A NOTE ON ADAPTIVE APPROXIMATION

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Abstract

One of the main results of this note is to characterize the set of functions which have the order of approximation $O(n^{-a})$ *with n the number of parameters for the approximating family. Then a limiting version of the Bironan–Solomjak result with* $K=d/b-d/q$ *is given.*

1. Introduction

There are many theorems which show that non-linear methods of approximation can perform markedly better than their linear counterparts. For example, we have the striking result of Newman [5] on the rational approximation of $|x|$, the work of Popov [6] on rational approximation and various theorems on adaptive and optimal knot spline approximation. One of the main goals of these subjects is to characterizie the set of functions which have the order of approximation $O(n^{-a})$ with n the number of parameters for the approximating family.

While much attention has been given recently to this problem for rational and optimal knot spline approximation, there has been little note of the simpler (and in some sense less nonlinear) adaptive approxima- tion. Part of our interest in adaptive approximation is because of its application to rational approximation (see e.g. [4]).

In their seminal work, Birman and Solomjak [3] gave an estimate for the degree of adaptive approximation on the unit cube Ω in R^d for functions in Sobolev spaces. Namely, they thow shat if $f \in W^k_p$, and $k>d/p-d/q$, then f can be approximated in the $L_q(\Omega)$ norm to an error of approximation $O(n^{-k})$ by an adaptive scheme which results in a piecewise polynomial with n dyadic pieces. A more precise description of this result is given in the next section. This result shows the general flavor of nonlinear approximation in that there is a gain in the order of approximation over the linear counterpart. For example, the typical linear schemes of polynomial or spline approximation would require "roughly speaking" that f is in W^k_∞ to get this order of approximation.

In this note, we want to give a limiting version of the Birman Solomjak result which allows k to equal $d/p-d/q$. The simplest example of our theorem is the univariate case and f' is in LlogL. We show then that f can be approximated in the uniform norm on Ω to an error $O(n^{-1})$ while using at most n dyadic pieces. This is to be compared with the Birman-Solomjak theorem which requires that f' be in L_p for some $p>1$. Our approach to this result is via maximal functions. Besides giving a stronger theorem, this technique is simpler than that given by Birman-Solomjak.

Our theorem will be formulated in terms of the Hardy-Littlewood maximal function,

$$Mg(x) := \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |g| \qquad (1.1)$$

where the sup is taken over all cubes Q which contain x and are contained in Ω. The connection between M and LlogL is that a function g is in LlogL if and only if Mg is in L_1 (see e.g. [2]). We shall also use the modified maximal function,

$$M_p(g) := M(|g|^p)^{1/p} \qquad (1.2)$$

which is defined for $p \geq 1$.

2. Adaptive approximation

We begin by describing the Birman-Solomjak theorem. We are interested in approximating functions f which are in $L_q(\Omega)$, $\Omega := [0,1]^d$, by piecewise polynomials of degree k. Recall that $f \in L_p(\Omega)$ is in $W^k_p(\Omega)$ if and only if the weak derivatives $D^v f$, $|v|=k$ are in $L_p(\Omega)$. In this case, we let

$$D_k(f) := \sum_{|v|=k} |D^v f| \qquad (2.1)$$

Then $D_k(f)$ is in $L_p(I)$. We let

$$I(Q) := \int_Q D_k(f)^p.$$

We now assume that we are given an error tolerance $\varepsilon>0$ and we wish to construct a partition $G := G_\varepsilon$ of Ω into dyadic cubes Q such that the error of best L_q approximation $E_k(f, Q)_q$ to f on Q by polynomials of order at most k (degree k-1) satisfies

$$Ek(f, Q)_q < \varepsilon, \quad Q \in G. \tag{2.2}$$

Then $S:P_Q$, $Q \in G$ satisfies $\|f-S\|_q < N^{1/q}\varepsilon$, where $N := N\varepsilon := |G|$ is the number of cubes in the partition G. We are therefore interested in estimating $N\varepsilon$ as a function of ε.

The Birman-Solomjak method constructs such a partition in an adaptive way as follows. It uses the fact that for each cube Q,

$$E_k(f, {}_Q)_q \leq \text{const.} \ |Q|^{k/d-1/q} I(Q)^{1/p}. \tag{2.3}$$

provided $k > d/p - d/q$. We call a cube Q "good" if the right side of (2.3) is $<\varepsilon$. Otherwise, we say Q is "bad". If Ω is a good cube, we let $G = \{\Omega\}$ and we have our set G. If Ω is not a good cube, we let $Bo := \{\Omega\}$ and $Go := \phi$, our sets of good and bad cubes respectively. Suppose that at stage j, we have a set of bad cubes B_j and good G_j with each cube of sidelength 2^{-j}. We subdivide each bad cube into 2^d cubes of equal size by "halving". The resulting cubes which are good are put into our set G_{j+1}. Any bad cubes are put into the set B_{j+1} and the whole process is repeated. If this process terminates, that is, if all cubes at a given stage are good, then the collection of all good cubes is our set G, i.e. $G := UG_k$.

The Birman-Solomjak theorem [3] gives that if $f \in W_p^k$ with $k > d/q - d/p$, then $N\varepsilon$ is $0(\varepsilon^{-\tau})$, $\tau := 1/(k/d + 1/q)$. By taking $\varepsilon := n^{-1/\tau}$, we see that f can be approximated to an order $0(n^{-k/d})$ with the resulting partition having at most n cubes.

We shall give a strengthening of this theorem based on the following.

Theorem. If $f \in W_p^k$ and $\|M_p(D_k f)\|_p = 1$, then there is a dyadic partition G of Ω with the following properties.

i) $|G| \leq \text{const.} n$ (2.4)

ii) $I(Q) < 1/n$, for all $Q \in G$,

Proof. We let $\varepsilon := 1/n$. We proceed as in the Birman-Solomjak method except that now we say a cube is "good" if $I(Q) < \varepsilon$; otherwise Q is bad. We construct the sets G_k based on this definition exactly as above.

It is clear that each cube Q in G satisfies ii). To verify i), we note that each cube $Q \in G_{m+1}$ comes from a parent cube R of B_m. By the definition of B_m, $I(R) > \varepsilon$. Since $|R| = 2^d|Q|$, we have

$$M_p(D_k f)(x) > (\varepsilon/|R|)^{1/p} > (c\varepsilon/|Q|^{1/p}, \ x \in Q.$$

It follows that

$$\int_Q M_p(D_k f)^p > c\varepsilon.$$

Now this is true for each good cube $Q \in G$ and since these cubes are disjoint, we have

$$c\varepsilon\ |G\varepsilon| < \int_\Omega M_p(f)^p = 1$$

which is ii).

We want to apply Theorem 2.1 to adaptive approximation. For this, we shall use the well known inequality,

$$E_k(f, Q)_q \leq \text{const.}\ \|D_k f\|_p(Q). \tag{2.5}$$

which holds for any cube $Q \subset \Omega$ and any k, p, q provided $k/d = 1/p - 1/q$. The inequality (2.5) follows from the Sobolev embedding theorem (see [1, p.105],

$$\|f\|_q \leq \text{const.}\ (\|f\|_p + \|D_k f\|_p). \tag{2.6}$$

In fact, we replace f in (2.6) by f-p with P the best L_p approximation to f on Q.

Then $D_k(f-p) = D_k f$ and $\|f-p\|_p \leq \text{const.}\ |Q|^{k/d}\ \|D_k f\|_p$ is a special case of Whitney's theorem (see [D-D-S]) in several variables.

It follows from (2.5) and our theorem that the adaptive scheme gives a partition G of at most const. n cubes Q for which $E_k(f, Q)_q \leq$ const. $n^{-1/p}$. Hence, if P_Q is the best approximation to f on Q for each $Q \in G$, then $S := P_Q$ on Q for each $Q \in G$, satisfies $\|f - s\|_q \leq \text{const.}\ n^{1/q - 1/p} = \text{const.}\ n^{-k/d}$. Hence, we have the following.

Corollary. Let $1 \leq p$, $q \leq \infty$ and the positive integer k be related by $k/d = 1/p - 1/q$. If $f \in W_p^k$, the adaptive scheme described above generates a partition G of at most const. n dyadic cubes and the spline function S defined above satisfies

$$\|f - s\|_q(\Omega) \leq \text{const.}\ n^{-k/d}\ \|Mp(D_k f)\|_p.$$

References

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