

INTERPOLATION OF APPROXIMATION SPACES

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Let $X_i, i=0,1, X_1, X_0$, be quasi-normed Abelian groups. Let $\{G_n\}_{n=0}^{\infty}$ be a normal approximating family in $X_i, i=0,1$. Let $A_q^\alpha(X_i)$ be the approximation spaces in X_i with respect to the family $\{G_n\}_{n=0}^{\infty}$. The paper considers the problem of interpolation of the approximation spaces $A_{q_0}^\alpha(X_0), A_{q_1}^\alpha(X_1)$ by the real method of interpolation.

Introduction. We shall consider approximations in quasi-normed Abelian groups (Q-spaces). This kind of spaces are very convenient for some nonlinear approximations, for example rational and spline free-knots approximations, as well as approximations in spaces like $L_p, 0 < p < 1$, see [1],[2],[3]. Let X be a Q-space. The quasi-norm (Q-norm) in $X : \| \cdot \|_X$, is characterized by the constant k in the triangle inequality $\|f+g\|_X \leq k(\|f\|_X + \|g\|_X)$, or, equivalently, by the power σ in the inequality $\|f+g\|_X^\sigma \leq \|f\|_X^\sigma + \|g\|_X^\sigma$ (see [2]). Some constants $c(\cdot, \dots, \cdot)$, which shall appear later on, shall depend on k or σ . We shall denote this dependence simply by $c(\cdot, \dots, X)$. If two Q-norms $\| \cdot \|$ and $\| \cdot \|$ in X are equivalent, we shall denote this by $\| \cdot \| \sim \| \cdot \|$.

Let $\{G_n\}_{n=0}^{\infty}$ be a normal approximating family in X (see [3],[4]). This means that i) $G_0 \subset G_n \subset G_{n+1}$; ii) $G_n + G_m \subset G_{n+m}$. Let

$$E_n(f) := E_n(f)_X := E_{G_n}(f)_X := \inf \{ \|f-g\|_X : g \in G_n \}$$

denote the best approximation of f by elements of G_n .

Let us remember, that the sequence spaces $l_q^\alpha(X)$ (see [1],[2],[5]) are defined in the following way:

$$l_q^\alpha(X) = \{ a : a = (a_n)_{n=0}^{\infty}, a_n \in X, \|a\|_{l_q^\alpha(X)} = \left(\sum_{n=0}^{\infty} (2^{n\alpha} \|a_n\|_X)^q \right)^{1/q} < \infty \}.$$

Using the Q-norm $\| \cdot \|_{1, \alpha(X)}$ we shall define the approximation spaces $A_q^\alpha(X), \alpha > 0, q > 0$, in the following way: Let $Ef = (E_n f)_{n=0}^\infty$, where $f \in X, E_0 f = f, E_1 f = E_1(f)_X, E_n f = E_{2^{n-1}}(f)_X, n=2,3,\dots$. Then the approximation space $A_q^\alpha(X)$ is the set of those elements $f \in X$, for which $\|Ef\|_{1, \alpha(X)}$ is finite. $A_q^\alpha(X)$ is a Q-space with respect to the Q-norm

$$(1) \quad \|f\|_{A_q^\alpha(X)} = \|Ef\|_{1, \alpha(X)} = \left\{ \|f\|_X + \sum_{n=0}^{\infty} (2^{(n+1)\alpha} E_{2^n}(f))^q \right\}^{1/q}$$

The problem, which we consider here, is the problem of interpolation of approximation spaces. More exactly, let $X_i, i=0,1, X_1 \subset X_0$, be Q-spaces with Q-norms $\| \cdot \|_{X_i} = \| \cdot \|_i$. Let $\{G_n\}_{n=0}^\infty$ be a normal approximating family in X_0 . Let us consider the approximation spaces $A_q^\alpha(X_i), i=0,1$. What we can say about their interpolation spaces with respect to the real method of interpolation (see [2],[5]), i.e. what is

$$(A_{q_0}^\alpha(X_0), A_{q_1}^\alpha(X_1))_{\theta, q} ?$$

The interpolation of the spaces $A_q^\alpha(X_0)$ by means of the real method of interpolation is well-known - the following theorem of J. Peetre and G. Sparr is valid ([1], see also [2] and [3]):

Theorem A. We have:

$$(2) \quad (A_{q_0}^\alpha(X_0), A_{q_1}^\alpha(X_0))_{\theta, q} = A_q^\alpha(X_0)$$

provided $\alpha = (1-\theta)\alpha_0 + \theta\alpha_1$.

We shall be interested here in the spaces $(A_{q_0}^\alpha(X_0), A_{q_1}^\alpha(X_1))_{\theta, q}$ when $X_0 \neq X_1$. We give a condition by which we have

$$(3) \quad (A_{q_0}^\alpha(X_0), A_{q_1}^\alpha(X_1))_{\theta, q} = A_q^\alpha((X_0, X_1)_{\theta, q})$$

provided

$$(4) \quad \alpha = (1-\theta)\alpha_0 + \theta\alpha_1, \quad 1/q = (1-\theta)q_0 + \theta/q.$$

The formula (3) is similar to the corresponding formula for sequence spaces (see [1],[2],[5]):

$$(5) \quad (l_{q_0}^\alpha(X_0), l_{q_1}^\alpha(X_1))_{\theta, q} = l_q^\alpha((X_0, X_1)_{\theta, q})$$

by the condition (4).

We shall use (5) in our proof.

1. Some notations. In order to prove (3) we must set some condition on the spaces $X_i, i=0,1$, and on the family $\{G_n\}_{n=0}^\infty$. Using this condition we shall obtain equivalent Q-norms in $A_q^\alpha(X_i)$.

We shall use the classical Hardy inequality in the following form:

Lemma 1 (Hardy inequalities). Let X be a Q-space, $a=(a_k)_{k=0}^\infty \in l_q^\alpha(X), b=(b_k)_{k=0}^\infty \in l_q^\alpha(X), \alpha > 0, q > 0$. If we have

$$(6) \quad \|b_k\|_X \leq M 2^{-k\lambda} \left(\sum_{j=0}^k (2^{j\lambda} \|a_j\|_X) \right), \lambda > \alpha,$$

or

$$(7) \quad \|b_k\|_X \leq M \left(\sum_{j=k}^\infty \|a_j\|_X^\mu \right)^{1/\mu}, \mu \leq q,$$

then

$$(8) \quad \|b\|_{l_q^\alpha(X)} \leq c(M, q, \alpha, \lambda, \mu) \|a\|_{l_q^\alpha(X)}.$$

Definition 1. M-approximant to f with respect to G we call every element $g \in G$ such that $\|f-g\|_X \leq ME_G(f)_X$.

Definition 2. Let $\{G_n\}_{n=0}^\infty$ be a normal approximating family in X . M-system for $f \in X$ we call every sequence $(g_n)_{n=0}^\infty, g_n \in G_n$, such that $\|f-g_n\|_X \leq ME_n(f)_X$.

Definition 3. M-mapping is a mapping $L=(L_n)_{n=0}^\infty, L_n: X \rightarrow G_n$, such that for every $f \in X$ the set $(L_n f)_{n=0}^\infty$ is M-system for f .

Remark. The operators L_n can be in the general case nonlinear operators.

Let us set $T_n f = L_{2^n-1} f - L_{2^n-2} f, n=2,3,\dots, T_1 f = L_1 f - f, T_0 f = f, T f = (T_n f)_{n=0}^\infty$.

Theorem 1. Every M-mapping defines an equivalent Q-norm in $A_q^\alpha(X)$, given by $\|f\|_M = \|T f\|_{l_q^\alpha(X)}$. The constants of equivalence depend only on M, X, q and α .

Proof. Since $\|T_n f\|_X \leq c(M, X) (E_{2^n}(f)_X + E_{2^n-1}(f)_X), n=2,3,\dots$, obviously

$$(9) \quad \|T f\|_{l_q^\alpha(X)} \leq c(M, X, q) \|E f\|_{l_q^\alpha(X)}.$$

To prove the converse inequality, let $f \in A_q^\alpha(X)$. Then, for some $\mu = \mu(X, q), \mu \leq q$, we have

$$(10) \quad E_{2^n}(f)_X \leq \left\| \sum_{j=n}^\infty T_j f \right\|_X \leq \left(\sum_{j=0}^\infty \|T_j f\|_X^\mu \right)^{1/\mu},$$

since by the condition iii) for the normal approximating

family $\{G_n\}_{n=0}^\infty$ we have $T_0 f + T_1 f + \dots + T_{2^n-1} f \in G_{2^n}$, and from $f \in A_q^\alpha(X)$ and (9) it follows that $f = \sum_{n=0}^\infty T_n f$ in X .

Using Hardy inequality (lemma 1, (7) and (8)), we obtain from (10):

$$(11) \quad \|Ef\|_{1_q^\alpha(X)} \leq c(\mu, \alpha, q, X) \|Tf\|_{1_q^\alpha(X)}.$$

From (1), (9) and (11) it follows the equivalence of the Q -norms $\|f\|_{A_q^\alpha(X)}$ and $\|Tf\|_{1_q^\alpha(X)}$ with constant of equivalence depending only on M, X, q and α .

Remark. From (9) it follows that $\|f\|_M$ is a Q -norm.

Now we set our basic assumption. The reasons of this assumption we shall discuss in section 3.

Let there exist a second normal approximating family $\{\tilde{G}_n\}_{n=0}^\infty$ in $X_i, i=0,1$. Let us denote by

$$E_n(f)_i := \inf \{ \|f-g\|_i : g \in \tilde{G}_n \}$$

the corresponding best approximations.

Assumption A. There exists a sequence of linear operators $\{L_n\}_{n=0}^\infty, L_n: \tilde{G}_n \rightarrow G_n, n=0,1,\dots$, which transform every M -mapping with respect to $\{\tilde{G}_n\}$ in X_0 into $N_i(M, X), i=0,1$, -mapping with respect to $\{G_n\}$ in $X_i, i=0,1$.

In other words if $\|f-g\|_0 \leq M E_n(f)_0, g \in \tilde{G}_n$, then

$$(12) \quad \|f - L_n g\|_i \leq N_i(M, X_i) E_n(f)_i, i=0,1,$$

where the constants $N_i(M, X_i)$ depend only on M and X_i .

From now we shall suppose that the assumption A holds true for X_0 and X_1 .

2. Main result. We shall make our basic construction, which shall give us convenient Q -norms in $X_i, i=0,1$, and their interpolation spaces. We shall use the following lemma, which is proved in [6]:

Lemma 2. Let X be a Q -space and G a subgroup of X . Let $f \in X$ be arbitrary and $Pf \in G$ an M -approximant to f with respect to G . For every $f_1 \in X$ there exists $c(M, X)$ -approximant $Pf_1 \in G$, such that $Pf - Pf_1$ is also $c(M, X)$ -approximant to $f - f_1$ with respect to G .

In other words Pf_1 satisfies

$$(13) \quad \begin{aligned} \|f_1 - Pf_1\|_X &\leq c(M, X) E_G(f_1)_X \\ \|f - f_1 - (Pf - Pf_1)\|_X &\leq c(M, X) E_G(f - f_1)_X \end{aligned}$$

Let now $f \in X_0$ be arbitrary. We shall fix this f and using this f we shall construct some Q -norms. Let $(P_n f)_{n=0}^\infty$, $P_n f \in \tilde{G}_n$, be arbitrary 2-system for f with respect to $\{\tilde{G}_n\}_{n=0}^\infty$ in X_0 . In view of lemma 2 for arbitrary $f_1 \in X_0$ there is a $c(X_0)$ -system $(P_n f_1)_{n=0}^\infty$ with respect to $\{\tilde{G}_n\}_{n=0}^\infty$ in X_0 such that $(P_n f - P_n f_1)_{n=0}^\infty$ is $c(X_0)$ -system for $f - f_1$. Using assumption A we conclude that $(L_n P_n f)_{n=0}^\infty$ and $(L_n P_n f_1)_{n=0}^\infty$ are $c_i := c_i(X_0, X_1)$ -systems to f and f_1 in X_1 , $i=0,1$, with respect to $\{G_n\}_{n=0}^\infty$ and $(L_n P_n f - L_n P_n f_1)_{n=0}^\infty$ is c_0 -system for $f - f_1$ in X_0 with respect to $\{G_n\}_{n=0}^\infty$.

Therefore the mapping $F = (F_n)_{n=0}^\infty$, $F_n = L_n P_n$, $n=0,1,\dots$, is c_i -mapping in X_i . Let us set: $T_n = F_{2n-1} - F_{2n-2}$, $n=2,3,\dots$, $T_1 = F_1 - I$, $T_0 = I$, $Ig = g$. Using theorem 1 and its proof, we obtain:

Lemma 3. a) The Q -norm $\|g\|_{\alpha, q}^i = \|Tg\|_{1, \alpha}(X_i)$ is an equivalent Q -norm in $A_q^\alpha(X_i)$ with constants of equivalence depending only on X_0, X_1, q and α .

b) if $f - f_1 \in A_q^\alpha(X_0)$ then

$$(14) \quad c_2(X_0) \|f - f_1\|_{A_q^\alpha(X_0)} \leq \|f - f_1\|_{\alpha, q}^* \leq c_3(X_0) \|f - f_1\|_{A_q^\alpha(X_0)},$$

where $\|f - f_1\|_{\alpha, q}^* = \|Tf - Tf_1\|_{1, \alpha}(X_0)$.

Lemma 4. For every $g \in X_i$, $i=0,1$, we have

$$(15) \quad \|T_n g\|_i \leq c_i(X_0, X_1) \|g\|_i,$$

$$(16) \quad \|F_n g\|_i \leq c_i(X_0, X_1) \|g\|_i, i=0,1.$$

Proof. We have

$$\|T_{2n+1} g\|_i \leq c(X_i) (\|F_{2n} g - g\|_i + \|F_{2n-1} g - g\|_i) \leq c(X_i) c_i(E_{2n}(g)_i + E_{2n-1}(g)_i) \leq c_i(X_0, X_1) \|g\|_i.$$

Let $K(f, t; X_0, X_1)$ denote the K -functional of J. Peetre [2], [5] for f with respect the Q -spaces X_0 and X_1 :

$$K(f, t; X_0, X_1) = \inf \|f_0\|_0 + t \|f_1\|_1 : f = f_0 + f_1, f_i \in X_i, i=0,1.$$

Lemma 5. For every $g \in G_n$ we have

$$K(f - F_n f, t; X_0, X_1) \leq c(X_0, X_1) K(f - g, t; X_0, X_1)$$

Proof. Let $\epsilon > 0$ be arbitrary. For some $f_1 \in X_1$, $f - f_1 \in X_0$,

we have:

$\epsilon + K(f-g, t; X_0, X_1) \geq \|f-f_1-g\|_0 + t\|f_1\|_1 \geq E_n(f-f_1)_0 + tE_n(f_1)_1 \geq c_0^{-1} \|f-f_1-(F_n f - F_n f_1)\|_0 + t c_1^{-1} \|f_1 - F_n f_1\|_1 \geq c(X_0, X_1) K(f - F_n f, t; X_0, X_1)$. We have used the fact that F is c_i -mapping in X_i , $i=0,1$, with respect to $\{G_n\}_{n=0}^\infty$.

Corollary 1. For every $g \in G_n$ we have

$$\|f - F_n f\|_{(X_0, X_1)_{\theta, q}} \leq c(X_0, X_1) \|f - g\|_{(X_0, X_1)_{\theta, q}}$$

Taking g equal to 2-approximant to f in $(X_0, X_1)_{\theta, q}$ we obtain:

Corollary 2. For every $X = (X_0, X_1)_{\theta, q}$ we have

$$\|f - F_n f\|_X \leq c(X_0, X_1) E_n(f)_X.$$

This corollary gives us that $F_n f$ is $c(X_0, X_1)$ -approximant to f in each $X = (X_0, X_1)_{\theta, q}$, $0 \leq \theta \leq 1, q > 0$.

Corollary 3. We have

$$\|Tf\|_{1_q^\alpha(X)} \sim \|f\|_{A_q^\alpha(X)}, X = (X_0, X_1)_{\theta, q}.$$

Let us denote $A_{q_i}^{\alpha_i}(X_i) = A_i, l_{q_i}^{\alpha_i}(X_i) = l_i, i=0,1$.

Lemma 6. We have

$c_0 K(f, t; A_0, A_1) \leq K(Tf, t; l_0, l_1) \leq c_1 K(f, t; A_0, A_1)$, where the constants $c_0 > 0$ and $c_1 > 0$ depend only on $X_i, q_i, \alpha_i, i=0,1$.

Proof. Let $\epsilon > 0$ be arbitrary. For some $f_1 \in A_1, f - f_1 \in A_0$, we have:

$$(17) \quad \epsilon + K(f, t; A_0, A_1) \geq \|f - f_1\|_{A_0} + t \|f_1\|_{A_1}.$$

Using lemma 3 we get from (17):

$$\epsilon + K(f, t; A_0, A_1) \geq c'(X_0, X_1) (\|Tf - Tf_1\|_{l_0} + t \|Tf_1\|_{l_1}) \geq c'(X_0, X_1) K(Tf, t; l_0, l_1).$$

To prove the converse inequality let $a = (a_n)_{n=0}^\infty \in l_1$ be such that $Tf - a \in l_0$. Since $a_n \in X_1 \subset X_0$, we can define $g_n = F_{2^n} a_n$. Lemma 4 gives us

$$(18) \quad \|g_n\|_i = \|F_{2^n} a_n\|_i \leq c_i(X_0, X_1) \|a_n\|_i.$$

Let us set $g = \sum_{n=0}^\infty g_n$. Since $a \in l_1$, in view of (18) we have $g \in X_i, i=0,1$. Again, since $a \in l_1$ we have for some $\mu = \mu(X_0, X_1), \mu \leq q_i$, similarly to (10):

$$E_{2^n}(g)_i \leq \left\| \sum_{j=n}^{\infty} g_j \right\|_i \leq c(X_i) \left(\sum_{j=n}^{\infty} \|g_j\|_i^\mu \right)^{1/\mu} \leq c_i(X_0, X_1) \left(\sum_{j=n}^{\infty} \|a_j\|_i^\mu \right)^{1/\mu} .$$

Using this inequality, we obtain from Hardy inequality (7) and (8):

$$(19) \quad \|g\|_{A_1} = \|Eg\|_{1_1} \leq c(\alpha_1, q_1, X_0, X_1) \|a\|_{1_1} .$$

In a similar way we have

$$(20) \quad E_{2^n}(f-g)_i \leq \left\| \sum_{j=n-1}^{\infty} (T_j f - g_j) \right\|_i \leq c(X_i) \left(\sum_{j=n-1}^{\infty} \|T_j f - g_j\|_i^\mu \right)^{1/\mu} .$$

Using the property of $F_{2^j} a_j$ to be c_0 -approximant to a_j with respect to G_{2^j} in X_0 , we get:

$$(21) \quad \|T_j f - g_j\|_0 \leq c(X_0) (\|T_j f - a_j\|_0 + \|a_j - F_{2^j} a_j\|_0) \leq c'(X_0) (\|T_j f - a_j\|_0 + E_{2^j} (a_j)_0) \leq c''(X_0) \|T_j f - a_j\|_0 ,$$

since $T_j f \in G_{2^j}$ and therefore $E_{2^j} (a_j)_0 \leq \|T_j f - a_j\|_0$.

From (20) and (21) we obtain, using again Hardy inequality (7) and (8):

$$(22) \quad \|f-g\|_{A_0} \leq c(\alpha_0, q_0, X_0, X_1) \|Tf-a\|_{1_0} .$$

From (19) and (22) we obtain:

$$K(f, t; A_0, A_1) \leq \|f-g\|_{A_0} + t \|g\|_{A_1} \leq c(\alpha_0, \alpha_1, q_0, q_1, X_0, X_1) \cdot (\|Tf-a\|_{1_0} + \|a\|_{1_1}) .$$

Since $a \in l_1$, $Tf-a \in l_0$, was arbitrary, the last inequality gives us

$$K(f, t; A_0, A_1) \leq c(\alpha_0, \alpha_1, q_0, q_1, X_0, X_1) K(Tf, t; l_0, l_1) .$$

Theorem 2. By the assumption A we have

$$(A_{q_0}^{\alpha_0}(X_0), A_{q_1}^{\alpha_1}(X_1))_{\theta, q} = A_q^{\alpha}((X_0, X_1)_{\theta, q})$$

provided

$$\alpha = (1-\theta)\alpha_0 + \theta\alpha_1, \quad 1/q = (1-\theta)/q_0 + \theta/q_1 .$$

Proof. We have, using (4), (5), corollary 3 and lemma 6:

$$\|f\|_{(A_0, A_1)_{\theta, q}} \sim \|Tf\|_{(l_0, l_1)_{\theta, q}} \sim \|Tf\|_{1_q((X_0, X_1)_{\theta, q})} \sim \|f\|_{A_q^{\alpha}((X_0, X_1)_{\theta, q})}$$

Since f was arbitrary and the constants of the equivalence depend only on $X_i, \alpha_i, q_i, i=0,1$, but not on f , we obtain the statement of the theorem.

3. Remarks and application. 1) Let us remark first, that all difficulties in the proof of theorem 2 come from the fact, that the operator Ef is non-linear.

2) In the trivial case of the assumption A, when $\tilde{G}_n = G_n, n=0,1,\dots$, and L_n is the identity operator: $L_n g = g, g \in G_n, n=0,1,\dots$, and $X_0 = X_1$, theorem 2 gives us theorem A (with the additional restriction $1/q = (1-\theta)/q_0 + \theta/q_1$).

3) If we take $\tilde{G}_n = X_0, n=0,1,\dots$, then the assumption A becomes the following ("linear assumption"):

There exist linear operators $L_n, L_n: X_0 \rightarrow G_n, n=0,1,\dots$, such that

$$(23) \quad \|f - L_n f\|_i \leq c_i E_n(f)_i, \quad i=0,1, n=0,1,\dots$$

Therefore we obtain a linear variant of theorem 2: the equality (3) holds true by the conditions (4) and (23).

4) We came to theorem 2 (more exactly to the assumption A), when we wanted to obtain interpolation theorem for Besov spaces $B_q^\alpha(L_p)$, $p > 0$, for the finite interval, when the Besov spaces are defined by means of the moduli of continuity. From the results of Z. Ciesielski [7] and P. Oswald [8] it follows, that the Besov spaces are approximation spaces for best diadic sufficiently smooth spline approximation. To obtain an interpolation theorem for these approximation spaces we used the following construction of the assumption A:

X_0 is the space L_{p_0} for p_0 sufficiently small, \tilde{G}_n is the set of all diadic piecewise polynomial splines and L_n is the so called "Quasi-interpolant" operator of de Boor and Fix [9], see also DeVore [10]. It is not difficult to see that the "quasi-interpolant" operator satisfies the assumption A with respect to all $X_1 = L_p, p > p_0$, see [6]. Therefore we can obtain from theorem 2 an interpolation theorem for Besov spaces, see [6], where such theorem is proved directly.

5) We can replace assumption A with "weak assumption A". This means the following:

We introduce the notions of (λ, M) -weak approximant, system, mapping simply replacing $E_n(f)$ on the right hand side in the corresponding inequalities by $n^{-\lambda} (\sum_{k=0}^n (k+1)^{\lambda-1} E_k(f))$. The assumption A becomes

Weak assumption A. There exists a sequence of linear operators $(L_n)_{n=0}^{\infty}$, $L_n: \tilde{G}_n \rightarrow G_n, n=0,1,\dots$, which transform every (λ, M) -weak mapping with respect to \tilde{G}_n in X_0 into $(\lambda, N_i(M, X))$ -weak mapping with respect to $\{G_n\}$ in $X_i, i=0,1$.

Then theorem 2 remains valid with weak assumption A if $\lambda \geq \alpha_i, i=0,1$. The proof remains the same with the following modifications: In theorem 1 we must use Hardy inequality (6), (8), lemma 5 and corollaries 1 and 2 must be written in "weak" form, corollary 3, lemma 6 and the proof of theorem 2 remain the same.

The linear case (see 2) in this section) also is valid by the weak linear assumption: (23) can be replaced by

$$\|f - L_n f\|_i \leq c_i n^{-\lambda} \sum_{k=0}^n (k+1)^{\lambda-1} E_k(f)_i, \lambda \geq \alpha_i, i=0,1, n=0,1,\dots$$

Since many operators in the theory of approximation are of weak type (for example the classical Jackson operator for trigonometrical approximation, the modify Jackson operator and others), we hope that theorem 2 in its weak form also shall have some applications.

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