Interpolation Spaces and Non-linear Approximation

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One of the central problems of approximation theory is to characterize the set of functions which have a prescribed order of approximation by a given method of approximation. The classical results in this subject go back to the turn of the century and the work of D. Jackson and S. Bernstein, who among other things showed that a continuous 2π-periodic function f is approximated in the uniform norm to an order $O(n^{-\alpha})$, $0<\alpha<1$, by trigonometric polynomials of degree n if and only if f is in Lip α. Many other results of this type have been obtained for various other types of approximation including algebraic polynomials and splines with fixed knots.

In the 1960's, the subject of approximation turned in large part to the study of non-linear methods. However, until recently, there was no really concrete characterization of the approximation spaces for the two most important types of non-linear approximation, namely, approximation by rational functions, and approximation by splines with free knots. One of the main purposes of this paper is to show how recent results of P. Petrushev [Pt, Pt_1] and P. Pekarskii [Pe_1] can be used to give characterizations of the approximations spaces for these two methods of approximation. In certain cases, the description of these approximation spaces is very concrete (they are Besov spaces).

Suppose we are given a sequence G_n of sets which will be used to approximate a function f in the metric of some space X. We let $E_n(f) := E_n(f)_X$ denote the error of approximation of f by the elements of G_n. We are interested in characterizing those f which have a given order of approximation. For example, we would like to describe the set, A_∞^α, $0<\alpha$, which consists of all f which have an order of approximation $E_n(f) = O(n^{-\alpha})$, $n\to\infty$. More general approximation classes are the A_q^α, $\alpha,q>0$, which consist of all functions f for which

$$(1.1)\qquad \sum_{n=1}^{\infty} [n^\alpha E_n(f)]^q\, n^{-1}$$

is finite. The primary index here is α which shows that the order c approximation is like $n^{-\alpha}$ while the q gives a finer gradation.

During the 1960's, much information [B-P, B, Bu] was given abou the classes A_q^α(splines of order k) for free knot spline approximatic in the space $L_p[0,1]$, $1\le p\le\infty$ (p is fixed and we do not indicate the dependence of the approximation spaces on p). In this type of approximation, G_n is the set of all piecewise polynomials of degree k which have at most n pieces on [0,1]. We mention in particular, t results of Bergh and Peetre [B-P] for approximation in the space C[0,1]. They introduce a space V_σ defined by means of a generalized variation and show that the spaces A_q^α are the real interpolation between C and V_σ. While this result gave important information abc A_q^α, it was not completely satisfactory since there was no descriptic of the resulting interpolation spaces.

Recently, P. Petrushev [Pt_1] studied free knot spline approximation in $L_p[0,1]$, p>0 and proved two fundamental inequalitie for this type of approximation (see §5) which are the anolgues of th Jackson and Bernstein inequalities for trigonometric polynomials. From these inequalities, it follows that the spaces A_q^α(splines of order k) can be characterized as the real interpolation spaces betwe L_p and a Besov space $B_\sigma^\lambda(L_\sigma)$, where λ is any real number larger than and σ is determined by the relation $1/\sigma = \lambda+1/p$. Here, the Besov spaces are defined using the modulus of smoothness of the function (see §5). Petruscev went further and anounced that for $0<\alpha<k$,

(1.2) A_σ^α(splines of order k) = $B_\sigma^\alpha(L_\sigma)$, when $1/\sigma= \alpha+1/p$ and $1\le p<\infty$.

Yu. Brudnyi also announced this result at a conference in Kiev in 19 but again with no indication of proof. Of course (1.2) is very interesting because it gives a classical description of the approximation spaces for these values of the parameters. We should mention that the first explicit characterization of this type for no linear approximation was the famous result of V. Peller [Pl] for approximation on the unit disc in the metric BMO by rational functions.

The research in this paper came about because it was not clear to us how the identity (1.2) followed from the Bernstein and Jackson inequalities, since in particular we did not know the interpolation spaces between L_p and B_σ^λ. This led us to investigate more thoroughl the connections between interpolation spaces and approximation space like the A_q^α and in particular to determine the role of the Jackson a Bernstein inequalities in such questions.

It turns out that it is possible to develop a systematic theory (§2,3) for deciding when approximation spaces are interpolation spac and to establish precisely the role of Jackson and Bernstein inequalities in this process. This is of course not unexpected since such results for linear approximation (see for example Butzer-Scherer [B-S]) are well known and as we have already mentioned, the techniques of interpolation have already been used for specific cases of non-linear approximation. We want also to mention that Peetre and Sparr [P-S] also have developed such a theory based on their E functional. In a sense (see §2,3), our approach is equivalent to theirs (we make this clearer in §3) but for the applications in approximation, our approach seems more natural.

In §2,3, we introduce approximation spaces for a rather general form of approximation and give some of their simple properties. In particular, we show that such approximation spaces are always interpolation spaces and we also show that such spaces always satisfy Bernstein and Jackson inequalities. The familiar reader will find this material rather "old hat" although the organization of these ideas may be useful.

Our main result given in §4 is what we call an extrapolation theorem. If we are given a family of interpolation spaces X_q^α (i.e. this family is invariant under the real method of interpolation) then our extrapolation theorem gives a condition which guarantees that the X_q^α are interpolation between some space X and one of the X_p^γ. The proof of this extrapolation theorem is quite simple and yet this theorem has important application to approximation theory.

Our main application of the extrapolation theorem is to free knot spline approximation and the interpolation of Besov spaces. For example, we use it to show that for any $0<p<\infty$,

$$B_\sigma^\alpha(L_\sigma) = (L_p, B_\gamma^\beta(L_\gamma))_{\alpha/\beta,\sigma}$$

where α and σ and likewise β and γ are related as in (1.2). As a consequence, we obtain (1.2). The extrapolation theorem was our first proof of (1.3). Subsequently, we have heard of two other ways to obtain (1.3), at least with some restrictions on the parameters. Michael Cwickel has shown us how to use retracts and the Littlewood-Paley theory for a proof of (1.3) provided $\sigma,p>1$. Additionally, Bjorn Jaewerth informed us at this conference that he has used an atomic decomposition to prove (1.3) when the Besov spaces $B_\sigma^\alpha(L_\sigma)$ are defined via Fourier transforms (this definition agrees with ours when $\sigma\geq 1$). We should mention that the case when $\sigma<1$ in (1.3) is most important for non-linear approximation.

It is also possible to use (1.3) to describe the approximation spaces A_q^α(rational) for rational approximation. In fact, there are equalities between rational and free knot spline approximation due to Petrushev [Pt] and Pekarskii [Pe_1] which show that (Theorem 5.2) A_q^α(rational) = A_q^α(splines) provided $1<p<\infty$.

Some additional applications of the extrapolation theorem are given in §5. Finally, in §6, we mention some related results and problems.

2. Approximation spaces. We wish to begin with a general theory whose main aim is to describe approximation spaces as interpolation spaces. We shall denote by X, the space in which approximation will take place. We wish to let X be as general as possible, in particular, we shall not assume that X is linear or normed. Instead, we follow Peetre and Sparr [P-S] and require only that X be what they call a quasi-normed Abelian group. This means that X is an Abelian group under addition with a neutral element denoted by 0 and for each $f \in X$, there is defined $||f|| := ||f||_X$ with the properties:

(2.1) i) $||f|| \geq 0$,

ii) $||-f|| = ||f||$

iii) $||f+g||^\mu \leq ||f||^\mu + ||g||^\mu$, for all $\mu>0$ sufficiently small.

It follows from iii) that $||0|| = 0$. However, there may be other f with $||f|| = 0$. Property iii) is equivalent to

(2.1) iv) $||f+g|| \leq c\ (||f|| + ||g||)$,

for some constant $c \geq 1$, (see [B-L, p.59]).

We let $G_o := \phi$ and let G_n, n=1,2,... denote a sequence of sets whose elements will be used in the approximation. We wish to assume as little as possible about the structure of the sets G_n. For example, we do not want to assume that G_n is a linear space since this is too restrictive for our intended applications. Instead, we shall assume that the sets G_n have the following properties:

(2.2) i) $0 \in G_n$, n=1,...,

ii) $G_n \subset G_{n+1}$, n=0,1,...,

ii) $G_n \pm G_m \subset G_{c(n+m)}$, n,m= 0,1,.... , with c an absolute constant.

iii) $\cup G_n$ is dense in X.

This is satisfied by all of the non-linear families mentioned above. For convenience, we shall also assume that

(2.2) iv) each $f \in X$ has a best approximation from G_n, n=1,2,... .

this assumption eliminates some technical difficulties and could be replaced by weaker assumptions.

We let $E_n := E_n(f) := E_n(f)_X$ denote the error of approximation

$$E_n := \inf_{g \in G_n} ||f-g||, \quad n=0,1,\ldots,$$

in particular $E_0(f)=||f||$.

We shall call a sequence of sets (G_n) which satisfies the conditions (2.2) a normal approximating family. Such sequences were first introduced by Petrushev and Popov [P-P, Chapter 3]. For such a sequence, we let A_q^α denote the set of $f \in X$ such th

$$\text{(2.3)} \qquad N(f) := \left(\sum_{n=0}^{\infty} [2^{n\alpha} E_{2^n}(f)]^q \right)^{1/q}$$

is finite. As usual, here and in the sequel, the l_q norm is replaced by the l_∞ norm when $q=\infty$. Since E_m is monoton non-increasing, we obtain an equivalent norm if we let the sum in (2.3) run over all $n\geq 0$ and introduce the weight n^{-1} as we did in (1). The set A_q^α is a normed Abelian group and N is a quasi-norm on A_q^α. However, there is another quasi-norm which is much more useful for our purposes which we now describe.

For each $f \in X$, we let $S_n := S_n(f)$ be a best approximation to f from G_{2^n}. S_n is generally not unique, so we fix S_n once and for all. In addition, we let $S_{-1}:=0$ and define $T_n :=$ (f)$:= S_n - S_{n-1}$. We shall use the space $l_q^\alpha(X)$ of sequences (a_n) of functions from X and its norm:

$$\text{(2.4)} \qquad ||(a_n)||_{l_q^\alpha(X)} := \left(\sum_c [2^{\alpha n} ||a_n||]^q \right)^{1/q},$$

THEOREM 2.1. <u>The following is an equivalent quasi-norm in</u> A_q^α:

$$\text{(2.5)} \qquad N_{q,\alpha}(f) := ||(T_n f)||_{l_q^\alpha(X)}.$$

<u>Proof</u>. From (2.1) iv), we have $||T_n|| \leq c[E_{2^n} + E_{2^{n-1}}]$. Therefore, applying the $l_q^\alpha(X)$ norm to (T_n) gives that $N_{q,\alpha}(f) \leq c\, N(f)$ with N(f) as in (2.3). To reverse this inequality, we note that $T_n \to f$, $n\to\infty$ by (2.1) iii). Hence, we have for $\mu > 0$ sufficiently small:

$$\text{(2.6)} \qquad E^{\mu}_{2^n} \leq \sum_{n+1}^{\infty} ||T_k||^{\mu}.$$

With $\beta := \alpha/2$, we write $||T_k||^\mu = 2^{-k\beta\mu}\, 2^{k\beta\mu}\, ||T_k||^\mu$ in (2.6) and then apply Holder's inequality to find

$$N(f)^q \le c \sum_{n=0}^{\infty} 2^{\alpha nq} \left(\sum_{n+1}^{\infty} ||T_k||^\mu \right)^{q/\mu} \le c \sum_{n=0}^{\infty} 2^{n\alpha q}\, 2^{-n\beta q} \sum_{n+1}^{\infty} 2^{k\beta q} ||T_k||^q.$$

A change in the order of summation shows that the right side of this inequality is less than $c\, N^q_{\alpha,q}(f)$ as desired.

Peetre and Sparr [P-S] have used a different approach to define approximation spaces. Namely, they consider a quasi-abelian group $Y \subset X$ and they introduce the E-functional

$$(2.7) \qquad E(f,t) := E(f,t,X,Y) := \inf_{||g||_Y \le t} ||f-g||_X .$$

Their approximation spaces are then defined as the set of all $f \varepsilon X$ for which The expression in (2.3) is finite when $E_{2^n}(f)$ is replaced by $E(f,2^n)$. Both approaches (theirs and ours) lead to the same approximation spaces.

For example, if $Y := \cup G_n$ with $||g||_Y := \inf\{n: g \varepsilon G_n\}$ then $E_n(f) = E(f,n)$ and therefore their approach gives our approximation spaces A^α_q with this choice. On the other hand, if we are given X and Y, we can take $G_n := \{g \varepsilon Y: ||g|| \le n\}$ and again obtain $E(f,n) = E_n(f)$, so that our approach gives their approximation spaces as well.

In the next section, we shall give various properties of the A^α_q spaces. Some of these can be found in [P-S].

3. Jackson and Bernstein inequalities. It has been understood for some time that one of the keys to characterizing the approximation spaces A^α_q in concrete cases is to establish certain inequalities known as the Jackson and Bernstein inequalities.

We suppose that for some (for the present fixed) number $\lambda > 0$, we have a subset X^λ of X which is itself an Abelian group with a quasi-norm $||\cdot||_{X^\lambda}$. We ask whether the following inequalities are valid

$$(3.1) \qquad E_n(f) \le c\, n^{-\lambda} ||f||_{X^\lambda}, \quad f \varepsilon X^\lambda \quad \text{(Jackson inequality)}.$$

$$(3.2) \quad ||g||_{X^\lambda} \le n^\lambda ||g||_X, \quad g \varepsilon G_n, \ n=1,2,\ldots \ \text{(Bernstein inequality)}$$

We now show that if these two inequalities are satisfied for some space X^λ, then the approximation spaces A^α_q, $0<\alpha<\lambda$, can be characterized as interpolation spaces between X and X^λ. Such results are well known for linear approximation (see [B-S]) and are implicit in the paper of Peetre-Spaar, although we have found no exact formulation as given in the next theorem. We let $(A,B)_{\theta,r}$ denote the interpolation spaces generated by the K method for the pair (A,B).

THEOREM 3.1. *If* X^λ *satisfies the Jackson and Bernstein inequalities, then for* $0<\alpha<\lambda$ *and* $q>0$, *we have* $A_q^\alpha = (X,X^\lambda)_{\alpha/\lambda,q}$.

Proof. We let $K(f,t) := K(f,t,X,X^\lambda)$ be the K-functional for (X,X^λ). We have for appropriate constants c_1, $c_2 > 0$, and for $n=0,1,\ldots$:

$$(3.3)\qquad c_1 E_{2^n}(f) \le K(f,2^{-n\lambda}) \le c_2\Big(E_{2^n}(f) + 2^{-n\lambda} \Big(\sum_{k=0}^{n} 2^{k\lambda\mu} E^\mu_{2^k}\Big)^{1/\mu}\Big).$$

Indeed, to prove the left inequality, we use property (2.1)iv) and the Jackson inequality to find that for each r in X^λ,

$$E_{2^n}(f) \le c\Big(||f-r|| + E_{2^n}(r)\Big) \le c\Big(||f-r|| + 2^{-n\lambda} ||r||_{X_\lambda}\Big).$$

Taking an infimum over all r gives the left inequality in (3.3).

For the right inequality, we have from Bernstein's inequality that for any $\mu>0$ sufficiently small:

$$K(f,2^{-n\lambda}) \le ||f-S_n|| + 2^{-n\lambda} ||S_n||_{X^\lambda} \le E_{2^n} + 2^{-n\lambda}\Big(\sum_{o}^{n} ||T_k||^\mu_{X^\lambda} \Big)^{1/\mu}$$

$$\le E_{2^n} + c\, 2^{-n\lambda} \Big(\sum_{o}^{n} 2^{k\lambda\mu} E^\mu_{2^k} \Big)^{1/\mu},$$

as desired.

To complete the proof of the theorem, we note that because $K(f,t)$ bounded $(X^\lambda\subset X)$, the q-th power of the norm in the space $(X,X^\lambda)_{\theta,q}$ equivalent to

$$(3.4)\qquad \sum_{n=0}^{\infty} \Big(2^{n\alpha} K(f,2^{-n\lambda})\Big)^q .$$

From the left inequality in (3.3), we get that the norm of f in A_q^α is smaller than the right side of (3.4). For the converse, we fix β satisfying $\alpha<\beta<\lambda$ and we write $2^{k\lambda\mu} = 2^{k(\lambda-\beta)\mu}\, 2^{k\beta\mu}$ and then apply Holder's inequality to find that the series on the right side of (3.3) is less than

$$2^{n(\lambda-\beta)} \Big(\sum_{o}^{n} 2^{k\beta q} E^q_{2^k} \Big)^{1/q} .$$

It follows from this and (3.3) that the sum in (3.4) is less than a constant multiple of

$$(3.5) \qquad \sum_{n=o}^{\infty} 2^{n\alpha q} E^q_{2^n} + \sum_{n=o}^{\infty} 2^{n(\alpha-\beta)q} \sum_{k=o}^{n} 2^{k\beta q} E^q_{2^k}.$$

Changing the order of summation, we see that (3.5) is less than a constant multiple of the q-th power of the norm in A^α_q. This proves the theorem.

It is rather easy to see that the spaces A^λ_q satisfies Jackson and Bernstein inequalities.

LEMMA 3.2. _If_ $q,\lambda > 0$, _then_ $X^\lambda := A^\lambda_q$ _satisfies the Jackson and Bernstein inequalities_.

Proof. If $f \varepsilon X^\lambda$, then from the monotonicity of E_m, we have that $m^\lambda E_{2m} \leq ||f||_{X^\lambda}$ which is the Jackson inequality. On the other hand, if $g \varepsilon G_{2^n}$, then $E_m(g)=0$ for $m \geq 2^n$. Hence,

$$||g||^q_{X^\lambda} = \sum_{o}^{n} [2^{m\lambda} E_{2^m}(g)]^q \leq c\ 2^{n\lambda q} ||g||^q_X .$$

which is the Bernstein inequality for g in G_{2^n}. The Bernstein inequality for all m=1,2,... follows easily from this.

From Lemma 3.2 and Theorem 3.1, it follows that approximation spaces are always an interpolation family. Namely, we have

COROLLARY 3.3. _If_ $q,\lambda,r>0$, and if $0<\alpha<\lambda$, _then we have_

$$A^\alpha_q = (X;A^\lambda_r)_{\alpha/\lambda,q}.$$

In particular, from the reiteration theorem for interpolation, we have the following result of Peetre-Sparr [P-S].

COROLLARY 3.4. _For any_ $q,\alpha_1,q_1,\alpha_2,q_2 > 0$, _and_ $\alpha = \theta\alpha_1+(1-\theta)\alpha_2$, _with_ $0<\theta<1$, _we have_

$$(A^{\alpha_o}_{q_o},A^{\alpha_1}_{q_1})_{\theta,q} = A^\alpha_q$$

The above results show that approximation spaces are interpolation spaces. The converse also holds. Namely, if Y is an Abelian subgroup of X which is a quasi-normed space then the interpolation spaces $(X,Y)_{\theta,q}$ are approximation spaces for the sets $G_n = \{g\varepsilon Y: ||g||_Y \leq n\ ||g||_X\}$. In fact, in view of Theorem 3.1, we only have to verify that the Jackson and Bernstein inequalities hold

It follows from this and (3.3) that the sum in (3.4) is less than a constant multiple of

$$(3.5) \qquad \sum_{n=o}^{\infty} 2^{n\alpha q} E^q_{2^n} + \sum_{n=o}^{\infty} 2^{n(\alpha-\beta)q} \sum_{k=o}^{n} 2^{k\beta q} E^q_{2^k}.$$

Changing the order of summation, we see that (3.5) is less than a constant multiple of the q-th power of the norm in A^{α}_q. This proves the theorem.

It is rather easy to see that the spaces A^{λ}_q satisfies Jackson and Bernstein inequalities.

LEMMA 3.2. <u>If</u> $q,\lambda>0$, <u>then</u> $X^{\lambda} := A^{\lambda}_q$ <u>satisfies the Jackson and Bernstein inequalities</u>.

<u>Proof</u>. If $f\epsilon X^{\lambda}$, then from the monotonicity of E_m, we have that $m^{\lambda}E_{2m} \le ||f||_{X^{\lambda}}$ which is the Jackson inequality. On the other hand, if $g\epsilon G_{2^n}$, then $E_m(g)=0$ for $m\geq 2^n$. Hence,

$$||g||^q_{X^{\lambda}} = \sum_{o}^{n} [2^{m\lambda} E_{2^m}(g)]^q \le c\ 2^{n\lambda q}\ ||g||^q_X .$$

which is the Bernstein inequality for g in G_{2^n}. The Bernstein inequality for all m=1,2,... follows easily from this.

From Lemma 3.2 and Theorem 3.1, it follows that approximation spaces are always an interpolation family. Namely, we have

COROLLARY 3.3. <u>If</u> $q,\lambda,r>0$, and if $0<\alpha<\lambda$, <u>then we have</u>

$$A^{\alpha}_q = (X, A^{\lambda}_r)_{\alpha/\lambda, q}.$$

In particular, from the reiteration theorem for interpolation, we have the following result of Peetre-Sparr [P-S].

COROLLARY 3.4. <u>For any</u> $q,\alpha_1,q_1,\alpha_2,q_2>0$, <u>and</u> $\alpha=\theta\alpha_1+(1-\theta)\alpha_2$, <u>with</u> $0<\theta<1$, <u>we have</u>

$$(A^{\alpha_o}_{q_o}, A^{\alpha_1}_{q_1})_{\theta,q} = A^{\alpha}_q$$

The above results show that approximation spaces are interpolation spaces. The converse also holds. Namely, if Y is an Abelian subgroup of X which is a quasi-normed space then the interpolation spaces $(X,Y)_{\theta,q}$ are approximation spaces for the sets $G_n = \{g\epsilon Y: ||g||_Y \le n\ ||g||_X\}$. In fact, in view of Theorem 3.1, we only have to verify that the Jackson and Bernstein inequalities hold for Y. But the Bernstein inequality is built into the definition of G_n. For the Jackson inequality, we write $||f||_Y =: \gamma\ ||f||_X$ for an appropriate γ. Then $E_m(f) = 0$ for $m>\gamma$ because $f\varepsilon G_m$. While if $m\leq\gamma$, then $E_m(f) \leq ||f||_X = \gamma^{-1}\ ||f||_Y \leq m^{-1}\ ||f||_Y$.

4. An extrapolation theorem. In the previous section, we have seen that the concepts of interpolation spaces and approximation spaces are identical. We want to exploit this to give a sort of extrapolation theorem for interpolation.

We call a family of spaces X_q^α, $0<q$: $\alpha_o<\alpha\leq\alpha_1$ an interpolation family if they are invariant under interpolation, i.e. if $(X_p^\alpha, X_r^\beta)_{\theta,q} = X_q^\gamma$ when $\gamma = \theta\alpha + (1-\theta)\beta$. Of course typically such a family comes from interpolation between a pair of spaces (X,Y). Namely, the spaces $X_q^\alpha := (X,Y)_{\alpha,q}$ are an interpolation family.

Given an interpolation family, we are interested in when there is a sort of limiting space X, as $\alpha\to 0$, for this family. We shall see that this is the case if the X_q^α satisfy Jackson and Bernstein inequalities with respect to X. While the proof of this fact is rather simple, it has several interesting applications.

In most of our applications, we shall not have a doubly indexed family of spaces but instead will have for each value of α, one space X^α. We shall say that (X^α) is selection of an interpolation family $(X^\alpha\ )$, if for each $0<\alpha<\alpha_o$, there is a $q=q(\alpha)$ such that $X^\alpha = X_q^\alpha$.

To formulate our extrapolation theorem, we suppose that G_n are approximation sets with the usual properties and A_q^α are its approximation spaces. We begin with the following simple lemma (see [P-S]).

LEMMA 4.1. If X^λ is a quasi-normed sub Abelian group of X which satisfies the Jackson and Bernstein inequalities (3.1) and (3.2), then we have the continuous embeddings:

$$A_\mu^\lambda \subset X^\lambda \subset A_\infty^\lambda, \tag{4.1}$$

provided μ is sufficiently small.

Proof The right inequality is immediate from the Jackson inequality. Over the left inequality, we use that $f\varepsilon X$ can be represented as $f = \sum_0^\infty T_k$ with our usual notation. Since $||.||_{X^\lambda}$ is a quasi-norm, for $\mu>0$ sufficiently small, we have from the Bernstein inequality

$$||f||_{X^\lambda}^\mu = ||\sum_0^\infty T_k||_{X^\lambda}^\mu \leq \sum_0^\infty ||T_k||_{X^\lambda}^\mu \leq c\sum_0^\infty [2^{k\lambda}||T_k||]^\mu,$$

as desired.

THEOREM 4.2. (Extrapolation theorem). Let X^λ, $0<\lambda<\lambda_o$ be a selectio of the interpolation family (X_q^λ) and suppose that X^λ, $0<\lambda<\lambda_o$ satisfy Jackson and Bernstein inequalities for some approximation sets G_n, $n=0,1,\ldots$ and some space X. Then,

$$(4.2)\qquad (X,X_r^\lambda)_{\theta,q} = X_q^{\lambda\theta} = A_q^{\lambda\theta} \text{ for all } r,q>0 \text{ and } 0<\lambda<\lambda_o;\ 0<\theta<1$$

Proof. Let A_q^α be the approximation spaces for G_n. From Lemma 4.1, we have that $A_\mu^\lambda \subset X^\lambda \subset A_\infty^\lambda$, provided $\mu>0$ is sufficiently is sufficiently small. Since the approximation spaces are interpolation spaces, they satisfy the reiteration theorem. Likewise for the spaces X_r^λ. Hence, if $0<\theta<1$,

$$A_q^\lambda = (A_r^\alpha, A_s^\beta)_{\theta,q} = (X^\alpha, X^\beta)_{\theta,q} = X_q^\lambda$$

for all $q>0$ and $0<\alpha<\beta\le\lambda_o$ with $\lambda = \theta\alpha + (1-\theta)\beta$. The theorem now follows from Corollary 3.2.

5. Applications. We want now to apply the above development to the study of the degree of non-linear approximation. We let $B_q^\alpha(L_\sigma)$, $\alpha,\sigma,q>0$, denote the Besov spaces on the interval $\Omega := [0,1]$. Here, we use the definition of Besov spaces with the modulus of smoothness $\omega_k(f,t)_\sigma$ of f in $L_\sigma(\Omega)$. Namely, the "quasi-norm" in $B_q^\alpha(L_\sigma)$ is given by

$$|f|_{B_q^\alpha(L_\sigma)} := \left(\int_0^\infty \left(t^{-\alpha}\omega_k(f,t)_\sigma\right)^q \frac{dt}{t} \right)^{1/q},$$

where $k := [\alpha]+1$.

When the value of σ is fixed, the Besov spaces are an interpolation family. In fact for $\sigma\ge1$, the $B_q^\alpha(L_\sigma)$, $0<\alpha<k$ are the interpolation spaces for L_σ and the Sobolev space W_σ^k. When $\sigma<1$, this is shown in [D-P]. This latter paper also describes the interpolation of Besov spaces for other values of the parameters.

A special case of the general interpolation for Besov spaces is very important in non-linear approximation. We let L_p, $0<p\le\infty$ be the space where the approximation is to take place. So p is fixed. We then relate the parameters, α,σ, and q by the equations

$$(5.1)\qquad 1/\sigma = \alpha + 1/p$$
$$q = \sigma.$$

Hence, we are talking about a one dimensional scale of spaces $B^\alpha := B_\sigma^\alpha(L_\sigma)$. In [D-P], it is shown that B^α is a selection from an interpolation family. Namely, if $0<\alpha<\beta$, then $(B^\alpha, B^\beta)_{\theta,q} = B^\gamma$ provided $\gamma = \theta\alpha+(1-\theta)\beta$ and $1/q = \gamma + 1/p$.

The spaces B^α appear naturally in non-linear approximation. For example, P. Petrushev [Pt] has recently shown that these spaces satisfy the Jackson and Bernstein inequalities for free knot spline approximation in $L_p(\Omega)$. Namely, if $0<p<\infty$, then

$$\text{(5.3)}\quad \text{i)}\quad s_n(f)_p \le c\, n^{-\alpha} \|f\|_{B^\alpha}$$

$$\text{ii)}\quad \|S\|_{B^\alpha} \le c\, n^\alpha \|S\|_p, \quad \text{for all } S \in S_n.$$

As was observed by Petrushev, it follows from (5.3), that the approximation spaces A_q^α(splines) for this type of approximation are the same as the interpolation spaces $(L_p, B^\beta)_{\alpha/\beta,q}$(see Theorem 3.1). From our extrapolation theorem (Theorem 4.2), we obtain the following characterization for free knot spline approximation

THEOREM 5.1. The approximation spaces A_q^α(splines), $0<\alpha<K$ for free knot approximation by splines of order k are characterized by

$$\text{(5.4)}\qquad A_q^\alpha = (L_p, B^\beta)_{\alpha/\beta,q}, \quad \text{whenever } 0<\alpha<\beta<k \text{ and } q>0.$$

In particular,

$$\text{(5.5)}\qquad A_\sigma^\alpha = B^\alpha = B_\sigma^\alpha(L_\sigma), \quad \text{whenever } 1/\sigma = \alpha + 1/p \text{ and } 0<\alpha<k.$$

The above results do not apply when $p=\infty$, that is for approximation L_∞ or in BMO. In fact, in these cases, the approximation classes cannot be characterized in terms of Besov spaces. Indeed, there is not a Bernstein inequality for such spline approximation. For example, a characteristic function of an interval is in L_∞ and BMO but not in $B_p^{1/p}$ for any $p>0$. Since, as we noted earlier in Lemma 3.2, approximation spaces always satisfy Jackson and Bernstein inequalities, the $B_p^{1/p}$ cannot be approximation spaces.

It is possible however to characterize A_q^α in terms of certain spaces which are closely related to Besov spaces when we assume that the approximating splines are continuous. We let $s_n^*(f) := s_{n,k}^*(f)$ be the error in approximating f in the uniform norm on Ω by continuous splines of degree < k which have n free knots. For this type of approximation, P. Petrushev [Pt_1] has given Jackson and Bernstein inequalities in terms of the spaces B_*^α, $\alpha>1$, which consists of all absolutely continuous functions f for which f' is in $B^{\alpha}_{1/\alpha}(L_{1/\alpha})$. That is,

$$(5.6)\quad \text{i)}\quad s_n^*(f)_C \le c\, n^{-\alpha} \|f\|_{B^{\alpha}_*},$$

$$\text{ii)}\quad \|S\|_{B^{\alpha}_*} \le c\, n^{\alpha} \|S\|_C.$$

The spaces B^{α}_*, $\alpha>1$ are invariant under interpolation because the corresponding Besov spaces $B^{\alpha-1}_{1/\alpha}(L_{1/\alpha})$ are. Hence, it follows from our extrapolation theorem and from (5.6) that for any $1<\beta\le k$ and all $q>0$ and $0<\alpha<\beta$ we have

$$(5.7)\qquad A^{\alpha}_q(\text{continuous splines}) = (C, B^{\beta}_*)_{\alpha/\beta, q}.$$

In addition,

$$(5.8)\qquad A^{\alpha}_{1/\alpha}(\text{continuous splines}) = B^{\alpha}_* \quad \text{for } \alpha>1.$$

The exact same results hold when the approximation is in the space BMO instead of C. That is, the Jackson and Bernstein inequalities (5.6) hold with BMO in place of C. Therefore (5.8) also holds with C replaced by BMO. As a consequence, we have the following interpolation theorem which is of independent interest:

$$(5.9)\qquad (C, B^{\alpha}_*)_{\theta,q} = (BMO, B^{\alpha}_*)_{\theta,q}, \quad \text{for all } 0<\theta<1;\ q>0 \text{ and } \alpha>1.$$

It is also possible to characterize the classes of functions A^{α}_q(rational) when $1<p<\infty$ (V. Peller [P_1] has solved the case BMO). For this, we use connections between the errors of rational and spline approximation. Petrushev [Pt] has shown that for $1\le p<\infty$,

$$(5.10)\quad r_n(f)_p \le c\, n^{-\lambda} \sum_{m=0}^{n} (m+1)^{\lambda-1} s_{m,k}(f)_p,$$

provided $k>\lambda$. This gives an estimate for the error in rational approximation once the spline approximation error is known. In the other direction, Pekarskii [Pe_1] has given the following inequality for $1<p\le\infty$:

$$(5.11)\qquad s_{n,k}(f)_p \le c\, n^{-k} \left(\sum_{m=0}^{n} m^{-1} [m^k r_m(f)]^{\sigma} \right)^{1/\sigma}$$

with $\sigma := (k+1/p)^{-1}$.

From these two inequalities, it is rather easy to obtain the following rather remarkable result.

THEOREM 5.2. <u>For approximation in</u> $L_p(\Omega)$ <u>with</u> $1<p<\infty$, <u>we have for any</u> α, q,

$$A^\alpha_q(\text{rational}) = A^\alpha_q(\text{splines of order } k) \tag{5.12}$$

<u>provided</u> $\alpha<k$. <u>In particular</u>,

$$A^\alpha_\sigma(\text{rational}) = A^\alpha_\sigma(\text{splines of order } k) = B^\alpha_\sigma(L_\sigma), \tag{5.13}$$

<u>when</u> $1/\sigma = \alpha + 1/p$.

Proof. It follows from (5.10) that if $s_{n,k}(f)_p = O(n^{-\alpha})$ then $r_n(f)_p = O(n^{-\alpha})$ and the inequality (5.11) gives the reverse implication. Hence, the classes A^α_∞(rational) = A^α_∞(splines of order k). Since each of these families of spaces A^α_q(rational) and A^α_q(splines of order k) is an interpolation family, (5.12) follows. The inequality (5.13) then follows from this and (5.5).

6. <u>Remarks</u>.

6.1. It still remains an open problem to characterize the spaces A^α_*(splines), if possible, in terms of classical function spaces. Brudnyi [B] has given a characterization of these spaces based on rearrangements of the local degree of polynomial approximation.

6.2. The spaces A^α_q(rational) and A^α_q(splines) are not the same when the approximation takes place in C or BMO since as we have already noted for spline approximation these spaces are not Besov spaces while for rational approximation, Peller has mentioned to us at this conference that he has used his results for the disc [P1] to show that $A^\alpha_{1/\alpha}$(rational in BMO) = $B^\alpha_{1/\alpha}(L_{1/\alpha})$. It has not yet been determined whether or not the approximation spaces for rationals and splines coincide when p=1.

6.3. It follows from Theorem 5.2 and the fact that Jackson and Bernstein inequalities are necessary for approximation spaces (Lemma 3.2) that the following inequalities hold for rational approximation:

$$r_n(f)_p \leq c\, n^{-\alpha}\, \|f\|_{B^\alpha}, \quad n=0,1,\ldots$$

$$||R||_{B^\alpha} \leq c\, n^\alpha\, ||R||_p, \quad R \text{ a rational function of degree } n.$$

The first of these inequalities follows from the work of Petrushev [Pt]. There is at present not a simple direct proof for either of these important inequalities.

6.4. For complex rational approximation on the disc, Pekarskii [Pe] has shown that $A^\alpha_\sigma(\text{rational in } L_p) = B^\alpha_\sigma(L_\sigma)$, provided $1/\sigma = \alpha + 1/p$ and $1<p<\infty$.

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