A proof of Borsuk's theorem

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1. Introduction: Different forms of the theorem

The remarkable *antipodality theorem of Borsuk* [3] has extensive applications in Analysis. It can be easily proved using advanced topological means (homotopy, cohomology), or using the notion of the degree of a mapping (see Dugundji [6], Amann [1]). We shall give here a direct, elementary proof, which uses combinatorial properties of triangulations of $\mathbb{R}^n$, and is based upon ideas of Tucker [8]. The main idea of the proof appears in §4; in §5 we mention some examples of application of Borsuk's theorem. Let $\Sigma_n := \{x = (x_1, \ldots, x_n) : \|x\| = 1\}$ be the n-sphere in the Euclidean space $\mathbb{R}^n$ and let P be a mapping of Σ_n into an $(n-1)$-dimensional Banach space X_{n-1}. (In other words, P is an $n-1$-dimensional vector field on Σ_n.) The *mapping P is odd* if $P(-x) = -P(x)$ *for all* $x \in \Sigma_n$.

Theorem 1. (Borsuk's theorem) *An odd continuous mapping P of Σ_n into X_{n-1} must vanish: $P(x) = 0$, for some $x \in \Sigma_n$.*

Since all spaces X_{n-1} are isomorphic, we can substitute for X_{n-1} one of them, $\mathbb{R}^{n-1}$.

We consider mappings R of the n-dimensional cube $Q_0 := [0,1]^n$ into its boundary B_n. If $x \in B_n$, its *antipodal point* x^*, which also belongs to B_n, is symmetric to x with respect to the center of Q_0. It other words, it is defined by $x^* := e - x$, where $e := (1, 1, \ldots, 1)$. More generally, if $A \subset B_n$, we define $A^* := \{x^* : x \in A\}$. A *mapping R of Q_0 into B_n is antipodal,* if

$$P(x^*) = P(x)^* \text{ for all } x \in B_n \,. \tag{1}$$

Theorem 2. *For $n = 1, 2, \ldots,$ there does not exist a continuous antipodal mapping R of Q_0 into its boundary B_n.*

We shall prove this theorem in §4. Here we show that Borsuk's theorem 1 follows easily from this.

Indeed, Theorem 2 remains true for any n-dimensional cube, for example for the cube $[-1,1]^n$. It is also true for antipodal mappings of the n-dimensional ball $U_n := \{x = (x_1, \ldots, x_n) : \|x\| \le 1\}$ in $\mathbb{R}^n$ into its boundary Σ_n. To see this, let S be the mapping which assigns to $x \in U_n$, $\|x\| = r$ its projection $S(x)$ onto the boundary of the cube $[-r,r]^n$ by the ray emanating from the origin and passing through x. Both S and S^{-1} are continuous and preserve antipodality. If there would exist a continuous antipodal mapping R of U_n into Σ_n, then SRS^{-1} would be a continuous antipodal mapping of $[-1,1]^n$ into its boundary, a contradiction.

We derive Borsuk's theorem from Theorem 2. If the former were not true, there would exist a continuous odd mapping P of Σ_{n+1}, $n \geq 1$ into $\mathbb{R}^n$, which does not vanish on Σ_{n+1}. Now if $y = (y_1, \ldots, y_n) \in U_n$, the point $z := (y_1, \ldots, y_n, \sqrt{1 - \|y\|^2})$ is on Σ_{n+1}. We define a mapping R of U_n into Σ_n by putting

$$R(y) = \frac{P(z)}{\|P(z)\|}, \quad y \in U_n . \tag{2}$$

This R is well defined and continuous. If $y \in \Sigma_n$, then $z = (y_1, \ldots, y_n, 0)$. Since P is odd, R is antipodal. This would contradict Theorem 2. Thus, P must vanish.

2. Properties of the "equators" B_k

We return to the cube Q_0. Its *k-dimensional facets* $k = 0, 1, \ldots, n-1$ are defined as intersections of Q_0 with some $n-k$ hyperplanes $x_{i_j} = c_j$, $j = 1, \ldots, n-k$, where $c_j = 0$ or 1, and $I := I_{n-k} := \{1 \leq i_1 < \cdots < i_{n-k} \leq n\} \in \mathcal{I}_{n-k}$ is a fixed set and $\mathcal{I}_k$, $k = 0, \ldots, n$ stands for the set of all subsets of $\{1, 2, \ldots, n\}$ of cardinality k. Facets of dimension $n-1$ are faces of Q_0, those of dimensions 1 and 0 are edges and vertices, respectively.

We shall need *special k-dimensional facets* $F_{I,k} := \{x \in Q_0 : x_{i_1} = 0, x_{i_2} = 1, x_{i_3} = 0, \ldots\}$, $I \in \mathcal{I}_{n-k}$ with alternating values of $x_{i_1}, x_{i_2}, \ldots,$ and also their antipodal sets $F^*_{I,k} = \{x \in Q_0 : x_{i_1} = 1, x_{i_2} = 0, \ldots\}$.

With the help of the $F_{I,k}$ and $F^*_{I,k}$ we construct something like a k-dimensional equator of Q_0 which separates its south pole $(0, \ldots, 0)$ from the north pole $(1, \ldots, 1)$. For $k = 0, \ldots, n-1$ let

$$H_k := \bigcup_{I \in \mathcal{I}_{n-k}} F_{I,k}, \quad H^*_k := \bigcup_{I \in \mathcal{I}_{n-k}} F^*_{I,k}, \quad B_k := H_k \cap H^*_k . \tag{3}$$

For example, H_0 consists of the single point $\{0,1,0,1,\ldots\}$, and H^*_0 of the point $\{1,0,1,0,\ldots\}$. We let $H_n := H^*_n := B_n$ be the boundary of Q_0. We call two k-dimensional polyhedrons *disjoint* if their interiors are disjoint. Thus, the facets F, F^* in (3) are all disjoint. Hence B_k is

$k-1$-dimensional.

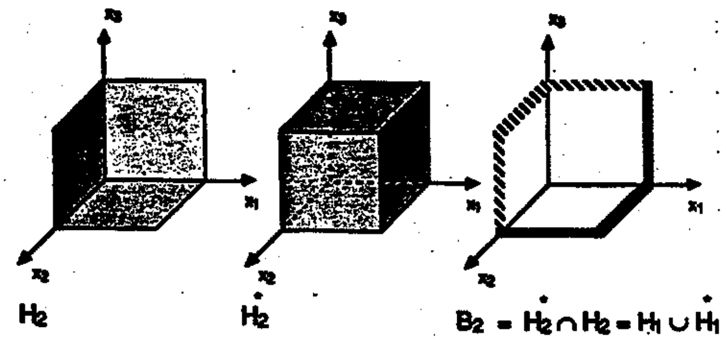

Figure 1. The sets H_2, H_2^*, and B_2 in $\mathbb{R}^3$.

Lemma 3. *We have (i)*

$$B_k = H_{k-1} \cup H_{k-1}^*\,, \quad k = 1, \ldots, n-1\,; \tag{4}$$

(ii) If F is a facet of dimension $k-1$ contained in H_k, $k \le n-1$, then either (a) F is a face of exactly two facets $\widetilde{F}$ of H_k and F and B_k are disjoint, or (b) F is a face of exactly one such $\widetilde{F}$, and $F \subset B_k$.

Proof. (i) For $x \in Q_0$, we examine x_j as a function of $j = 1, \ldots, n$. Thus, $x \in H_k$ (or $x \in H_k^*$), $k = 0, \ldots, n-1$ means that either x_j contains an alternating sequence of 0's and 1's of length $> n-k$, or that it has such a sequence of length $n-k$ beginning with 0 (or, correspondingly, with 1). Therefore $x \in H_k \cup H_k^*$ if and only if x has an alternating sequence of length $\ge n-k$. And $x \in H_k \cap H_k^* = B_k$ means that x has an alternating sequence of length $\ge n-k+1$. Thus we have (4).

(ii) Here, F is a face of some $\widetilde{F} := F_{I,k}$ from H_k, with $I \in \mathcal{I}_{n-k}$. The set $\widetilde{F}$ is described by the equations $x_{i_j} = c_j$, $j = 1, \ldots, n-k$ where the c_j are alternatively 0's and 1's, the remaining x_j being arbitrary. The face F is obtained by adding an equation $x_i = c$, $c = 0$, or $c = 1$, $i \notin I$. Let $c = 0$. If $i < i_{n-k}$, or if $i > i_{n-k}$ and $c_{n-k} = 0$, the new sequence of c's, of length $n-k+1$, would have two adjacent zeros. Thus F would be a face of exactly two facets from H_k, obtained by omitting one of the zeros, and would not belong to any of the $F_{I,k}^*$. But if the last $c_{n-k} = 1$ and $i > i_{n-k}$, then F would be the face of the original $F_{I,k}$, and of a facet from H_k^*, obtained by omitting $c_{i_1} = 0$, and would not belong to other facets. Moreover, $F \subset B_k$. The case $c = 1$ is similar. ∎

Another formulation of (ii) is that B_k is the common boundary of H_k and of H_k^* in B_{k+1}.

3. Partition and triangulation of the cube Q_0

For the proof of Theorem 2 we need a decomposition of the B_k into simplices, not cubes, and they must be sufficiently small. The last aim is achieved by selecting a large integer N and by decomposing Q_0 into N^n small cubes Q of side length $h = N^{-1}$. Each of the small cubes Q consists of points $y_Q + hx$, where y_Q is the smallest element in Q, and x is an arbitrary point of Q_0.

A k-dimensional facet $\widetilde{R}$ of Q is obtained by setting $n-k$ of the coordinates of x to 0 or 1. A face R of $\widetilde{R}$ has an additional coordinate set to 0 or 1. For example the $\widetilde{R}$ contained in $\widetilde{F} := F_{I,k}$, $I : i_1, \dots, i_{n-k}$, are obtained by setting $x_{i_1} = 0$, $x_{i_2} = 1, \dots$. Hence,

(6) if R is interior to $\widetilde{F}$, then R appears as a face of exactly two $\widetilde{R}$.

For example, if R is obtained from $\widetilde{R}$ by setting $x_j = 0$, then R is also a face of the k-dimensional facet $\widetilde{R}'$ of Q', $y_{Q'} = y_Q - he_j$, obtained by setting $x_j = 1$.

We further partition each Q into simplices. We use the following procedure (known as the Kuhn triangulation). By e_i, $i = 1, \dots, n$, we denote the i-th unit vector of $\mathbb{R}^n$, with i-th coordinate equal to one and all other coordinates equal to zero. Let $\sigma := (\sigma(1), \dots, \sigma(n))$ be any permutation of the integers $1, \dots, n$. For each σ, the simplex $T_\sigma(Q)$ has the vertices $y_Q + hx$, where y_Q is the smallest vector in Q, and x is one of the vectors

$$0,\ e_{\sigma(1)},\ e_{\sigma(1)} + e_{\sigma(2)},\ \dots\ ,\ e_{\sigma(1)} + \dots + e_{\sigma(n)} = (1, \dots, 1)\,.$$

Equivalently, $T_\sigma(Q)$ consists of all points $y = y_Q + hx$ with $x = (x_1, \dots, x_n)$ satisfying

$$0 \le x_{\sigma(n)} \le \dots \le x_{\sigma(1)} \le 1\,; \tag{7}$$

and y will be in the interior of T_σ if and only if all these inequalities are strict. Since for any x with distinct coordinates x_i there is a unique decreasing rearrangement of the x_i, the T_σ will have disjoint interiors, thus $Q = \bigcup_\sigma T_\sigma(Q)$ is a disjoint decomposition of Q.

The faces T of $T_\sigma(Q)$ are obtained by changing one of the inequalities of (7) to equality. Changing an outer inequality gives a face contained in the boundary (in one of the faces) of Q. Such a T is contained in only one $T_\sigma(Q)$. Changing an interior inequality $x_{\sigma(i)} < x_{\sigma(i+1)}$ to equality produces an interior face T which is shared by one other simplex $T_{\sigma'}(Q)$; one gets σ' by interchanging $\sigma(i)$ and $\sigma(i+1)$.

The Kuhn triangulation of a Q induces also a Kuhn triangulation of the facets F of Q of dimensions $1 \le k \le n-1$. It is given by those facets of the $T_\sigma(Q)$ of dimension k which are contained in F. For example, let F be the face of $Q : y = y_Q + hx$ with $x_i = 0$. Then a T_σ has a face contained in F precisely when $x_{\sigma(n)} = 0$ and $\sigma(n) = i$ in (6). Since $y_F = y_Q$, this face of T_σ is given by the relations

$$0 = x_{\sigma(n)} \le x_{\sigma(n-1)} \le \cdots \le x_{\sigma(1)} \le 1 . \tag{8}$$

The faces of the T_σ with this property produce the Kuhn triangulation of F. The same is true if F is given by $x_i = 1$, but then $y_F = y_Q + e_i$. This proof also applies to facets of Q of lower dimensions:

We let $\mathcal{T}$ denote the set of all facets T of all $T_\sigma(Q)$, for all small cubes $Q \subset Q_0$.

Lemma 4. *Let $T \in \mathcal{T}$ be a $k-1$-dimensional simplex in $\mathcal{T}$ such that $T \subset H_k$, $1 \le k \le n-1$. Then either (A) T is a face of exactly two k-dimensional simplices $\widetilde{T} \subset H_k$, and T is disjoint with B_k, or (B) T is a face of exactly one such $\widetilde{T}$, and $T \subset B_k$.*

Proof. Let $T \subset \widetilde{F} := F_{I,k}$, then T is contained in some k-dimensional cube $\widetilde{R}$ produced by the h partition of Q_0, and is a face of some $\widetilde{T} \in \mathcal{T}$ with $\widetilde{T} \subset \widetilde{R}$. It can happen that T is interior to $\widetilde{R}$, (Figure 2(i)) then by what was said above, T is the face of exactly two $\widetilde{T} \in \mathcal{T}$, $\widetilde{T} \subset H_k$ and we have A). We can also have T is contained in a face R of $\widetilde{R}$ which is interior to $\widetilde{F}$ (Figure 2(ii)). Then by (6), exactly two $\widetilde{R}$ contain R as a face and hence T is a face of exactly two $\widetilde{T} \in \mathcal{T}$, $\widetilde{T} \subset H_k$ and again we have A).

It remains to consider the case when T is contained in a face R of $\widetilde{R}$ and $R \subset F$, where F is a face of $\widetilde{F}$. In case (a) of Lemma 3, T is contained in two different facets of H_k (Figure 2(iii)) and by (a) of the lemma, we again have (A).

However, in case (b) of Lemma 3, T is contained in exactly one $\widetilde{T} \subset \widetilde{F}$ and $T \subset \widetilde{F} \subset B_k$ (Figure 2(iv)). . This yields (B). ∎

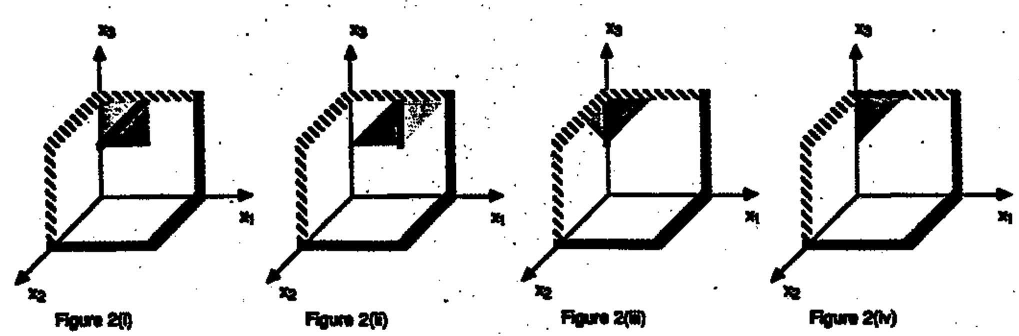

Figure 2.

4. Proof of Theorem 2

We assume that R of Theorem 2 exists and derive a contradiction. For $x \in Q_0$ and $y = R(x)$, let i be the smallest integer such that $y_i = 0$ or $y_i = 1$. We assign to x the "color" i of $y_i = 0$, the "color" $-i$ if $y_i = 1$. In this way, each $x \in Q_0$ is assigned one of the colors $\pm 1, \dots, \pm n$. Antipodal elements of Q_0 are assigned opposite colors.

We assume that h is so small that no two points of a simplex $T \in \mathcal{T}$ are mapped onto opposite faces of Q_0. This guarantees that no two vertices of T get opposite colors. As a consequence, the colors of the vertices of any k-dimensional $T \in \mathcal{T}$ can be uniquely written as a sequence $c_0, \dots, c_k$ with

$$1 \le |c_0| \le \cdots \le |c_k| \le n ;$$

there is strict inequality $|c_i| < |c_{i+1}|$, if c_i, c_{i+1} are of opposite signs. With this ordering, we call $C = (c_0, \dots, c_k)$ *the color of* T. We also need *special colors* $\widehat{C}$ of order k (and their set C_k) whose components satisfy

$$1 \le |c_0| < \cdots < |c_k| \le n \quad \text{and} \quad \text{sign } c_i = (-1)^i .$$

Then also $-\widehat{C} = (-c_0, \dots, -c_k)$ is a color with alternating signs. For example, if $k = 0$, T is single point of some color c_0, then $C = (c_0)$, and either C or $-C$ is special.

By $N_k(C)$ we denote the number of k-dimensional $T \in \mathcal{T}$, contained in H_k, which have color C.

Let $A_k(C)$ be the number of incidences of k dimensional $T \in \mathcal{T}$ of color C as a face of $k+1$ dimensional $\widetilde{T} \in \mathcal{T}$ with $\widetilde{T} \subset H_{k+1}$. From Lemma 4, if $T \subset B_{k+1}$ then T appears exactly once as a face of a $\widetilde{T}$. Otherwise T appears in two $\widetilde{T}$. Thus to compute $A_k(\widehat{C})$, modulo 2, we have to count the number of T of color $\widehat{C}$ contained in $B_{k+1} = H_k \cup H_k^*$. Either $T \subset H_k$ and has color $\widehat{C}$, or $T \subset H_k^*$ and then $T^* \subset H_k$ has color $-\widehat{C}$. Thus $A_k(\widehat{C}) \equiv N_k(\widehat{C}) + N_k(-\widehat{C})$ mod 2. We therefore obtain, summing over all possible $\widehat{C} \in C_k$, for $k = 0, \dots, n-1$,

$$\sigma_k := \sum_{\widehat{C} \in C_k} A_k(\widehat{C}) \equiv \sum [N_k(\widehat{C}) + N_k(-\widehat{C})] \mod 2 \tag{9}$$

For example, $\sigma_n = 0$, for there are no special colors of order n. Also σ_0 can be easily computed. Since H_0 is a single point with a definite color, the sum on the right in (9) reduces to a single term 1, so that $\sigma_0 \equiv 1 \mod 2$.

We can count σ_k a different way. Namely, if $\widetilde{T} \subset H_{k+1}$ is a simplex of dimension $k+1$ which contributes to σ_k, and $\widetilde{T}$ has color $D = (d_0, \dots, d_{k+1})$, then the color of T is a subsequence of D, so that D has either k or $k+1$ changes of sign. In the first case, for some i, $|d_0| <$

$\cdots < |d_i| \le |d_{i+1}| < \cdots < |d_{k+1}|$ and sign $d_i =$ sign d_{i+1}. Then exactly two faces of $\widetilde{T}$ will contribute to σ_k. On the other hand, if D has $k+1$ changes of sign, then *exactly one* face of $\widetilde{T}$ has a color from C_k. This happens when $D \in C_{k+1}$ or $-D \in C_{k+1}$. Hence for $0 \le k < n$,

$$\sigma_k \equiv \sum_{\widehat{C} \in C_{k+1}} \left[N_{k+1}(\widehat{C}) + N_{k+1}(-\widehat{C})\right] \mod 2.$$

It follows that $\sigma_k \equiv \sigma_{k+1}$, $k = 0, \ldots, n-1$. However, this is a contradiction, since $\sigma_n = 0$ and $\sigma_0 \equiv 1$. ∎

5. Some applications

We give a selection of results that can be proved using Borsuk's theorem.

1. Kolmogorov widths.

Let B be a subset of a linear normed space X. The Kolmogorov width $d_n(B, X)$ of B in X is defined as the infimum of distances of B from all possible n-dimensional subspaces X_n of X. Borsuk's theorem has been used here in many different ways by Tihomirov, Makovoz, Pinkus and others. The most famous case is perhaps the formula of Gohberg and Krein

$$d_n(B_{n+1}, X) = 1 ,$$

where B_{n+1} is the unit ball of a subspace X_{n+1} of X. This is trivial if $X = X_{n+1}$, but far from it in the general case (see [7]).

2. Spline interpolation.

The ordinary polynomial Lagrange interpolation operator L_n at the interpolation points $Y_n : y_0^{(n)} < \cdots < y_n^{(n)}$ in $[a, b]$ in the uniform norm satisfies $\|L_n\| \ge \text{Const} \log n$ for all selections of Y_n. In contrast, for the spline interpolation operator S_n by splines from a Schoenberg spline space $S_r(T)$, with knots T and smoothness r, one has $\|S_n\| \le$ Const for properly selected interpolation points Y_n and all r and T (Demko [5]). A very natural proof of this uses Borsuk's theorem.

3. Gaussian quadrature formulas.

The existence of these formulas can often be proved (see Bojanov, Braess, Dyn [2]) by means of Borsuk's theorem.

4. Nonuniqueness of rational approximation.

Let $\mathcal{R}_{m,n}(I)$, $I = [a, b]$, be the class of all rational functions $R = P/Q$, where P and Q are polynomials of degrees $\le m$ and $\le n$, respectively, and $Q(x) \ne 0$ on I. According to Walsh, a continuous function has exactly one best approximant from $\mathcal{R}_{m,n}(I)$ in the uniform norm.

The situation is different in the spaces L_p. Here, Braess [4] uses the antipodality theorem to obtain functions $f \in L_p(I)$ with several best approximants.

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