Nonlinear n-widths in Besov Spaces

Ronald A. DeVore and Xiang Ming Yu

Abstract. We determine the nonlinear n-width of the unit ball of certain Besov spaces in $L_p[0,1]$, $1 \le p < \infty$.

§1. Introduction and Results

We refer the reader to the survey article of DeVore in this volume for the notation in this paper. We fix $1 \le p < \infty$ and determine the nonlinear n-width $d_n(K_\alpha)_{L_p(\Omega)}$ of the unit ball $K_\alpha := U(B^\alpha)$ in $L_p(\Omega)$, $\Omega := [0,1]$. Here, we use the abbreviated notation $B^\alpha := B^\alpha_\tau(L_\tau)$, where $\tau := (\alpha + 1/p)^{-1}$. The set K_α consists of all $f \in B^\alpha$ for which $\|f\|_{B^\alpha} \le 1$. The approximation order for the elements of B^α in L_p is known to be $O(n^{-\alpha})$ for the classical methods of nonlinear approximation.

Theorem. *For $K_\alpha := U(B^\alpha)$,*

$$C_1 n^{-\alpha} \le d_n(K_\alpha)_{L_p(\Omega)} \le C_2 n^{-\alpha}, \quad n = 1, 2, \ldots \tag{1}$$

with positive constants C_1, C_2 independent of n.

Proof: The lower estimate in (1) has been shown in [1]. We shall show that the upper estimate holds for the manifold $\mathcal{M}_n$ of piecewise polynomials of degree $< r$ with n pieces. The breakpoints are free parameters. Here r is a fixed integer larger than α. To prove the upper estimate of (1), we must find a parameterization a for $\mathcal{M}_n$ such that $\mathcal{M}_n = \{M_n(a)\}$ and a continuous selection $\bar{a}$ defined on K such that

$$\|f - M_n(\bar{a}(f))\|_{L_p(\Omega)} \le C n^{-\alpha}, \quad f \in K_\alpha. \tag{2}$$

Now $\mathcal{M}_n$ has dimension $nr + n - 1$. To parametrize $\mathcal{M}_n$, we use the vector a whose first $n-1$ components $0 \le a_{-n+1} \le \cdots \le a_{-1} \le 1$ are the breakpoints of the piecewise polynomial $M_n(a)$. It will be convenient to let $t_j := a_{-n+j}$, $j = 1, \ldots, n-1$ and $t_0 := 0$, $t_n := 1$ and $I_j := [t_j, t_{j+1}]$, $j = 0, \ldots, n-1$.. Notice that we allow equality in the t_j; this corresponds

Approximation Theory VI: Volume 1
C. K. Chui, L. L. Schumaker and J. D. Ward (eds.), pp. 203–206.

ISBN 0-12-174586-4.

to a degenerate interval. The other coordinates $a_0, \dots, a_{nr-1}$ of a denote the coefficients of the polynomials P_j which serve to define $M_n(a)$ on I_j. Thus, $P_j := a_{rj} + a_{rj+1}x + \dots + a_{rj+r-1}x^{r-1}$, $j = 0, \dots, n-1$.

For each function $f \in K_\alpha$, we have $\int_0^\infty t^{\alpha\tau - 1} w_r(f,t)_\tau^\tau \, dt \le 1$, where

$$w_r(f,t)_\tau := \left(\frac{1}{t} \int_0^t \int_{I(rs)} |\Delta_s^r(f,x)|^\tau \, dx \, ds \right)^{1/\tau} \tag{3}$$

is the averaged modulus of smoothness of f and $I(rs) := [0, 1 - rs]$. We shall use the generic notation

$$\begin{gathered} G(x) := \int_0^\infty \int_0^\infty g(x,s,t) \, ds \, dt, \\ g(x,s,t) := t^{-\alpha\tau - 2} \chi_{(0,t]}(s) \chi_{I(rs)}(x) |\Delta_s^r(f,x)|^\tau. \end{gathered} \tag{4}$$

Then, $|f| := \int_0^1 G(x)\,dx$ is equivalent to the $|f|_{B^\alpha}^\tau$, and for each $f \in K$, $\int_0^1 G(x)\,dx \le 1$ because $w_r \le \omega_r$. We shall complete the proof of (2) in the case that $\tau \le 1$ (a similar proof applies when $\tau > 1$.) The point of this assumption on τ is that $|.|$ is subadditive: $|f_1 + f_2| \le |f_1| + |f_2|$. In the case $\tau > 1$ one uses the subadditivity of $\|.\|_{B^\alpha}$ for the proof of (2). We also let $\|.\| := \|.\|_{L_\tau(\Omega)} + |.|^{1/\tau}$. Then, $\|.\| \le \|.\|_{B^\alpha}$. Now let $\eta := (8n)^{-1/\tau}$. Since B^α is separable, there are functions $f_1, f_2, \dots$ such that $\{B(f_j, \eta)\}$ is a cover for B^α, where $B(f_j, \eta)$ denotes the ball centered at f_j of radius η (with respect to $\|.\|_{B^\alpha}$.) By the paracompactness of Y, there is a refinement $\{U\}$ of $\{B(f_j, \eta)\}$ where the U are open sets and a partition of unity $\{\alpha_U\}$ subordinate to $\{U\}$. The functions α_U are continuous, nonnegative, and supported on U, and satisfy $\sum_U \alpha_U \equiv 1$. A point x is in at most a finite number of U.

For each U, $U \cap K \ne \emptyset$, we take $f_U \in U \cap K$. Since $f_U \in K$, we have $|f_U| \le 1$. Now, there is an integer k with $0 \le k < n$ and points $\xi_j := \xi_j(f_U)$ with $0 =: \xi_0 < \xi_1 < \dots < \xi_k \le 1$ such that

$$\begin{gathered} \int_0^{\xi_j} G_U(x)\,dx = \frac{j}{n}, \quad j = 1, \dots, k, \\ \text{and} \\ \int_0^1 G_U(x)\,dx = \theta + \frac{k}{n}, \quad 0 \le \theta < 1/n. \end{gathered} \tag{5}$$

Let $\xi_j(f_U) := 1$, $j = k+1, \dots, n$ and let $\xi(f_U) := (\xi_0(f_U), \dots, \xi_n(f_U))$.

For each $f \in K$ we define $t(f) := (t_0(f), \dots, t_n(f)) := \sum_U \alpha_U(f) \xi(f_U)$. Then obviously $t(f)$ is continuous on K. We next prove that for each U with $\alpha_U(f) \ne 0$, we have

$$\xi_{i-1}(f_U) \le t_i(f) \le t_{i+1}(f) \le \xi_{i+2}(f_U). \tag{6}$$

To prove (6), let $\tilde{U}$ be any other set with $\alpha_{\tilde{U}}(f) \neq 0$. Then $f \in U \cap \tilde{U}$ and therefore $|f - f_U| \leq |f - f_j| + |f_j - f_U| \leq 2\eta^\tau$, where $U \subset B(f_j, \eta)$. A similar estimate holds for $f_{\tilde{U}}$, and therefore $|f_U - f_{\tilde{U}}| \leq 4\eta^\tau$. Since $\big| |\Delta_s^r(f_U)|^\tau - |\Delta_s^r(f_{\tilde{U}})|^\tau \big| \leq |\Delta_s^r(f_U - f_{\tilde{U}})|^\tau$, we have

$$\left| \int_0^{\xi_i(f_{\tilde{U}})} G_{\tilde{U}}(x)\,dx - \int_0^{\xi_i(f_{\tilde{U}})} G_U(x)\,dx \right| \leq \int_0^{\xi_i(f_{\tilde{U}})} |G_{\tilde{U}}(x) - G_U(x)|\,dx \leq 4\eta^\tau \leq (2n)^{-1}.$$

Now, if $\xi_i(f_{\tilde{U}}) < 1$, then since $\int_0^{\xi_i(f_{\tilde{U}})} G_{\tilde{U}}(x)dx = i/n$, we have

$$\frac{i-1}{n} < \int_0^{\xi_i(f_{\tilde{U}})} G_U(x)\,dx < \frac{i+1}{n}.$$

This shows that

$$\xi_{i-1}(f_U) \leq \xi_i(f_{\tilde{U}}) \leq \xi_{i+1}(f_U). \tag{7}$$

Similarly, if $\xi_i(f_{\tilde{U}}) = 1$ and

$$\int_0^{\xi_i(f_{\tilde{U}})} G_{\tilde{U}}(x)\,dx = \frac{\tilde{k}}{n} + \tilde{\theta}, \quad 0 \leq \tilde{\theta} < \frac{1}{n},$$

then

$$\frac{\tilde{k}-1}{n} + \tilde{\theta} < \int_0^1 G_U(x)\,dx < \frac{\tilde{k}+1}{n} + \tilde{\theta}$$

and so $\xi_{\tilde{k}+1}(f_U) = 1$ and (7) still holds. Now, since $\sum_{\tilde{U}} \alpha_{\tilde{U}} \equiv 1$, we obtain from (7) that

$$\xi_{i-1}(f_U) \leq t_i(f) := \sum_{\substack{\tilde{U} \\ \alpha_{\tilde{U}}(f) \neq 0}} \alpha_{\tilde{U}}(f)\xi_i(f_{\tilde{U}}) \leq \xi_{i+1}(f_U),$$

and (6) follows from this.

Now, let $I_j := [t_j(f), t_{j+1}(f)]$, $j = 0, \ldots, n-1$ and for each U such that $\alpha_U(f) \neq 0$, we let $J_j(U) := [\xi_{j-1}(f_U), \xi_{j+2}(f_U)]$, $j = 1, \ldots, n-2$, $J_0(U) := [0, \xi_2(f_U)]$, $J_{n-1}(U) := [\xi_{n-2}(f_U), 1]$. From (6), we have $I_j \subset J_j(U)$, for all U and $j = 0, \ldots, n-1$. Then, by the definition of the points $\xi_j(f_U)$, we have

$$|f_U|(J_j(U)) \leq \int_{J_j(U)} G_U(x)\,dx \leq 3/n. \tag{8}$$

We let $P_j(f_U)$ be a polynomial of best L_p approximation to f_U on $J_j(U)$. We shall need the following embedding of B^α into L_p (see (5.16) of the article of DeVore in this volume):

$$E_r(f, I)_p \leq C|f|_{B^\alpha(I)} \tag{9}$$

with C here and later depending only on α, p, and r. From this and (8), we obtain

$$\|f_U - P_j(f_U)\|_{L_p(I_j)} \le \|f_U - P_j(f_U)\|_{L_p(J_j(U))} \le C|f_U|(J_j(U))^{1/\tau} \le Cn^{-1/\tau}. \tag{10}$$

Now define

$$P_j(f) := \sum_U \alpha_U(f)P_j(f_U), \quad j = 0, \dots, n-1.$$

This is a continous selection of polynomials and the coefficients of the $P_j(f)$ are the continuous selection (with respect to $f \in K$) that we want for the remaining coordinates of $\bar{a}$.

Finally, we check that $M_n(f)$ has the correct approximation properties. We first note that on $\Omega = [0,1]$, we have from (9) that if $\alpha_U(f) \neq 0$, then for some polynomial Q of degree $< r$ we have

$$\|f - f_U - Q\|_{L_\tau(\Omega)} \le \|f - f_U - Q\|_{L_p(\Omega)} \le C|f - f_U|_{B^\alpha} \le C\eta. \tag{11}$$

Since $\|f - f_U\|_{L_\tau(\Omega)} \le \eta$, we have from elementary inequalities (see [2]) for polynomials that $\|Q\|_{L_p(\Omega)} \le C\|Q\|_{L_\tau(\Omega)} \le C\{\|f - f_U - Q\|_{L_\tau(\Omega)} + \|f - f_U\|_{L_\tau(\Omega)}\} \le C\eta$. Using (11), we have

$$\|f - f_U\|_{L_p(I_j)} \le \|f - f_U\|_{L_p(\Omega)} \le \|f - f_U - Q\|_{L_p(\Omega)} + \|Q\|_{L_p(\Omega)} \le C\eta. \tag{12}$$

Now, since $M_n(f) := P_j(f)$ on I_j, we have by (9) and (12) that

$$\begin{aligned}\|f - P_j(f)\|_{L_p(I_j)} &\le \sum_U \alpha_U(f)\{\|f - f_U\|_{L_p(I_j)} + \|f_U - P_j(f_U)\|_{L_p(I_j)}\} \\ &\le C\{\eta + n^{-1/\tau}\} \le Cn^{-1/\tau}.\end{aligned}$$

If we add up these estimates, we obtain

$$\|f - M_n(f)\|^p_{L_p(\Omega)} = \sum_{j=0}^{n} \|f - P_j(f)\|^p_{L_p(I_j)} \le Cn^{1-p/\tau} = Cn^{-\alpha p}$$

as desired. ∎

References

1. DeVore, R., R. Howard, and C. Micchelli, Optimal nonlinear approximation, Manuskripta Mathematika (1989).
2. DeVore, R. and R. Sharpley, *Maximal Functions Measuring Smoothness*, Memoirs, Amer. Math. Soc., Vol. 283, Providence, R.I., 1984.