INTERPOLATION OF APPROXIMATION SPACES

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Let X_i, $i=0,1$, $X_1\ X_0$, be quasi-normed Abelian groups. Let $\{G_n\}_{n=0}^{\infty}$ be a normal approximating family in X_i, $i=0,1$. Let $A_q^{\alpha}(X_i)$ be the approximation spaces in X_i with respect to the family $\{G_n\}_{n=0}^{\infty}$. The paper considers the problem of interpolation of the approximation spaces $A_{q_o}^{\alpha_o}(X_o)$, $A_{q_1}^{\alpha_1}(X_1)$ by the real method of interpolation.

Introduction. We shall consider approximations in quasi-normed Abelian groups(Q-spaces). This kind of spaces are very convinient for some nonlinear approximations, for example rational and spline free-knots approximations, as well as aproximations in spaces like L_p, $0<p<1$, see [1],[2],[3]. Let X be a Q-space. The quasi-norm(Q-norm) in X : $\|\cdot\|_X$, is characterized by the constant k in the triangle inequality $\|f+g\|_X \leq k(\|f\|_X + \|g\|_X)$, or, equivalently, by the power σ in the inequality $\|f+g\|_X^{\sigma} \leq \|f\|_X^{\sigma} + \|g\|_X^{\sigma}$ (see [2]). Some constants $c(\cdot,\dots,\cdot)$, which shall appear later on, shall depend on k or σ. We shall denote this dependence simply by $c(\cdot,\dots,X)$. If two Q-norms $\|\cdot\|$ and $|||\cdot|||$ in X are equivalent, we shall denote this by $\|\cdot\| \sim |||\cdot|||$.

Let $\{G_n\}_{n=0}^{\infty}$ be a normal approximating family in X (see [3],[4]). This means that i) $G_o=0\subset G_n\subset G_{n+1}$; ii) $G_n \pm G_m \subset G_{n+m}$. Let

$$E_n(f):=E_n(f)_X:=E_{G_n}(f)_X:=\inf\{\|f-g\|_X : g\in X\}$$

denote the best approximation of f by elements of G_n.

Let us remember, that the sequence spaces $l_q^{\alpha}(X)$ (see [1],[2],[5]) are defined in the following way:

$$l_q^{\alpha}(X)=\{a : a=(a_n)_{n=0}^{\infty}, a_n\in X, \|a\|_{l_q^{\alpha}(X)}=\{\sum_{n=0}^{\infty}(2^{n\alpha}\|a_n\|_X)^q\}^{1/q}<\infty\}.$$

Using the Q-norm $\|\cdot\|_{l^\alpha_q(X)}$ we shall define the approximation spaces $A^\alpha_q(X),\alpha>0,q>0$, in the following way: Let $Ef=(E_nf)_{n=0}^\infty$, where $f\epsilon X, E_o f=f, E_1 f=E_1(f)_X, E_n f=E_{2^{n-1}}(f)_X$, $n=2,3,\ldots$. Then the approximation space $A^\alpha_q(X)$ is the set of those elements $f\epsilon X$, for which $\|Ef\|_{l^\alpha_q(X)}$ is finite. $A^\alpha_q(X)$ is a Q-space with respect to the Q-norm

(1) $\|f\|_{A^\alpha_q(X)}=\|Ef\|_{l^\alpha_q(X)}=\{\ \|f\|_X+\sum_{n=0}^{\infty}(2^{(n+1)\alpha}E_{2^n}(f))^q\}^{1/q}$

The problem, which we consider here, is the problem of interpolation of approximation spaces. More exactly, let $X_i, i=0,1, X_1 c X_0$, be Q-spaces with Q-norms $\|\cdot\|_{X_i}=\|\cdot\|_i$. Let $\{G_n\}_{n=0}^\infty$ be a normal approximating family in X_0. Let us consider the approximation spaces $A^\alpha_q(X_i), i=0,1$. What we can say about ~~their~~ interpolation spaces with respect to the real method of interpolation (see [2],[5]), i.e. what is

$$(\ A^{\alpha_o}_{q_o}(X_o), A^{\alpha_1}_{q_1}(X_1)\ )_{\theta,q}\ ?$$

The interpolation of the spaces $A^\alpha_q(X_o)$ by means of the real method of interpolation is well-known - the following theorem of J.Peetre and G.Sparr is valid([1], see also [2] and [3]):

<u>Theorem A</u>. We have:

(2) $$(\ A^{\alpha_o}_{q_o}(X_o), A^{\alpha_1}_{q_1}(X_o)\ )_{\theta,q} = A^\alpha_q(X_o)$$

provided $\alpha=(1-\theta)\alpha_o+\theta\alpha_1$.

We shall be interested here in the spaces $(\ A^{\alpha_o}_{q_o}(X_o), A^{\alpha_1}_{q_1}(X_1)\ )_{\theta,q}$ when $X_o \neq X_1$. We give a condition by which we have

(3) $$(\ A^{\alpha_o}_{q_o}(X_o), A^{\alpha_1}_{q_1}(X_1)\ )_{\theta,q} = A^\alpha_q((X_o,X_1)_{\theta,q})$$

provided

(4) $$\alpha=(1-\theta)\alpha_o+\theta\alpha_1,\ 1/q = (1-\theta)q_o + \theta/q\ .$$

The formula (3) is similar to the corresponding formula for sequence spaces(see [1],[2],[5]):

(5) $$(\ l^{\alpha_o}_{q_o}(X_o), l^{\alpha_1}_{q_1}(X_1)\ )_{\theta,q} = l^\alpha_q((X_o,X_1)_{\theta,q})$$

by the condition (4).

We shall use (5) in our proof.

1.Some notations. In order to prove (3) we must set some condition on the spaces X_i, $i=0,1$, and on the family $\{G_n\}_{n=0}^{\infty}$. Using this condition we shall obtain equivalent Q-norms in $A_q^{\alpha}(X_i)$.

We shall use the classical Hardy inequality in the following form:

Lemma 1 (Hardy inequalities). Let X be a Q-space, $a=(a_k)_{k=0}\in l_q^{\alpha}(X)$, $b=(b_k)_{k=0}\in l_q^{\alpha}(X)$, $\alpha>0$, $q>0$. If we have

$$(6)\qquad \|b_k\|_X \leqq M2^{-k\lambda}\left(\sum_{j=0}^{k}(2^{j\lambda}\|a_j\|_X)\right),\ \lambda>\alpha\ ,$$

or

$$(7)\qquad \|b_k\|_X \leqq M\left(\sum_{j=k}^{\infty}\|a_j\|_X^{\mu}\right)^{1/\mu},\ \mu\leqq q\ ,$$

then

$$(8)\qquad \|b\|_{l_q^{\alpha}(X)} \leqq c(M,q,\alpha,\lambda,\mu)\|a\|_{l_q^{\alpha}(X)}\ .$$

Definition 1. M-approximant to f with respact to G we call every element $g\in G$ such that $\|f-g\|_X \leqq ME_G(f)_X$.

Definition 2. Let $\{G_n\}_{n=0}^{\infty}$ be a normal approximating family in X. M-system for $f\in X$ we call every sequence $(g_n)_{n=0}^{\infty}$, $g_n\in G_n$, such that $\|f-g_n\|_X \leqq ME_n(f)_X$.

Definition 3. M-mapping is a mapping $L=(L_n)_{n=0}^{\infty}$, L_n: $X\to G_n$, such that for every $f\in X$ the set $(L_nf)_{n=0}^{\infty}$ is M-system for f.

Remark. The operators L_n can be in the general case nonlinear operators.

Let us set $T_nf= L_{2^{n-1}}f- L_{2^{n-2}}f$, $n=2,3,\ldots$, $T_1f=L_1-f$, $T_0f=f$, $Tf=(T_nf)_{n=0}^{\infty}$.

Theorem 1. Every M-mapping defines an equivalent Q-norm in $A_q^{\alpha}(X)$, given by $\|f\|_M=\|Tf\|_{l_q^{\alpha}(X)}$. The constants of equivalence depend only on M,X,q and α.

Proof. Since $\|T_nf\|_X\leqq c(M,X)(E_{2^n}(f)_X+E_{2^{n-1}}(f)_X)$, $n=2,3,\ldots$, obviously

$$(9)\qquad \|Tf\|_{l_q^{\alpha}(X)} \leqq c(M,X,q)\|Ef\|_{l_q^{\alpha}(X)}\ .$$

To prove the converse inequality, let $f\in A_q^{\alpha}(X)$. Then, for some $\mu=\mu(X,q)$, $\mu\leqq q$, we have

$$(10)\qquad E_{2^n}(f)_X \leqq \left\|\sum_{j=n}^{\infty}T_jf\right\|_X\leqq\left(\sum_{j=0}^{\infty}\|T_jf\|_X^{\mu}\right)^{1/\mu}\ ,$$

since by the condition iii) for the normal approximating family $\{G_n\}_{n=0}^{\infty}$ we have $T_0f+T_1f+\ldots+T_{2^n-1}f \in G_{2^n}$, and from $f\in A_q^\alpha(X)$ and (9) it follows that $f=\sum_{n=0}^{\infty} T_nf$ in X.

Using Hardy inequality(lemma 1,(7) and (8)),we obtain from (10):

$$\|Ef\|_{l_q^\alpha(X)} \leq c(\mu,\alpha,q,X)\|Tf\|_{l_q^\alpha(X)}. \tag{11}$$

From (1),(9) and (11) it follows the equivalence of the Q-norms $\|f\|_{A_q^\alpha(X)}$ and $\|Tf\|_{l_q^\alpha(X)}$ with constant of equivalence depending only on M,X,q and α.

Remark.From (9) it follows that $\|f\|_M$ is a Q-norm.

Now we set our basic assumpton.The reasons of this assumption we shall discuss in section 3.

Let there exist a second normal approximationg family $\{\tilde{G}_n\}_{n=0}^{\infty}$ in X_i,i=0,1.Let us denote by

$$\tilde{E}_n(f)_i:=\inf\ \{\|f-g\|_i : g\in\tilde{G}_n\}$$

the corresponding best approximations.

Assumption A.There exists a sequence of linear operators $\{L_n\}_{n=0}^{\infty}$, $L_n:\ \tilde{G}_n\to G_n$, n=0,1,...,which transform every M-mapping with respect to $\{\tilde{G}_n\}$ in X_o into $N_i(M,X)$,i=0,1,-mapping with respect to $\{G_n\}$ in X_i,i=0,1.

In other words if $\|f-g\|_o\leq M\tilde{E}_n(f)_o$, $g\in\tilde{G}_n$, then

$$\|f-L_ng\|_i\leq N_i(M,X_i)E_n(f)_i,\ i=0,1, \tag{12}$$

where the constants $N_i(M,X_i)$ depend only on M and X_i.

From now we shall suppose that the assumption A holds true for X_o and X_1.

2.Main result. We shall make our basic construction,which shall give us convinient Q-norms in X_i,i=0,1,and their interpolation spaces.We shall use the following lemma,which is proved in [6]:

Lemma 2.Let X be a Q-space and G-a subgroup of X. Let $f\in X$ be arbitrary and $Pf\in G$-an M-approximant to f with respect to G.For every $f_1\in X$ there exists c(M,X)-approximant $Pf_1\in G$,such that $Pf-Pf_1$ is also c(M,X)-approximant to $f-f_1$ with respect to G.

In other words Pf_1 satisfies

$$\begin{aligned} &\|f_1-Pf_1\|_X\leq c(M,X)E_G(f_1)_X \\ &\|f-f_1-(Pf-Pf_1)\|_X\leq c(M,X)E_G(f-f_1)_X \end{aligned} \tag{13}$$

Let now $f\varepsilon X_o$ be arbitrary. We shall fix this f and using this f we shall construct some Q-norms. Let $(P_n f)_{n=0}^{\infty}$, $P_n f\,\varepsilon\tilde{G}_n$, be arbitrary 2-system for f with respect to $\{\tilde{G}_n\}^{\infty}, n=0$ in X_o. In view of lemma 2 for arbitrary $f_1\varepsilon X_o$ there is a $c(X_o)$-system $(P_n f_1)_{n=0}^{\infty}$ with respect to $\{\tilde{G}_n\}_{n=0}^{\infty}$ in X_o such that $(P_n f-P_n f_1)_{n=0}^{\infty}$ is $c(X_o)$-system for $f-f_1$. Using assumption A we conclude that $(L_n P_n f)_{n=0}^{\infty}$ and $(L_n P_n f_1)_{n=0}^{\infty}$ are $c_i:=c_i(X_o,X_1)$-systems to f and f_1 in X_i, $i=0,1$, with respect to $\{G_n\}_{n=0}^{\infty}$ and $(L_n P_n f-L_n P_n f_1)_{n=0}^{\infty}$ is c_o-system for $f-f_1$ in X_o with respect to $\{G_n\}_{n=0}^{\infty}$.

Therefore the mapping $F=(F_n)_{n=0}^{\infty}, F_n=L_n P_n, n=0,1,\ldots,$ is c_i-mapping in X_i. Let us set: $T_n=F_{2^{n-1}}-F_{2^{n-2}}, n=2,3,\ldots, T_1=F_1-I, T_o=I, Ig=g$. Using theorem 1 and its proof, we obtain:

Lemma 3. a) The Q-norm $\|g\|_{\alpha,q}^{i}=\|Tg\|_{l_q^{\alpha}(X_i)}$ is an equivalent Q-norm in $A_q^{\alpha}(X_i)$ with constants of equivalence depending only on X_o,X_1,q and α.

b) if $f-f_1\varepsilon A_q^{\alpha}(X_o)$ then

$$(14)\qquad c_2(X_o)\|f-f_1\|_{A_q^{\alpha}(X_o)}\leq\|f-f_1\|_{\alpha,q}^{*}\leq c_3(X_o)\|f-f_1\|_{A_q^{\alpha}(X_o)},$$

where $\|f-f_1\|_{\alpha,q}^{*}=\|Tf-Tf_1\|_{l_q^{\alpha}(X_o)}$.

Lemma 4. For every $g\,\varepsilon X_i\, i=0,1$, we have

$$(15)\qquad \|T_n g\|_i\leq c_i(X_o,X_1)\|g\|_i\ ,$$

$$(16)\qquad \|F_n g\|_i\leq c_i(X_o,X_1)\|g\|_i, i=0,1.$$

Proof. We have

$$\|T_{2^{n+1}}g\|_i\leq c(X_i)(\|F_{2^n}g-g\|_i+\|F_{2^{n-1}}g-g\|_i)\leq c(X_i)c_i(E_{2^n}(g)_i+E_{2^{n-1}}(g)_i)\leq c_i(X_o,X_1)\|g\|_i.$$

Let $K(f,t;X_o,X_1)$ denote the K-functional of J. Peetre [2],[5] for f with respect the Q-spaces X_o and X_1:

$$K(f,t;X_o,X_1)=\inf\ \|f_o\|_o+t\|f_1\|_1 : f=f_o+f_1, f_i\varepsilon X_i, i=0,1\ .$$

Lemma 5. For every $g\,\varepsilon G_n$ we have

$$K(f-F_n f,t;X_o,X_1)\leq c(X_o,X_1)K(f-g,t;X_o,X_1)$$

Proof. Let $\varepsilon>0$ be arbitrary. For some $f_1\varepsilon X_1, f-f_1\varepsilon X_o$, we have:

$\varepsilon+K(f-g,t;X_o,X_1)\geq \|f-f_1-g\|_o+t\|f_1\|_1\geq E_n(f-f_1)_o+tE_n(f_1)_1\geq$

$c_o^{-1}\|f-f_1-(F_nf-F_nf_1)\|_o+tc_1^{-1}\|f_1-F_nf_1\|_1\geq c(X_o,X_1)K(f-F_nf,t;$

$X_o,X_1)$ .We have used the fact that F is c_i-mapping in X_i, $i=0,1$,with respect to $\{G_n\}_{n=0}^{\infty}$.

Corollary 1.For every $g\varepsilon G_n$ we have

$$\|f-F_nf\|_{(X_o,X_1)_{\theta,q}}\leq c(X_o,X_1)\|f-g\|_{(X_o,X_1)_{\theta,q}}$$

Taking g equal to 2-approximant to f in $(X_o,X_1)_{\Theta,q}$ we obtain:

Corollary 2.For every $X=(X_o,X_1)_{\Theta,q}$ we have

$$\|f-F_nf\|_X\leq c(X_o,X_1)E_n(f)_X .$$

This corollary gives us that F_nf is $c(X_o,X_1)$-approximant to f in each $X=(X_o,X_1)_{\Theta,q}$,$0\leq\Theta\leq 1,q>0$.

Corollary 3.We have

$$\|Tf\|_{l_q^\alpha(X)}\sim\|f\|_{A_q^\alpha(X)} ,X=(X_o,X_1)_{\Theta,q} .$$

Let us denote $A_{q_i}^{\alpha_i}(X_i)=A_i,l_{q_i}^{\alpha_i}(X_i)=l_i,i=0,1$.

Lemma 6.We have

$$c_oK(f,t;A_o,A_1)\leq K(Tf,t;l_o,l_1)\leq c_1K(f,t;A_o,A_1) ,$$

where the constants $c_o>0$ and $c_1>0$ depend only on X_i,q_i,α_i, $i=0,1$.

Proof.Let $\varepsilon>0$ be arbitrary.For some $f_1\varepsilon A_1,f-f_1\varepsilon A_o$, we have:

(17) $\varepsilon+K(f,t;A_o,A_1)\geq\|f-f_1\|_{A_o}+t\|f_1\|_{A_1}$.

Using lemma 3 we get from (17):

$$\varepsilon+K(f,t;A_o,A_1)\geq c'(X_o,X_1)(\|Tf-Tf_1\|_{l_o}+t\|Tf_1\|_{l_1})\geq$$
$$c'(X_oX_1)K(Tf,t;l_o,l_1) .$$

To prove the converse inequality let $a=(a_n)_{n=0}^{\infty}\varepsilon l_1$ be such that $Tf-a\varepsilon l_o$.Since $a_n\varepsilon X_1 c X_o$,we can define $g_n=F_{2^n}a_n$. Lemma 4 gives us

(18) $\|g_n\|_i=\|F_{2^n}a_n\|_i\leq c_i(X_o,X_1)\|a_n\|_i$.

Let us set $g=\sum_{n=0}^{\infty}g_n$.Since $a\varepsilon l_1$,in view of (18) we have $g\varepsilon X_i$,$i=0,1$.Again,since $a\varepsilon l_1$ we have for some $\mu=\mu(X_o,X_1)$, $\mu\leq q_i$,similarly to (10):

$$E_{2^n}(g)_i \leq \| \sum_{j=n}^{\infty} g_j \|_i \leq c(X_i)(\sum_{j=n}^{\infty} \|g_j\|_i^{\mu})^{1/\mu} \leq$$
$$c_i(X_o,X_1)(\sum_{j=n}^{\infty} \|a_j\|_i^{\mu})^{1/\mu} .$$

Using this inequality, we obtain from Hardy inequality (7) and (8):

(19) $\|g\|_{A_1} = \|Eg\|_{l_1} \leq c(\mu,\alpha_1,q_1,X_o,X_1)\|a\|_{l_1}.$

In a similar way we have

(20) $E_{2^n}(f-g)_i \leq \| \sum_{j=n-1}^{\infty} (T_j f-g_j \|_i \leq cX_i)(\sum_{j=n-1}^{\infty} \|T_j f-g_j\|_i^{\mu})^{1/\mu} .$

Using the property of $F_{2^j}\, a_j$ to be c_o-approximant to a_j with respect to G_{2^j} in X_o, we get:

(21) $\|T_j f-g_j\|_o \leq c(X_o)(\|T_j f-a_j\|_o + \|a_j - F_{2^j}\, a_j\|_o) \leq$
$c'(X_o)(\|T_j f-a_j\|_o + E_{2^j}(a_j)_o) \leq c''(X_o)\|T_j f-a_j\|_o ,$

since $T_j f \in G_{2^j}$ and therefore $E_{2^j}(a_j)_o \leq \|T_j f-a_j\|_o$.

From (20) and (21) we obtain, using again Hardy inequality (7) and (8):

(22) $\|f-g\|_{A_o} \leq c(\alpha_o,q_o,X_o,X_1)\|Tf-a\|_{l_o} .$

From (19) and (22) we obtain:

$$K(f,t;A_o,A_1) \leq \|f-g\|_{A_o} + t\|g\|_{A_1} \leq c(\alpha_o,\alpha_1,q_o,q_1,X_o,X_1).$$
$$(\|Tf-a\|_{l_o} + \|a\|_{l_1}) .$$

Since $a\in l_1$, $Tf-a\in l_o$, was arbitrary, the last inequality gives us

$$K(f,t;A_o,A_1) \leq c(\alpha_o,\alpha_1,q_o,q_1,X_o,X_1)K(Tf,t;l_o,l_1) .$$

<u>Theorem 2</u>. By the assumption A we have

$$(A_{q_o}^{\alpha_o}(X_o), A_{q_1}^{\alpha_1}(X_1))_{\theta,q} = A_q^{\alpha}((X_o,X_1)_{\theta,q})$$

provided

$$\alpha = (1-\theta)\alpha_o + \theta\alpha_1 , \quad 1/q = (1-\theta)/q_o + \theta/q_1 .$$

Proof. We have, using (4), (5), corollary 3 and lemma 6:

$$\|f\|_{(A_o,A_1)_{\theta,q}} \sim \|Tf\|_{(l_o,l_1)_{\theta,q}} \sim \|Tf\|_{l_q((X_o,X_1)_{\theta,q})} \sim$$
$$\sim \|f\|_{A_q^{\alpha}((X_o,X_1)_{\theta,q})} ,$$

Since f was arbitrary and the constants of the equivalence depend only on $X_i,\alpha_i,q_i,i=0,1$,but not on f,we obtain the statement of the theorem.

3.Remarks and application.1)Let us remark first,that all dificulties in the proof of theorem 2 come from the fact, that the operator Ef in non-linear.

2)In the trivial case of the assumption A,when $\tilde{G}_n=G_n,n=0,1,\ldots$,and L_n is the identity operator:$L_ng=g,g\in G_n,n=0,1,\ldots$,and $X_o=X_1$,theorem 2 gives us theorem A(with the additional restriction $1/q=(1-\Theta)/q_o+\Theta/q_1$).

3)If we take $\tilde{G}_n=X_o,n=0,1,\ldots$,then the assumption A becames the following("linear assumption"):

There exist linear operators $L_n,L_n:X_o\to G_n,n=0,1,\ldots$, such that

(23) $$\|f-L_nf\|_i \leq c_iE_n(f)_i \quad ,i=0,1,n=0,1,\ldots.$$

Therefore we obtain a linear variant of theorem 2: the equality (3)holds true by the conditions (4) and (23).

4)We came to theorem 2(more exactly to the assumption A), when we wanted to obtain interpolation theorem for Besov spaces $B^{\alpha}_q(L_p),p>0$,for the finite interval,when the Besov spaces are defined by means of the moduli of continuity. From the results of Z.Ciesielski [7] and P.Osvald [8[it follows,that the Besov spaces are approximation spaces for best diadic sufficiently smooth spline approximation.To obtain an interpolation theorem for these approximation spaces we used the following construction of the assumtion A: X_o is the space L_{p_o} for p_o sufficiently small, $\tilde{G}_2n$ is the set of all diadic piecewise polynomial splines and L_n is the so called "Quasi-interpolant" operator of be Boor and Fix [9],see also DeVore [10].It is not difficult to see that the"quasi-interpolant" operator satisfies the assumption A with respect to all $X_1= L_p,p>p_o$,see [6].Therefore we can obtain from theorem 2 an interpolation theorem for Besov spaces,see [6],where such theorem is proved direct.

5)We can replace assumption A with"weak aasumption A".This means the following:

We introduce the notions of (λ,M)-weak approximant, system, mapping simply replacing $E_n(f)$ on the right hand sind in the corresponding inequalities by $n^{-\lambda}(\sum_{k=0}^{n}(k+1)^{\lambda-1}E_n(f))$. The assumption A becames

Weak assumption A. There exists a sequence of linear operators $(L_n)_{n=0}^{\infty}$, $L_n: \tilde{G}_n \to G_n$, $n=0,1,\ldots$, which transform every (λ,M)-weak mapping with respect to $\tilde{G}_n$ in X_o into $(\lambda, N_i(M,X))$-weak mapping with respect to $\{G_n\}$ in X_i, $i=0,1$.

Then theorem 2 remains valid with weak assumption A if $\lambda \geqq \alpha_i$, $i=0,1$. The proof remains the same with the following modifications: In theorem 1 we must use Hardy inequality (6), (8), lemma 5 and corollaries 1 and 2 must be written in "weak" form, corollary 3, lemma 6 and the proof of theorem 2 remain the same.

The linear case (see 2) in this section) also is valid by the weak linear assumption: (23) can be replaced by

$$\|f-L_nf\|_i \leqq c_i n^{-\lambda}\sum_{k=0}^{n}(k+1)^{\lambda-1}E_k(f)_i, \lambda \geqq \alpha_i, i=0,1, n=0,1,\ldots$$

Since many operators in the theory of approximation are of weak type (for example the classical Jackson operator for trigonometrical approximation, the modify Jackson operator and others), we hope that theorem 2 in its weak form also shall have some applications.

REFERENCES

!.J.Peetre, G.Sparr, Interpolation of normed Abelian groups, Ann.Mat.Pura Appl.92,1972,217-262.

2.J.Bergh, J.Löfström, Interpolation spaces, Springer Verlag, Grundlehren, v.223, Berlin, 1976.

3.R.A.DeVore, V.A.Popov, Interpolation and nonlinear approximation, to appear in Proceedings of the Conference on Interpolation and Allied Topics in Analysis, Lund, 1986.

4.P.P.Petrushev, V.A.Popov, Rational Approximation of real Functions, Cambridge University Press, Enciclopedia of Mathematics and its Applications, v.28, 1987.

5.J.Peetre,New Thoughtson Besov Spaces,Duke University Mathematics Series I,Durham,N.C.,1976.

6.R.A.DeVore,V.A.Popov,Interpolation of Besov Spaces,Trans.of Amer.Math.Soc.(to appear).

7.Z.Ciesielski,Constructive function theory and spline systems,Studia Math.,52,1973,277-302.

8.P.Oswald,Approximation by splines in the metric L_p,$0<p<1$, Math.Nachr.,94,1980,69-83(Russian).

9.C.de Boor,G.J.Fix,Spline approximation by quasi-interpolants, J.of Approxiamation Theory,8,1973,19-45.

10.R.DeVore,Degree of approximation,in"Approximation theory II" (G.G.Lorentz,C.K.Chui,L.L.Schumaker eds),Academic Press,N.J., 1976.

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