \sqrt{k}$. This gives by (iii) of Lemma 1,

$$e_k \le \frac{4k}{n}\phi_I(x) \le \frac{4k}{n}(a^2+1)^{-m} \le \frac{16k}{n}(5/4)^{-\sqrt{k}}. \tag{8}$$

Case $0 < a < 1/2$. We write $a =: 2^{-m/\lambda}$ with $0 < \lambda < m$. Similar to the second case, for $u \in I$, we have $M(f')(u) \ge (k-1)/n(u-x)$. Therefore,

$$m \ge 4n\int_I M(f') \ge 4(k-1)\int_{2^{-m/\lambda}|I|}^{|I|} \frac{du}{u} \ge 2k\left(\frac{m}{\lambda}\right)\log 2.$$

This shows that $\lambda \ge 2k\log 2 \ge k$. Hence by (ii) of Lemma 1, we have

$$e_k \le \frac{4k}{n}\phi_I(x) \le \frac{32k}{n}2^{-\sqrt{k}/4}. \tag{9}$$

The estimates (7)–(9) serve to show that $\sum_1 = \sum e_k \le cn^{-1}$, with c an absolute constant. This combined with (5) and the corresponding estimate for $\sum_{-1}$ when placed in (4) proves the theorem.

We turn now to the proof of Lemma 1. For this, we shall use the following:

LEMMA 2. *For each even integer $m \ge 8$ there is a rational function R of degree $\le 2m$ with the following properties:*

(i) $R(x) \ge 1$, $x \in [-1, 0]$,
(ii) $0 \le R(x) \le 2$, *for* $-\infty < x < \infty$,
(iii) $|R(x)| \le 2 \cdot 2^{-m/4j}$, *if* $2^{-(j+1)^2/m} \le x \le 1/2$, *with* $\sqrt{m} - 1 \le j < m$.

PROOF. With $a := 2^{-1/m}$ and $a_k := a^{k^2}$, we define $p(x) := \prod_1^m (x + a_k)$. We first estimate $\pi(x) := p(-x)/p(x) = \prod_1^m (-x + a_k)/(x + a_k)$ when $x \ge 0$. Since each term in π has absolute value at most 1, we have

$$|\pi(x)| \le 1, \qquad x \ge 0. \tag{10}$$

When $a_m \le x \le 1/2$, we take j so that $a_{j+1} \le x \le a_j$; so $\sqrt{m} - 1 \le j < m$. Then,

$$|\pi(x)| \le \pi_1(x) := \prod_1^j \frac{a_k - x}{a_k + x}.$$