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CONSTRUCTIVE APPROXIMATION

Free Multivariate Splines

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Abstract. We introduce a definition of free multivariate splines which generalizes the univariate notion of splines with free knots. We then concentrate on the simplest case, piecewise constant functions and characterize some classes of functions which have a prescribed order of approximation in L_p by these splines. These characterizations involve the classical Besov spaces.

1. Introduction

One of the most interesting types of nonlinear approximation is the univariate approximation by splines (piecewise polynomials) of a given order r (degree $r-1$) which have n free knots. The word free means that the knots are allowed to vary with the function being approximated; only the number of knots is held fixed. This type of approximation has been widely studied since the 1970s with much emphasis placed on trying to describe the class A^α of all functions which have the order of approximation $O(n^{-\alpha})$, $n \to \infty$, by such splines. An important breakthrough on this problem has recently been obtained by P. Petruschev [6]. In particular, using his work, it is now possible to give a characterization of these approximation spaces in terms of the classical Besov spaces (see [3]).

There has been no useful formulation of free knot splines in several variables. Apparently, this is at least in part due to the fact that a univariate spline has been viewed as coming from a partitioning of the interval [0, 1] into n subintervals. It is not clear how to carry over such partitioning to the multivariate case in a meaningful way. It turns out that it is more useful to think of a univariate spline with n free knots as a linear combination of n simple functions of a prescribed type. For example, when $r=1$ such a spline is a linear combination of n characteristic functions of intervals. When $r>1$, the characteristic functions can be replaced by truncated power functions or by B-splines. From this viewpoint, the definition of free multivariate splines is more clear; it should be a linear combination of n elementary piecewise polynomials of a given generic type. This is the definition we shall adopt in this paper.

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Let Φ be a collection of piecewise polynomials defined on $\Omega := [0, 1]^d$. We say that S is a free spline with respect to Φ of freedom n if

$$S = \sum_{i=1}^{n} c_i \varphi_i \tag{1.1}$$

with $c_1, \ldots, c_n$ real constants and $\varphi_1, \ldots, \varphi_n \in \Phi$. For example, for splines of order 1, we can take Φ to be a collection of all characteristic functions of sets of some prescribed type, e.g., cubes or rectangles. To get splines of higher degree, we could replace characteristic functions by some type of B-spline, such as the tensor product B-splines, box splines, or simplicial B-splines. Each of these choices leads to a different type of free spline which is interesting in its own right and worthy of study.

In this paper, we wish to prove first results for this type of multivariate free spline approximation. We shall take for Φ the collection of all characteristic functions χ_I of dyadic cubes $I \subset \Omega$. We shall give a characterization of the approximation classes for this type of approximation in terms of the classical Besov spaces in a way that is analogous with the univariate case.

We have chosen to concentrate only on this elementary case. While it is possible to derive theorems for more general classes Φ from the arguments put forward in this paper, we believe that such generality would obscure the essential points of our proofs. A second point is that the study of free multivariate approximation brings out features not seen in the univariate case which are most clearly pointed out in the simple case that we study. For example, in contrast to the univariate case, the smoothness of the spline is important when discussing the approximation classes (see the remarks after Theorem 2.1). Also, it is interesting to see how the dimension d appears in the characterization of the approximation classes (see again the remarks after Theorem 2.1).

Another feature of this paper which deserves some explanation is why we consider dyadic cubes and not just arbitrary cubes in the definition of our free piecewise constant functions. It is clear that the estimates we obtain (Section 3) for the degree of approximation in the dyadic case automatically hold also for the case of arbitrary cubes. However, our inverse theorems of approximation (Section 4) are proved only for dyadic cubes. It is not clear to us whether or not it is possible to characterize the approximation classes for the case of arbitrary cubes by using an approach like the one presented here. It turns out, that in one dimension, these two types of approximation, i.e., whether we use arbitrary intervals or just dyadic intervals in the definition (1.2), have the same approximation classes. Hence, at least as far as the order of approximation $O(n^{-\alpha})$ is concerned, these two types of approximation are equivalent.

We should also mention that there is some relation between our free spline approximation and adaptive approximation by piecewise polynomials. Adaptive approximation is more restrictive than the free spline approximation since the resulting partitions do not permit cubes of arbitrarily small size; for example, if there are to be no more than n cubes in the final partition, then in adaptive approximation, no cube can have measure smaller than 2^{-nd}. Using adaptive approximation, Birman and Solomjak [2] have proven a result which is similar to (3.8). They require somewhat more about the function (it should have more smoothness). It is not possible to prove (3.8) by adaptive approximation because of the restrictions on the cubes mentioned above.

2. Approximation Classes

We shall consider approximation on the unit cube $\Omega := [0, 1]^d$ in d-dimensional Euclidean space. We let D denote the collection of all dyadic subcubes of Ω. We say that S is a free dyadic spline of order 1 and freedom n if there is a collection $\Gamma = \{I_1, \ldots, I_n\}$ from D such that

(2.1) $$S = \sum_{I \in \Gamma} c_I \chi_I,$$

with the c_I real constants. We let Σ_n denote the collection of all such S of freedom n.

We are interested in approximating functions in the metric $L_p(\Omega)$ where $0 < p \leq \infty$. We let

(2.2) $$s_n(f) := s_n(f)_p := \inf_{S \in \Sigma_n} \|f - S\|$$

denote the error in approximating $f \in L_p(\Omega)$ by splines of freedom n. When $n = 0$, we let $s_0(f) := \|f\|_p(\Omega)$. We fix the value of p and therefore usually do not indicate dependence on p. For example, in (2.2) and what follows $\|\cdot\|$ denotes the L_p norm on Ω.

We are interested in characterizing the functions for which $s_n(f) = O(n^{-\alpha})$, $n \to \infty$. It turns out to be useful to discuss a more general approximation class which depends not only on α but on a second more subtle parameter $q > 0$. We let $A_q^\alpha := A_q^\alpha(L_p)$ denote the class of all functions $f \in L_p(\Omega)$ for which

(2.3) $$\sum_{n=1}^{\infty} [n^\alpha s_{n-1}(f)]^q n^{-1} < \infty.$$

Let us mention at the outset that since s_n is monotone decreasing, (2.3) can be replaced by the equivalent and more convenient condition

(2.4) $$\sum_{n=-1}^{\infty} [2^{\alpha n} s_{2^n}(f)]^q < \infty,$$

with $s_{1/2}(f) := s_0(f)$.

When $q = \infty$, the space A_∞^α consists of all functions f for which $s_n(f) = O(n^{-\alpha})$. When $q < \infty$, we get a slightly smaller space, since the condition (2.3) gets stronger as q gets smaller.

In the univariate case, the corresponding spaces A_q^α for free knot spline approximation have been given various characterizations; usually, as interpolation spaces between L_p and a second space Y. For example, when $p = \infty$, such a characterization was given by Bergh and Peetre [1] with the space Y a certain space of functions satisfying a variation condition. The more recent work of P. Petruschev [6] allows one to prove (see [3]) that for any $0 < p < \infty$ (the case

$p=\infty$ is conspicuously absent from these results), A_q^α is an interpolation space between L_p and a Besov space. Namely, we let $B_q^\alpha(L_\sigma)$ denote the space of functions f in $L_\sigma(\Omega)$ for which the modulus of smoothness $\omega_k(f,t)_\sigma$, $k:=[\alpha]+1$ of order k in L_σ satisfies

$$|f|^q_{B_q^\alpha(L_\sigma)} := \int_0^1 [t^{-\alpha}\omega_k(f,t)_\sigma]^q \, dt/t < \infty.$$

Important in what follows is the special case where the parameters α, q, σ are restricted by the equations

$$(2.5) \qquad 1/\sigma = \alpha/d + 1/p; \qquad q = \sigma,$$

where d is the dimension of the Euclidean space. We denote the resulting space $B_\sigma^\alpha(L_\sigma)$ more simply by B^α. Then, when $d=1$, we have (see [6], [3]) for all $0<\beta\le r$,

$$(2.6) \qquad A_q^\alpha \text{ (univariate free splines of order } r) = (L_p, B^\beta)_{\alpha/\beta, q},$$

whenever $0\le\alpha<\beta$ and $q>0$. It is also possible to characterize the spaces A_q^α when $\alpha\ge r$ by using other spaces (similar to the Besov spaces, [6]). More remarkably, it turns out that in certain cases, through (2.6), we can describe the spaces A_q^α in classical terms. For example, from the interpolation of Besov spaces [4], it follows that when σ and α are related as in (2.5), we have [3] for all $0<\alpha<r$,

$$(2.7) \qquad A_\sigma^\alpha \text{ (univariate free knot splines)} = B^\alpha.$$

For the reader not familiar with nonlinear approximation, it might be useful to make a few remarks in explanation of (2.6) and (2.7). First notice that as α varies so does the index σ (see (2.5)) of the space L_σ where the smoothness is measured. This is a common feature of nonlinear approximation and it shows the improvement in nonlinear methods over their linear counterparts. In fact, for α fixed, σ is the smallest value so that $B_\sigma^\alpha(L_\sigma)$ is continuously embedded in L_p. Therefore, as soon as we have such an embedding, we obtain an order of approximation at least $O(n^{-\alpha})$.

In this paper, we want to show that (2.6) and (2.7) hold for free spline approximation of order 1 (as described in Section 1) in Ω. It has long been recognized that at the heart of identifying approximation classes like the A_q^α is the establishment of two inequalities which are similar to the Jackson and Bernstein inequalities for polynomial approximation. In this spirit, we shall prove the following estimates for $0<\alpha<\tau:=\min(1, d/(d-1)p)$:

$$(2.8) \qquad s_n(f)_p \le cn^{-\alpha/d}|f|_{B^\alpha}, \qquad n=1,2,\ldots,$$

$$(2.9) \qquad |S|_{B^\alpha} \le cn^{\alpha/d}\|s\|_p, \qquad S\in\Sigma_n.$$

Here and throughout, c denotes a constant which may depend on p and α but does not depend on f or n.

From (2.8) and (2.9), we can use the general principles in [3] and the interpolation of Besov spaces on Ω as developed in [4] to obtain

Theorem 2.1. *If $0<\beta<\tau$, we have for approximation by free multivariate splines of order* 1 *in $L_p(\Omega)$:*

(2.10) $$A_q^{\alpha/d}=(L_p, B^\beta)_{\alpha/\beta,q}, \qquad 0<\alpha<\beta \quad\text{and}\quad q>0,$$

and if α, p, σ are related as in (2.5), *then*

(2.11) $$A_\sigma^{\alpha/d}=B^\alpha, \qquad 0<\alpha<\tau.$$

This theorem has certain features not seen in the univariate case. For example, the dimension d plays a role in the range of α for which we obtain the characterizations (2.10) and (2.11). The reason for this is that a piecewise polynomial is in each of the spaces B^α, $\alpha>0$, for univariate functions but for multivariate functions, it is only in B^α when $\alpha<d/(d-1)p$. For larger α in the multivariate case, one needs to assume some smoothness (depending on α) for the piecewise polynomial to be in B^α. It appears that smoothness is also important in the univariate case when $p=\infty$ which may in part explain why this case is not covered by (2.7). We could increase the range of allowable α in (2.10) by replacing τ by $d/(d-1)p$, if we would use modified Besov spaces (defined by always using ω_1 rather than ω_k in the definition of B^α) as is done in [6].

The remainder of this paper will be devoted to proving the fundamental inequalities (2.8) and (2.9).

3. The Jackson Estimate

To prove (2.8), we must construct a good approximation to f from Σ_n. For this, we shall use an equivalent norm for B^α which was given in [4]. If I is a dyadic cube, we let $\Pi_k:=\Pi_k(I)$ be the dyadic partition of I into 2^{kd} subcubes of equal size; this partition is obtained by "halving" the cubes in Π_{k-1}. We let P_k denote a piecewise constant function on the partition Π_k which is a best $L_\sigma(I)$ approximation to f from the class $\mathcal{P}_k$ of all such functions, i.e.,

(3.1) $$\|f-P_k\|_\sigma=\inf_{P\in\mathcal{P}_k}\|f-P\|_\sigma.$$

We then have the following two equivalent semi-norms for B^α when $\alpha<1/\sigma$:

(3.2) $$|f|_{B^\alpha}^\sigma \asymp \sum_{k=0}^{\infty} 2^{k\alpha\sigma}\|f-P_k\|_\sigma^\sigma,$$

(3.3) $$|f|_{B^\alpha}^\sigma \asymp \sum_{k=0}^{\infty} 2^{k\alpha\sigma}\|P_{k+1}-P_k\|_\sigma^\sigma = 2^{-\alpha\sigma}\sum_{k=1}^{\infty}\sum_{J\in\Pi_k}|b_J|^\sigma|J|^{\sigma/p},$$

where $b_J:=a_J-a_{J'}$ (a_J the coefficient of χ_J in P_k) and J' is the parent of J, i.e., J' is the cube in Π_{k-1} which contains J and $b_\Omega:=a_\Omega$.

We shall also need the fact that B^α, $\alpha<\tau$, is continuously embedded in L_p modulo constants. That is, if $f\in B^\alpha(I)$ and if $a_I:=a_I(f)$ is the constant of best L_p approximation to f, then

(3.4) $$\|f-a_I\|_p\le C|f|_{B^\alpha(I)}.$$

The univariate case of (3.4) can be found in [5] and the general case in [4].

In what follows, we will frequently have summations over dyadic intervals and therefore it is convenient to introduce the notation $\sum_I$ to denote the sum over all dyadic intervals $J \subseteq I$. Now suppose that g is a function which can be written as

$$g = \sum_{\Omega} \beta_I \chi_I, \tag{3.5}$$

with convergence in the $L_\sigma(\Omega)$ norm. We introduce the notation

$$\lambda(g, I) := \left(\sum_{J \subsetneq I} |\beta_J|^\sigma |J|^{\sigma/p} \right)^{1/\sigma}. \tag{3.6}$$

In view of (3.4), if $\lambda(g, I)$ is finite for some I, then

$$\|g - a_I(g)\|_p \leq c\lambda(g, I) \tag{3.7}$$

for each cube $I \subset \Omega$. Indeed, if Π_k are the dyadic partitions for I and

$$S_k := \sum_{j=0}^{k} \sum_{J \in \Pi_j} \beta_J \chi_J,$$

then clearly $\|g - P_k\|_\sigma \leq \|g - S_k\|_\sigma$, $k = 0, \ldots$. Hence, by (3.2),

$$|g|^\sigma_{B^\alpha(I)} \leq c \sum_{k=0}^{\infty} 2^{\alpha k \sigma} \|g - S_k\|^\sigma_\sigma(I) \leq c \sum_{k=0}^{\infty} 2^{\alpha k \sigma} \|S_{k+1} - S_k\|^\sigma_\sigma = c\lambda(g, I).$$

Here, the right-most inequality is proved by writing $g - S_k$ as a telescoping sum of $S_{j+1} - S_j$, $j \geq k$ (we omit the details). Now, (3.7) follows from our last inequality and (3.4).

Theorem 3.1. *Let $0 < p < \infty$ and α, σ, and p be related as in (2.5). If $f \in B^\alpha$ and $0 < \alpha < \tau$, then there is a spline $S \in \Sigma_n$ such that*

$$\|f - S\|_p \leq cn^{-\alpha/d} |f|_{B^\alpha}. \tag{3.8}$$

Proof. We first write $f = \sum_{k=0}^{\infty} (P_{k+1} - P_k) = \sum_{k=0}^{\infty} \sum_{I \in \Pi_k} b_I \chi_I$ with P_k as in (3.1) and b_I as in (3.3). Since $\lambda(f, \Omega)$ is equivalent to $|f|_{B^\alpha}$, we can assume without loss of generality that $\lambda(f, \Omega)^\sigma = 2^{-d}$. We fix f and let $\varepsilon := n^{-1/\sigma}$. We shall define inductively a sequence of functions f_j and a sequence of dyadic cubes I_j, $j = 0, 1, \ldots$, as follows.

For $j = 0$, we let $I_0 := \varnothing$ and $f_0 := f$ and $\beta^0_I := b_I$. Suppose now that f_j has already been defined and f_j can be represented as in (3.5) with $\beta_I = \beta^j_I$. We choose a minimal cube I_{j+1} for which $\lambda(f_j, I_{j+1}) < \varepsilon$ and $\lambda(f_j, I'_{j+1}) > \varepsilon$ for the parent I'_{j+1} of I_{j+1}. If there is no such maximal cube, we let $I_{j+1} := \Omega$.

Once I_{j+1} has been defined, we let

$$\beta^{j+1}_I := \begin{cases} 0, & I \subseteq I_{j+1}, \\ \beta^j_I, & \text{else}, \end{cases} \tag{3.9}$$

and

$$f_{j+1} := \sum_{\Omega} \beta^{j+1}_I \chi_I. \tag{3.10}$$

We terminate this process if $I_{j+1} = \Omega$, otherwise we continue in the above manner.

We first show that this process terminates. For this, we suppose that we have generated cubes $I_0, \ldots, I_m$ with $I_m \neq \Omega$. Now for each j, we have from the definition of I_j and its parent I'_j that

$$\sum_{I'_j} |\beta^j_I|^\sigma |I|^{\sigma/p} = \lambda(f_j, I'_j)^\sigma > \varepsilon^\sigma, \qquad j = 1, \ldots, m. \tag{3.11}$$

Each coefficient β^j_I is either 0 or b_I and for any given I, there are at most 2^d values of j for which $I \subseteq I'_j$ and $\beta^j_I \neq 0$ (one for each time I or a brother of I is an I_j). Hence, if we add together the inequalities in (3.11), we obtain

$$1 = 2\lambda(f, \Omega)^\sigma > m\varepsilon^\sigma; \tag{3.12}$$

that is, $m \leq n$. This shows that the above process terminates and it generates the cubes $I_0, \ldots, I_{m+1} = \Omega$ with $m \leq n+1$.

We note for further use a couple of the important properties of the cubes I_j and the functions f_j. The cubes $I_0, \ldots, I_{m+1}$ are partially ordered by inclusion, i.e., if $j < k$ then either $I_j \subset I_k$ or I_j and I_k are disjoint. We let $A_j := I_j \setminus \bigcup_{k<j} I_k$, $j = 1, \ldots, m$. Then the sets A_j are pairwise disjoint and their union is Ω.

Since $\beta^j_I = b_I$ whenever I is not contained in $\bigcup_{k<j} I_k$, we have $f_j \equiv f$ on A_{j+1}. Now, let $a_j := a_{I_{j+1}}(f_j)$ be the constant of best L_p approximation to f_j on I_{j+1}. We then have from (3.7) and the definition of I_{j+1} that

$$\begin{aligned} \|f - a_j\|_p(A_{j+1}) = \|f_j - a_j\|_p(A_{j+1}) &\leq \|f_j - a_j\|_p(I_{j+1}) \\ &\leq c\lambda(f_j, I_{j+1}) \leq c\varepsilon, \end{aligned} \tag{3.13}$$

Hence if S is the piecewise constant function which is defined to be a_j on A_{j+1}, $j = 0, \ldots, m$, then we have

$$\|f - S\|_p^p(\Omega) = \sum_{j=0}^{m} \|f - a_j\|_p^p(A_{j+1}) \leq c(m+1)\varepsilon^p \leq cn\varepsilon^p = cn^{-\alpha p/d}. \tag{3.14}$$

Therefore, S provides the desired estimate (3.7). Finally, we show that $S \in \Sigma_{n+2}$. We let $\tau(j)$ denote the smallest integer larger than j such that I_{j+1} is contained in $I_{\tau(j)+1}$. We claim that

$$S = a_m \chi_\Omega + \sum_{j=0}^{m-1} (a_j - a_{\tau(j)}) \chi_{I_{j+1}}, \tag{3.15}$$

Indeed, if T denotes the right side of (3.15), then for $x \in A_{j+1}$, we have

$$T(x) = (a_j - a_{\tau(j)}) + (a_{\tau(j)} - a_{\tau(\tau(j))}) + \cdots + a_m = a_j = S(x),$$

as desired. ∎

4. The Bernstein Inequality

We now want to prove (2.9). For this, we shall need another representation for S. Let $\Gamma := \{I_1, \ldots, I_n\}$ be a finite collection of dyadic cubes. If I is one of these cubes, we let B_I denote the collection of all cubes $J \in \Gamma$ such that $J \subset I$, $J \neq I$, and J is maximal (that is, J is not contained in another cube with these properties). It may happen that some of the sets B_I have large cardinality, however, as we now show, we can imbed Γ in a larger family $\Gamma' = \{A_1, \ldots, A_m\}$ where $|B_A| \leq 2^d$ (with respect to Γ') for all $A \in \Gamma'$.

Lemma 4.1. *Let $\Gamma=\{I_1,\ldots,I_n\}$ be an arbitrary collection of dyadic cubes, with $\Omega\in\Gamma$. Then, there is a second collection $\Gamma'=\{A_1,\ldots,A_m\}$ of dyadic cubes with the following properties:*

$$\text{(4.1)}\qquad \begin{aligned} &\text{(i)}\quad \Gamma\subset\Gamma',\\ &\text{(ii)}\quad |B_A|\le 2^d \qquad \textit{for all}\quad A\in\Gamma'.\\ &\text{(iii)}\quad m\le 2^d n. \end{aligned}$$

Proof. We say a dyadic cube $J\neq\varphi$ is minimal if no proper dyadic subcube of J contains exactly the same cubes from Γ as does J. We let Γ' denote the collection of all minimal cubes. Since each $I\in\Gamma$ is clearly minimal, we have (i). If $A\in\Gamma'$ and $A_1,\ldots,A_s$, $s:=2^d$ are the children of A, then any $J\in B_A$ is contained in one of the A_i, $i=1,\ldots,s$. We also claim that a given A_i can contain at most one $J\in B_A$. Indeed, if $J,J'\in B_A$ are both contained in A_i, then there is a smallest dyadic cube $J''\subset A$ which contains J and J'. Clearly, $J''\in\Gamma'$. But then this contradicts the maximality of J required for membership in B_A.

To prove (iii), for $I\in\Gamma'$, we let r_I be the number of $J\in\Gamma$ with $J\subseteq I$ and let r'_I be the number of $J'\in\Gamma'$ with $J'\subseteq I$. We claim

$$\text{(4.2)}\qquad r'_I\le 2^d r_I-1 \qquad \text{for all}\quad I\in\Gamma'.$$

We establish (4.2) by induction on the size of I. If I is a smallest cube in Γ' then $I\in\Gamma$ and clearly $r_I=r'_I=1$. Suppose now that $I\in\Gamma'$ and we have shown (4.2) for all cubes $J\in\Gamma'$ with $J\subset I$. If we consider only maximal J, we have

$$\text{(4.3)}\qquad \begin{aligned} r'_1&=\sum_{\substack{J\subset I\\ J\text{ maximal}}} r'_J+1\le \sum_{\substack{J\subset 1\\ J\text{ maximal}}}(2^d r_J-1)+1\\ &\le 2^d r_I-(\#\text{ of maximal }J)+1. \end{aligned}$$

Since there must be at least two maximal J contained in I (otherwise we contradict the criteria for membership of I in Γ'), we have (4.2). If we now apply (4.2) to Ω, we have (iii). ■

Now let $S\in\Sigma_n$ and let $\Gamma=\{I_1,\ldots,I_n\}$ be the cubes in the representation (2.1). If Ω is not in Γ, then we adjoin it to Γ and we let Γ' be a set of dyadic cubes given by Lemma 4.1. Clearly, with respect to this larger set Γ', we can also represent S as in (2.1) with constants c'_I. If $I\in\Gamma'$, we let $I':=I\setminus\bigcup\{J: J\in B_I\}$ and $\varphi_I:=\chi_{I'}$. The functions φ_I have disjoint supports and we have

$$\text{(4.4)}\qquad S=\sum_{I\in\Gamma'}\gamma_I\varphi_I,\qquad \gamma_I:=\sum_{\substack{J\supseteq I\\ J\in\Gamma'}} c'_J.$$

We now want to estimate the norm of φ_I in B^α. We first note that

$$\text{(4.5)}\qquad |\varphi_I(x+h)-\varphi_I(x)|=\begin{cases}1 & \text{if } x\in I'(h),\\ 0 & \text{else,}\end{cases}$$

where for any set T, we let $T(h):=\{x:\text{ either } x\in T,\ x+h\notin T \text{ or } x\notin T,\ x+h\in T\}$.

Lemma 4.2. *If φ_I is not identically zero and if l_I is the side-length of I, then for any h, we have*

$$|I'(h)| \le C l_I^{d-1} \min(|h|, l_I) \tag{4.6}$$

with C depending only on d.

Proof. We have $|I'(h)| \le 2|I'| \le 2|I|$ and so (4.6) is obvious when $|h| \ge l_I$. When $|h| \le l_I$, we note that $x \in I'(h)$ implies that $\operatorname{dist}(x, \operatorname{bdr}(I')) \le |h|$. Now, the boundary of I' is a union of faces F of dyadic cubes with side-length smaller than or equal to l_I. Hence, the set of points whose distance to F does not exceed $|h|$ has measure $|F||h| \le C l_I^{d-1}|h|$. Since there are at most 2^d cubes in B_I, there are at most $(2^d+1)2^d$ faces and we obtain (4.6). ∎

It follows from (4.6) that

$$\omega(\varphi_I, t)_\sigma^\sigma \le C l_I^{d-1} \min(t, l_I). \tag{4.7}$$

Therefore, if $0 < \alpha < 1/\sigma$ (i.e., $\alpha < d/(d-1)p$),

$$|\varphi_I|_{B^\alpha}^\sigma \le C l_I^{d-1} \int_0^\infty t^{-\alpha\sigma-1} \min(t, l_I)\, dt \le C l_I^{-\alpha\sigma+1} = C|I|^{\sigma/p}. \tag{4.8}$$

Theorem 4.3. *Each spline S of the type* (2.1) *satisfies inequality* (2.9) *provided $0 < \alpha < \tau$.*

Proof. We can write

$$S(x+h) - S(x) = \sum_{I \in \Gamma'} \gamma_I [\varphi_I(x+h) - \varphi_I(x)]. \tag{4.9}$$

Since the φ_I have disjoint supports at most two terms in (4.9) are nonzero for each x and h. Hence,

$$\int_\Omega |S(x+h) - S(x)|^\sigma\, dx \le 2^\sigma \sum_{I \in \Gamma'} |\gamma_I|^\sigma \int_\Omega |\varphi_I(x+h) - \varphi_I(x)|^\sigma\, dx.$$

Hence,

$$\omega(S, t)^\sigma \le 2^\sigma \sum_{I \in \Gamma'} |\gamma_I|^\sigma \omega(\varphi_I, t)^\sigma.$$

Multiplying this by $t^{-\alpha\sigma-1}$ and integrating with respect to t, gives

$$|S|_{B^\alpha}^\sigma \le \sum_{I \in \Gamma'} |\gamma_I|^\sigma |\varphi_I|_{B^\alpha}^\sigma \le C \sum_{I \in \Gamma'} |\gamma_I|^\sigma |I|^{\sigma/p}, \tag{4.10}$$

where the last inequality uses (4.8). We now apply Holder's inequality in (4.10) to find

$$|S|_{B^\alpha} \le C n^{1/\sigma - 1/p} \left(\sum_{I \in \Gamma'} |\gamma_I|^p |I| \right)^{1/p} \le C n^{\alpha/d} \|S\|_p,$$

where the last inequality uses (2.5) and the fact that the support of φ_I has measure between $4^{-d}|I|$ and $|I|$. Indeed, since each child of I contains at most one cube from B_I, and not all children are in B_I, at least one of the grandchildren of I is contained in I'. ■

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