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On monotone extensions of boundary data

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Summary. A function $f \in C(\Omega)$, $\Omega \subseteq \mathbb{R}^s$ is called monotone on Ω if for any $x, y \in \Omega$ the relation $x - y \in \mathbb{R}^s_+$ implies $f(x) \geqq f(y)$. Given a domain $\Omega \subseteq \mathbb{R}^s$ with a continuous boundary $\partial\Omega$ and given any monotone function f on $\partial\Omega$ we are concerned with the existence and regularity of *monotone extensions* i.e., of functions F which are monotone on all of Ω and agree with f on $\partial\Omega$. In particular, we show that there is no linear mapping that is capable of producing a monotone extension to arbitrarily given monotone boundary data. Three nonlinear methods for constructing monotone extensions are then presented. Two of these constructions, however, have the common drawback that regardless of how smooth the boundary data may be, the resulting extensions will, in general, only be Lipschitz continuous. This leads us to consider a third and more involved monotonicity preserving extension scheme to prove that, when Ω is the unit square $[0, 1]^2$ in $\mathbb{R}^2$, strictly monotone analytic boundary data admit a monotone analytic extension.

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1 Introduction

During the past few years, *shape preserving* approximation and interpolation have been attracting considerable attention, see the survey article [1] by Utreras and the references therein. 'Shape preserving' typically means that the interpolant or approximant is monotone or convex whenever the given *discrete* data are monotone or convex in an appropriate sense. Specifically, various piecewise polynomial interpolation schemes have been proposed for the purpose of monotonically interpolating (discrete) data on regular grids, while variational approaches are employed for scattered data problems. In this paper we consider the rather different

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problem of constructing monotone surfaces under *transfinite* interpolation constraints given in terms of traces of functions with respect to (sufficiently smooth) boundaries. To our knowledge, this problem has not been addressed yet in the literature.

We say a function f is *monotone* (nondecreasing) on $\Omega \subseteq \mathbb{R}^s$ if for any $x, y \in \Omega$ the relation $x - y \in \mathbb{R}^s_+ := \{x = (x_1, \ldots, x_s) \in \mathbb{R}^s : x_i \geqq 0, \ i = 1, \ldots, s\}$ implies $f(x) \geqq f(y)$. The monotonicity is called *strict* if in addition $f(x) > f(y)$ whenever $x \neq y, x - y \in \mathbb{R}^s_+$. Given any domain Ω and some monotone function f on $\partial\Omega$ any function F that agrees with f on $\partial\Omega$ and is monotone on Ω is called a monotone *extension* of f (to Ω).

The motivation for the present investigation arose from the practical problem of modeling charge distributions for semiconductor design. In this particular setting, exact analytic representations of the charge distributions in terms of bivariate expressions were available everywhere except on a region Ω which could be transformed into a rectangle. The objective then was to determine a monotone transition between these two regions separated by a strip in such a way that the boundary values given by the known analytic expressions were matched precisely.

In the case $\Omega = [0, 1]^2$ one realizes rather quickly that neither a Boolean transfinite interpolant nor the solution of a Dirichlet problem for Laplace's equation with the prescribed boundary data will work in all cases. In fact, we show in Sect. 2 that, whatever domain Ω is considered, there never exists a *linear* mapping $\mathscr{L}$ such that $\mathscr{L}f$ is a monotone extension of f to Ω for all possible monotone boundary data f on $\partial\Omega$. This fact makes the question of existence of monotone extensions more interesting than it might have appeared at first glance. So in Sect. 3 we propose nonlinear ways of constructing monotone extensions for general domains. Unfortunately, these methods suffer from the common drawback that, regardless of how smooth the boundary functions are, the extensions will in general be at most Lipschitz continuous. Therefore we focus in Sect. 4 on the construction of monotone extensions for the unit square in the plane that have the same smoothness as the functions on the boundary.

2 Can monotone extensions be generated by linear operators?

As mentioned above, any attempt to construct monotone extensions of boundary data, which is linear, is bound to fail. Our first theorem gives a result along these lines for any bounded open set $\Omega \subset \mathbb{R}^s$, $s \geqq 2$. To this end, we designate $\mathscr{L}$ to be any mapping from monotone functions on $\partial\Omega$ into monotone functions on Ω. We require $\mathscr{L}$ to be continuous where on the boundary data we have the topology of uniform convergence and for functions on Ω we put pointwise convergence.

Theorem 2.1. *Suppose that $\Omega \subset \mathbb{R}^s$, $s \geqq 2$, is a bounded open set with boundary $\partial\Omega$. Then there is no continuous linear map $\mathscr{L}$ which yields for every monotone boundary data f on $\partial\Omega$, a monotone extensions $\mathscr{L}f$ on all of Ω.*

Proof. We can assume without loss of generality that $0 \in \Omega$. Suppose there exists a linear extension operator $\mathscr{L}$ such that for any monotone f on $\partial\Omega$, the function

$\mathscr{L}f$ is monotone on Ω. Since $\partial\Omega$ is compact, the linear functional $\lambda(f) := \mathscr{L}f(0)$ has the representation

$$\lambda(f) = \int_{\partial\Omega} f\,d\mu\,,$$

where $d\mu$ is a regular (signed) Borel measure on $\partial\Omega$. We shall show that $\mu(Q \cap \partial\Omega) = 0$ for any cube Q. Since μ is a Borel measure, this implies that μ is identically zero and gives a contradiction which proves the theorem.

To this end, let $a \in \mathbb{R}^s$ and let $h > 0$. The cubes $Q_\nu := a + \nu h + [0, h)^s$, $\nu \in \mathbb{Z}^s$ are a tiling of $\mathbb{R}^s$. We assume that $0 \in Q_0$. We consider h so small that Q_0 is completely contained in Ω. Let $f := \sum_\nu c_\nu \chi_{Q_\nu}$, with $c_\nu = 0$ or 1, be a monotone function on $\mathbb{R}^s$. The condition for monotonicity of f is simply that whenever $c_\nu = 1$, then $c_{\nu'} = 1$ for all $\nu' \geqq \nu$.

We shall now show that $\mu(Q_\nu \cap \partial\Omega) = 0$ for all $\nu \in \mathbb{Z}^s$ by using the above functions f. By our previous remarks this will complete the proof of the theorem. Of course, if $Q_\nu \cap \partial\Omega = \emptyset$ then $\mu(Q_\nu \cap \partial\Omega) = 0$. In particular, $\mu(Q_0 \cap \partial\Omega) = 0$.

Let $\nu \neq 0$. We consider first the case where there is an $i = 1, \ldots, s$ such that $\nu_i < 0$. We fix this i and define $c_{\nu'} := 1$ if either $\nu'_i \geqq 0$ or if $\nu'_i < 0$ and $\nu' \geqq \nu$. Otherwise, we define $c_{\nu'} := 0$. Then the function f is monotone nondecreasing. Let $j \in \{1, \ldots, s\}$ be chosen so that $j \neq i$. The rays $L_+ := \{te_j : t > 0\}$ and $L_- := \{te_j : t < 0\}$ both intersect $\partial\Omega$ (because $0 \in \Omega$ and Ω is bounded). Since $f \equiv 1$ on L_+ and L_-, we have $\lambda(f) = 1$. Moreover, if g is obtained from f by changing the value of c_ν from 1 to 0, then g is also monotone and $\lambda(g) = 1$. Hence

$$0 = \lambda(f - g) = \mu(Q_\nu \cap \partial\Omega)\,.$$

as desired.

In the remaining case, we have $\nu \geqq 0$, and $\nu_i > 0$ for some $i = 1, \ldots, s$. In this case, we define $c_{\nu'} := 1$, if $\nu' \geqq \nu$ and $c_{\nu'} := 0$ otherwise. We again let $j \in \{1, \ldots, s\}$ be chosen so that $j \neq i$. Then, since $L_+ := \{te_j : t > 0\}$ and $L_- := \{te_j : t < 0\}$ both intersect $\partial\Omega$ and $f \equiv 0$ on L_+ and L_-, we have $\lambda(f) = 0$. If g is obtained from f by changing c_ν from 1 to 0, then likewise $\lambda(g) = 0$. Hence, as before $0 = \lambda(f - g) = \mu(Q_\nu \cap \partial\Omega)$. □

3 Construction of monotone extensions

This section is concerned with the construction of monotone extensions for bounded domains in $\mathbb{R}^s$. In order to give the simplest possible arguments, we shall assume that Ω is convex; however some of the results of this section also hold without this assumption. We begin by describing a relatively simple method for monotone extensions that will also motivate subsequent discussions of the regularity of monotone extensions. We shall assume that f is continuous on $\partial\Omega$ and nondecreasing there.

If $x \in \Omega$, we let $A_x^\pm := x \pm \mathbb{R}_+^s$ and define

(3.1) $$m_+(x) := \inf\{f(y) : y \in \partial\Omega \cap A_x^+\}$$

$$m_-(x) := \sup\{f(y) : y \in \partial\Omega \cap (A_x^-)\}\,.$$

Then, any monotone extension F of f must satisfy

$$m_-(x) \leqq F(x) \leqq m_+(x), \quad x \in \Omega . \tag{3.2}$$

A natural candidate for F is a convex combination of $m_\pm$:

$$F(x) := (1 - a(x))m_+(x) + a(x)m_-(x) \tag{3.3}$$

where a is some appropriate nonnegative function taking values in [0, 1]. We begin by discussing the following choice for a:

$$a(x) := \begin{cases} \dfrac{v_+(x)}{v_+(x) + v_-(x)} ; & v_+(x) + v_-(x) > 0 , \\ 1 ; & v_+(x) + v_-(x) = 0 . \end{cases} \tag{3.4}$$

with

$$v_\pm(x) := \mathrm{vol}_s(\Omega \cap A_x^\pm) .$$

We first note some elementary properties of the functions $m_\pm, v_\pm$ on Ω and $\bar{\Omega} := \Omega \cup \partial\Omega$.

Lemma 3.1. *If Ω is convex and f is continuous on $\partial\Omega$, then (i) each of the functions $v_\pm$, is continuous on $\bar{\Omega}$, (ii) each of the function $m_\pm$ is continuous on Ω, (iii) the function $m_\pm$ is continuous at each point $x \in \partial\Omega$ where $v_\pm(x) = 0$, (iv) the functions $m_\pm$ are continuous at each point $x \in \partial\Omega$ where $v_+(x)v_-(x) > 0$.*

Proof. For $x, y \in \bar{\Omega}$, we have

$$v_\pm(x) - v_\pm(y) \leqq \mathrm{vol}_s((A_x^\pm \backslash A_y^\pm) \cap \Omega) . \tag{3.5}$$

Since Ω is a bounded domain the right side of (3.5) tends to zero uniformly as $x \to y$. Since x, y are arbitrary, we can interchange their roles and thereby introduce absolute values on the left side of (3.5) and deduce the continuity of $v_\pm$. This proves (i).

We shall prove (ii), (iii), and (iv) for m_+, a similar argument applies for m_-. For each $y \in \Omega$, let $z(y) \in \partial\Omega \cap A_y^+$ be a point where $m_+(y) = f(z(y))$. The existence of such points follows from the compactness of $A_y^+ \cap \partial\Omega$ and the continuity of f. We first show that for any $x \in \bar{\Omega}$, we have

$$m_+(x) \leqq \liminf_{y \to x} m_+(y) . \tag{3.6}$$

We fix x and for each y we can write $z(y) := y + v(y)$ with $v(y) \geqq 0$. We take a subsequence $\{y_n\}_{n \in \mathbb{N}}$ such that the limit infimum in (3.6) is attained. By choosing a further subsequence if necessary, we can suppose that $v(y_n) \to v_0, n \to \infty$ (because the $v(y)$ come from a bounded set). The point $x + v_0$ is in $A_x^+ \cap \partial\Omega$ and $x + v_0 = \lim_{n \to \infty} z(y_n)$. Therefore,

$$m_+(x) \leqq f(x + v_0) = \lim_{n \to \infty} f(z(y_n)) = \liminf_{y \to x} m_+(y) ,$$

which proves (3.6).

To complete the proof of (ii), (iii), and (iv), we shall show in each of these cases that

$$\limsup_{y \to x} m_+(y) \leqq m_+(x) \tag{3.7}$$

with the limit supremum taken over $y \in \Omega$.

Consider first the case of (ii), that is $x \in \Omega$. We fix one of the points $z(x)$ for x. We can write $z := z(x) = x + v$ with $v \geqq 0$. Because Ω is convex, the ray $y + tv, t > 0$ intersects $\partial\Omega$ at a unique point $\xi(y)$. We claim that $\lim_{y \to x} \xi(y) = z$. Indeed, if this were not the case, there would be a subsequence $y_n \to x$ such that $\xi(y_n) \to w$ with $w \neq z$. But clearly $w = x + t_0 v$ for some $t_0 \geqq 0$ and $w \in \partial\Omega$ (because $\partial\Omega$ is closed) and this contradicts the fact that the ray $x + tv, t \geqq 0$ intersect $\partial\Omega$ at the unique z. Hence our claim is established. Since $m_+(y) \leqq f(z(y))$, it follows that

$$\limsup_{y \to z} m_+(y) \leqq \lim_{y \to x} f(z(y)) = f(z) = m_+(x)$$

and hence we have proven (3.7) in this case.

To verify (3.7) in case (iii), we suppose that $x \in \partial\Omega$ and $v_+(x) = 0$. Then $m_+(x) = f(x)$. If $y \in \bar{\Omega}$, we write $y = x + h$, with $h = h(y) = (h_1, \ldots, h_s)$. Then, $w(y) := y + v(y)$, $v(y) := \sum_{j=1}^{s} |h_j| e_j$, is in A_y^+ and also in A_x^+. Since $v_+(x) = 0$, we must have $w(y) \notin \Omega$. Hence, the ray $y + tv(y)$ intersects $\partial\Omega$ for some $t = t(y)$, $0 \leqq t(y) \leqq 1$. It follows that $m_+(y) \leqq f(y + tv(y))$. Since, $v(y) \to 0$, we have

$$\limsup_{y \to x} m_+(y) \leqq \lim_{y \to x} f(y + t(y)v(y)) = f(x) = m_+(x)$$

which establishes (3.7) in this case.

Finally, we consider (iv) with $x \in \partial\Omega$. Given $\varepsilon > 0$, we choose $\delta > 0$ to be so small that $|w - x| \leqq \delta$, $w \in \partial\Omega$, implies $|f(w) - f(x)| < \varepsilon$. Further denote by B_δ the ball of radius δ centered at x and let $S_\delta := B_\delta \cap (\mathrm{int}(x + \mathbb{R}_+^s)$. We claim that S_δ contains points y from $\partial\Omega$. Indeed, since $v_-(x) \neq 0$, there is a point $z \in \Omega \cap \mathrm{int}(x - \mathbb{R}_+^s)$. The ray emenating from z and passing through x enters into S_δ. Points on this ray in S_δ must be in Ω^c since otherwise x would also be in Ω. Hence there are points in S_δ from Ω^c and also points from Ω (because $v_+(x) \neq 0$). Therefore there are points in S_δ from $\partial\Omega$ which establishes our claim. Now let w be one of those points. If y is sufficiently close to x then $w \in A_y^+$. Therefore,

$$m_+(y) \leqq f(w) \leqq f(x) + \varepsilon \leqq m + (x) + \varepsilon\,.$$

Since ε is arbitrary, we have proved (3.7). □

Theorem 3.1. *Suppose $\Omega \subset \mathbb{R}^s$ is any bounded domain. Then for any monotone function f on $\partial\Omega$, the function F defined by* (3.3), (3.4) *is a monotone extension of f to $\bar{\Omega}$. Moreover, if Ω is convex and f is continuous on $\partial\Omega$, then F is continuous on $\bar{\Omega}$.*

Proof. If $x, y \in \Omega$ with $y \leqq x$ (that is $y - x \in \mathbb{R}_+^s$), then

$$A_x^+ \subset A_y^+, \quad A_y^- \subset A_x^- \tag{3.8}$$

and therefore $m_+(x) \geqq m_+(y)$ and $m_-(x) \geqq m_-(y)$. Since $m_+ \geqq m_-$, the monotonicity of F will follow as soon as we have shown that

$$a(x) \leqq a(y), \qquad x \geqq y . \tag{3.9}$$

To this end, we first note that (3.8) implies

$$v_+(x) \leqq v_+(y), \qquad v_-(y) \leqq v_-(x), \qquad y \leqq x . \tag{3.10}$$

Hence, when $v_+(x) > 0$, we have

$$\frac{v_-(x)}{v_+(x)} \geqq \frac{v_-(y)}{v_+(y)}$$

which readily confirms (3.9). If $v_+(x) = 0$, then either $a(x) = 1$ or $a(x) = 0$. In the first case, we must also have $v_-(x) = 0$, which, in view of (3.10) implies that $v_-(y) = 0$ so that $a(y) = 1$ and (3.9) has been verified in this case as well. Finally, if $a(x) = 0$, then (3.9) holds trivially since $a(y) \geqq 0$. This confirms (3.9) and the monotonicity of F. Clearly for $x \in \partial\Omega$, we have

$$m_+(x) = m_-(x) = f(x)$$

so that the first assertion of the theorem follows.

Suppose now that Ω is convex and f is continuous on $\partial\Omega$. It follows from Lemma 3.1 that $a, m_\pm$ are all continuous at each point $x \in \Omega$ and also at any point $x \in \partial\Omega$ where $v_+(x)v_-(x) > 0$. Therefore, F is also continuous at such points. Therefore, we need only check continuity of F at points $x \in \partial\Omega$ where either $v_-(x) = 0$ or $v_+(x) = 0$. We shall only consider the first case since the second is completely symmetric. If in addition $v_+(x) = 0$, then Lemma 3.1 gives that both $m_\pm$ are continuous at x and so the continuity of F at x follows from the very definition of F. On the other hand, if $v_+(x) > 0$, then $a(x) = 1$ and it follows from the continuity of $v_\pm$ that a is continuous at x. Therefore, since m_+ is bounded, the term $(1 - a(y))m_+(y)$ in the definition of F tends to 0 as $y \to x$. Moreover, by Lemma 3.1, m_- is continuous at x. □

Note that, due to the properties of the functions $m_\pm$, the extensions constructed above will generally be only Lipschitz-continuous regardless of how smooth the boundary data might be. Moreover, locations where such an extension is not differentiable may occur anywhere in Ω. In the following section we will propose and discuss alternative schemes which reduce the occurrence of such singularities or even avoid them totally.

4 Regular extensions to the unit square

This section is devoted to the construction of monotone extensions to the unit square. In particular, we are interested in finding extensions that exhibit the same regularity as the boundary data.

We will describe first another extension scheme which does not quite accomplish this goal but is nonetheless interesting in its own right. Suppose f is strictly monotone on $\Gamma = \partial[0, 1]^2$. In particular, this implies that the functions

$$f_1(t) = f(t, 0), \qquad f_2(t) = f(0, t), \qquad f_3(t) = f(t, 1), \qquad f_4(t) = f(1, t)$$

are strictly increasing. Hence,

$$f_u(t) = \begin{cases} f_2(t) & ; \quad t \in [0, 1] , \\ f_3(t-1); & t \in [1, 2] , \end{cases}$$

and

$$f_l(t) = \begin{cases} f_\ell(t) & ; \quad t \in [0, 1] , \\ f_4(t-1); & t \in [1, 2] , \end{cases}$$

are strictly increasing functions on $[0, 2]$ satisfying, in view of the continuity of f,

$$f_u(0) = f_\ell(0), \qquad f_u(2) = f_\ell(2) . \tag{4.1}$$

Therefore for any $x = (x_1, x_2) \in [0, 1]^2$ one has

$$f_u(x_1 + 1) > f_\ell(x_1), \qquad f_u(x_2) < f_\ell(x_2 + 1) . \tag{4.2}$$

Due to the strict monotonicity of f_u and f_l, and by (4.1), (4.2), there exists a unique line L_x through x with strictly negative slope such that its intersections x_u, x_ℓ with the upper and lower part of Γ, respectively, satisfy $f(x_u) = f(x_\ell)$. In terms of the corresponding parameters $t_u(x)$, $t_\ell(x) \in [0, 2]$ this may be expressed as

$$f_u(t_u(x)) = f_\ell(t_\ell(x))$$

where

$$1 + x_1 > t_u(x) > x_2, \qquad 1 + x_2 > t_l(x) > x_1 .$$

Defining

$$F(x) := f(x_u) = f(x_\ell) , \tag{4.3}$$

it is clear that whenever $y \in L_x$ one has $F(y) = F(x)$. To see that F is monotone let $x, y \in [0, 1]^2$, $x - y \in \mathbb{R}^2_+$. Since $F(y) = f_u(t_u(y))$ the fact that the lines L_x and L_y do not cross immediately reveals that $t_u(x) > t_u(y)$ so that $f_u(t_u(y)) < f_u(t_u(x)) = F(x)$ confirming the monotonicity of F.

As for the smoothness of F, let $\alpha(t)$ be defined by

$$f_u(\alpha(t)) = f_\ell(t), \quad t \in [0, 2] .$$

Thus, defining the mappings

$$b_u(t) := \begin{cases} (0, t) \ ; & 0 \leqq t \leqq 1 , \\ (t-1, 1) \ ; & 1 \leqq t \leqq 2 , \end{cases} \qquad b_l(t) := (t, t) - b_u(t)$$

we have

$$F(x) = f_u(t)$$

for any point x on the line spanned by $b_u(t)$ and $b_l(\alpha^{-1}(t))$. The smoothness of F is determined by the smoothness of the strictly increasing function $\alpha : [0, 2] \to [0, 2]$. In fact, since at any point $x \in [0, 1]^2$ gF possesses continuous derivatives of order l if and only if there exist continuous directional derivatives of order l in some direction not parallel to L_x. This, in turn, requires f_u and $f_u^{-1} \circ f_\ell$ to possess that many derivatives. This is ensured by the strict monotonicity of the functions $f_i(t)$, $i = 1, \ldots, 4$ and their smoothness as long as the line L_x does not contain any of the corner points (0, 1), (1, 0), since f_u and f_l may not join with higher order continuity at $t = 1$. One may summarize these observations as follows.

Proposition 4.1. *Suppose the functions f_i, $i = 1, \ldots 4$ have continuous derivatives of order k. Then the monotone extension F defined by* (4.3) *is k times continuously differentiable in any point x which is not located on the lines spanned by the points* (0, 1), $b_l(\alpha^{-1}(1))$ *and* $b_u(\alpha(1))$, (1, 0), *respectively.*

The expected lack of regularity across the level lines of F intersecting the above corner points is illustrated in Fig. 1.

Nevertheless, we will prove the following fact

Theorem 4.1. *For $f \in C([0, 1]^2)$ let the functions $f_i(t)$, $i = 1, \ldots, 4$ be defined as before by*

$$f_1(t) := f(t, 0), \qquad f_2(t) := f(0, t), \qquad f_3(t) := f(t, 1), \qquad f_4(t) := f(1, t) .$$

Suppose that f is strictly monotone on Γ and that the $f_i(t)$ are differentiable functions satisfying

$$f_i'(t) > 0, \quad t \in [0, 1], \quad i = 1, \ldots, 4 .$$

Then for any order of smoothness possessed by the functions f_i, $i = 1, \ldots, 4$, there exists a monotone extension F possessing the same order of smoothness on Ω. More precisely, if f_1, f_3 and f_2, f_4 have continuous derivatives up to order k_1, k_2,

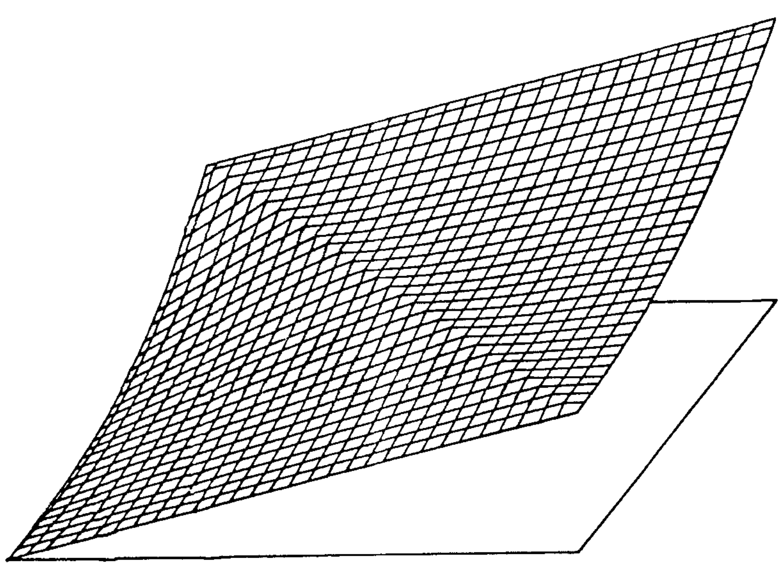

Fig. 1

respectively, then $\dfrac{\partial^{i+j}}{\partial^i x \partial^j y} F(x, y)$ *is continuous for* $i \leqq k_1$ *and* $j \leqq k_2$. *Moreover, if the functions* f_i, $i = 1, \ldots, 4$, *are analytic on* $[0, 1]$ *then* F *is analytic on* $[0, 1]^2$.

The remainder of this section is devoted to the proof of Theorem 4.1. The key idea is to combine a blending technique based on judiciously chosen blending functions with a suitable parameter transformation.

To this end, we wish to find suitable functions $\varphi, \psi \in C^1[0, 1]$ satisfying

$$\varphi(0) = \psi(0) = 1, \qquad \varphi(1) = \psi(1) = 0 \tag{4.4}$$

such that the blending interpolant

$$\begin{aligned} L(x, y) &:= L(x, y; f, \varphi, \psi) \\ &:= \varphi(x) f(0, y) + (1 - \varphi(x)) f(1, y) + \psi(y) f(x, 0) \\ &\quad + (1 - \psi(y)) f(x, 1) - \{ f(0, 0) \varphi(x) \psi(y) \\ &\quad + f(0, 1) \varphi(x)(1 - \psi(y)) + f(1, 0)(1 - \varphi(x)) \psi(y) \\ &\quad + f(1, 1)(1 - \varphi(x))(1 - \psi(y)) \} \end{aligned} \tag{4.5}$$

is monotone.

In view of Theorem 2.1 the functions φ and ψ will have to depend on f. To derive suitable conditions on φ and ψ let

$$\Delta := -f(0, 0) - f(1, 1) + f(0, 1) + f(1, 0) .$$

One readily verifies that

$$\begin{aligned} \frac{\partial L}{\partial x}(x, y) &= \psi(y) \frac{\partial f}{\partial x}(x, 0) + (1 - \psi(y)) \frac{\partial f}{\partial x}(x, 1) \\ &\quad + \varphi'(x) [\Delta \psi(y) + f(0, y) - f(1, y) - f(0, 1) + f(1, 1)] \end{aligned} \tag{4.6}$$

and

$$\begin{aligned} \frac{\partial L}{\partial y}(x, y) &= \varphi(x) \frac{\partial f}{\partial y}(0, y) + (1 - \varphi(x)) \frac{\partial f}{\partial y}(1, y) \\ &\quad + \psi'(y) [\Delta \varphi(x) + f(x, 0) - f(x, 1) - f(1, 0) + f(1, 1)] . \end{aligned} \tag{4.7}$$

This leads to the following observation

Lemma 4.1. *Suppose* $\Delta \neq 0$ *and set*

$$\psi(y) := \frac{f(1, y) - f(0, y) - f(1, 1) + f(0, 1)}{\Delta}$$

$$\varphi(x) := \frac{f(x, 1) - f(x, 0) - f(1, 1) + f(1, 0)}{\Delta} .$$

If the functions φ, ψ *map* $[0, 1]$ *into* $[0, 1]$ *then the function* L *given by* (4.5) *is a monotone extension of* f.

Proof. The fact that L is an extension of f follows immediately from the Boolean sum construction (4.5) and (4.4). Since the above choice of φ and ψ ensures that

$$\frac{\partial L}{\partial x}(x, y) = \psi(y)\frac{\partial f}{\partial x}(x, 0) + (1 - \psi(y))\frac{\partial f}{\partial x}(x, 1)$$

$$\frac{\partial L}{\partial x}(x, y) = \varphi(x)\frac{\partial f}{\partial y}(0, y) + (1 - \varphi(x))\frac{\partial f}{\partial y}(1, y)$$

the assertion follows immediately. □

Lemma 4.2. *Suppose* $\Delta \neq 0$. *If*

$$f(x, 1) - f(x, 0) \in [f(1, 1) - f(1, 0), f(0, 1) - f(0, 0)] \tag{4.8}$$

and

$$f(1, y) - f(0, y) \in [f(1, 1) - f(0, 1), f(1, 0) - f(0, 0)] \tag{4.9}$$

then the hypotheses of Lemma 4.1 *are fulfilled, i.e. the functions* φ, ψ *map* $[0, 1]$ *onto* $[0, 1]$.

Proof. We consider first (4.8) and assume

$$f(1, 1) - f(1, 0) < f(0, 1) - f(0, 0)\,.$$

Therefore, it is enough to show that

$$0 \leqq f(x, 1) - f(x, 0) - f(1, 1) + f(1, 0) \leqq \Delta$$

which holds by (4.8). If $f(0, 1) - f(0, 0) < f(1, 1) - f(1, 0)$ we have $\Delta < 0$ and (4.8) implies

$$f(0, 1) - f(0, 0) \leqq f(x, 1) - f(x, 0) \leqq f(1, 1) - f(1, 0)$$

so that

$$\Delta \leqq f(x, 1) - f(x, 0) - f(1, 1) + f(1, 0) \leqq 0\,.$$

Hence

$$1 \geqq \frac{f(x, 1) - f(x, 0) - f(1, 1) + f(1, 0)}{\Delta} \geqq 0$$

which again confirms that $\varphi(x) \in [0, 1]$, $x \in [0, 1]$.

Interchanging the roles of x and y, the rest of the assertion follows by symmetry. □

Clearly, the conditions (4.8) and (4.9) will generally not be fulfilled. To simplify the following discussion we may assume without loss of generality (by adding a constant, by scaling f and/or by interchanging the roles of x and y) that $f(0, 0) = 0, f(1, 1) = 1$ and that the values

$$f(1, 0) = a, \qquad f(0, 1) = b$$

satisfy

$$0 < a \leqq b < 1\,. \tag{4.10}$$

Our goal is to construct a strictly increasing analytic function $\Phi:[0,1]\to\mathbb{R}$, $\Phi(0)=0$ such that $\Phi\circ f$ satisfies (4.8) and (4.9), i.e.

$$\Phi(f(x,1))-\Phi(f(x,0))\in[\Phi(b),\Phi(1)-\Phi(a)]\,. \tag{4.11}$$

and

$$\Phi(f(1,y))-\Phi(f(0,y))\in[\Phi(a),\Phi(1)-\Phi(b)] \tag{4.12}$$

Once we have found a function Φ satisfying (4.11), (4.12), Lemma 4.1 and Lemma 4.2 imply that the function $L(x,y;\Phi\circ f,\tilde{\varphi},\tilde{\psi})$ defined by (4.5) is a monotone extension of $\tilde{f}:=\Phi\circ f$ where $\tilde{\varphi},\psi$ are defined as in Lemma 4.1 with respect to $\tilde{f}$.

Hence

$$F(x,y):=\Phi^{-1}\circ L(x,y,\Phi\circ f,\tilde{\varphi},\tilde{\psi})$$

is a monotone extension of f to $[0,1]^2$ which, in veiw of the analyticity of Φ exhibits the same regularity as the boundary functions $f_i(t)$, $i=1,\ldots,4$.

Thus, to complete the proof of Theorem 4.1, it remains to construct an analytic function Φ satisfying (4.11), (4.12). To do so we will make use of the following quantities derived from the boundary data f_i, $i=1,\ldots,4$:

$$\delta:=\min_{x,y}\,(f(1,y)-f(0,y),f(x,1)-f(x,0))\,,$$

$$m:=\inf_{i,t}\,(f_i'(t))\,,$$

$$M:=\max_{i,t}\,(f_i'(t))\,.$$

Moreover, since $a\geqq\delta>0$, we can choose $\mu_1>0$ such that

$$f(x,0)\leqq a-\frac{\delta}{4}\,,\quad\text{for } x\leqq\frac{2\mu_1}{m}\,. \tag{4.13}$$

With these, we let μ be any fixed real number satisfying

$$0<\mu<\min\left\{b-a+\frac{\delta}{4},\frac{1-b}{4},\mu_1\right\} \tag{4.14}$$

and define

$$\gamma:=\min\left\{\frac{\delta}{2},\frac{m}{M},\mu,\frac{1-b}{2b}\right\}. \tag{4.15}$$

Lemma 4.3. *Let $\Phi\in C^1([0,1])$ be a strictly increasing convex function satisfying $\Phi(0)=0$. If*

$$\max\left\{\Phi'(t):0\leqq t\leqq a-\frac{\delta}{4}\right\}\leqq\gamma\min\{\Phi'(t):a\leqq t\leqq 1\}\,, \tag{4.16}$$

$$\max\{\Phi'(t):0\leqq t\leqq b-\mu\}\leqq\gamma\min\left\{\Phi'(t):b\leqq t\leqq\frac{b+1}{2}\right\}, \tag{4.17}$$

$$\max\{\Phi'(t):0\leqq t\leqq b\}\leqq\gamma\min\left\{\Phi'(t):\frac{b+1}{2}\leqq t\leqq 1\right\}, \tag{4.18}$$

then Φ satisfies (4.11) *and* (4.12).

Before turning to the proof of Lemma 4.3 let us point out how to construct a function Φ which is analytic in a neighborhood of the interval [0, 1] and satisfies the requirements (4.16)–(4.18). Fixing $c_1 > 0$, we choose $\varepsilon > 0$ such that

$$\gamma c_1 - \varepsilon > 0 .$$

Let $h(t)$ be the continuous piecewise linear function with breakpoints $a - \delta/4, a, b - \mu, b, b + 2\mu$ determined by

$$h(t) = \begin{cases} c_1 ; & 0 \leqq t \leqq a - \delta/4 ; \\ c_2 := \gamma^{-1}(c_1 + \varepsilon) ; & a \leqq t \leqq b - \mu , \\ c_3 := \gamma^{-1}(c_2 + \varepsilon) ; & b \leqq t \leqq b + \mu , \\ c_4 \geqq \gamma^{-1}(c_3 + \varepsilon) ; & b + 2\mu \leqq t \leqq 1 \end{cases}$$

when $a < b$, whereas we set

$$h(t) = \begin{cases} c_1 ; & 0 \leqq t \leqq b - \mu , \\ c_2 := \gamma^{-1}(c_1 + \varepsilon) ; & b \leqq t \leqq b + \mu , \\ c_3 \geqq \gamma^{-1}(c_2 + \varepsilon) ; & b + 2\mu \leqq t \leqq 1 , \end{cases}$$

when $a = b$. Noting that by (4.14), $b + \mu \leqq \frac{b+1}{2}$, one readily verifies that $\Phi'(t) = h(t)$ fulfills (4.16), (4.17) and (4.18) with strict inequalities. Since $h(t)$ is increasing the Bernstein polynomial

$$B_n(h)(t) = \sum_{j=0}^{n} h\left(\frac{j}{n}\right)\binom{n}{j} t^j(1 - t)^{n-j}$$

is increasing, too. Since Bernstein polynomials approximate continuous functions arbitrarily well we conclude that for sufficiently large n the function $\Phi'(t) = B_n(h)(t)$ still satisfies (4.16), (4.17) and (4.18). Hence,

$$\Phi(t) = \int_0^t B_n(h)(x)dx, \quad t \in [0, 1] ,$$

is an analytic function satisfying (4.16)–(4.18). Moreover, by choosing the constants c_3 or c_4, depending on whether $a = b$ or $a < b$, in the above definition of $h(t)$ sufficiently large, we can ensure that $\Phi(1) > \Phi(a) + \Phi(b)$. Hence the quantity Δ for the function $\Phi \circ f$ appearing in Lemma 4.1 and Lemma 4.2 is different from zero. Therefore $\Phi \circ f$ satisfies all the assumptions in Lemma 4.1 and Lemma 4.2.

Thus to finish the proof of Theorem 4.1, it remains to complete the following.

Proof of Lemma 4.3. Let

$$\Delta(x) := \Phi(f(x, 1)) - \Phi(f(x, 0))$$

$$\Delta(y) := \Phi(f(1, y)) - \Phi(f(0, y)) .$$

First we consider the range

$$f(0, y) \leqq a - \frac{\delta}{4}. \tag{4.19}$$

Writing

$$\Delta(y) = \Phi(a) + \Phi(f(1, y)) - \Phi(f(1, 0)) - \Phi(f(0, y))$$

we note that, since $f(0, 0) = 0$

$$\begin{aligned} \Phi(f(0, y)) &= \Phi(f(0, 0)) + (f(0, y) - f(0, 0))\Phi'(f(0, 0) + t_y) \\ &\leqq yM\Phi'(t_y) \end{aligned}$$

for some $t_y \in \left(0, a - \frac{\delta}{4}\right)$, while for some $\xi_y \in (a, f(1, y))$

$$\begin{aligned} \Phi(f(1, y)) - \Phi(f(1, 0)) &= (f(1, y) - f(1, 0))\Phi'(\xi_y) \\ &\geqq ym \min\{\Phi'(t) : a \leqq t \leqq 1\}. \end{aligned}$$

Thus setting $\varepsilon_1 := \max\left\{\Phi'(t) : 0 \leqq t \leqq a - \frac{\delta}{4}\right\}$ and $\eta_1 := \min\{\Phi'(t) : a \leqq t \leqq 1\}$ we obtain

$$\Delta(y) \geqq \Phi(a) + ym\eta_1 - yM\varepsilon_1.$$

From (4.15) and (4.16) we infer that $m\eta_1 \geqq M\varepsilon_1$ and therefore

$$\Delta(y) \geqq \Phi(a)$$

whenever (4.19) holds.

Suppose next that $f(0, y) \in \left[a - \frac{\delta}{4}, a\right]$ so that

$$\begin{aligned} f(1, y) &= f(1, y) - f(0, y) + f(0, y) \geqq \delta + f(0, y) \\ &\geqq a + \tfrac{3}{4}\delta = f(1, 0) + \tfrac{3}{4}\delta. \end{aligned}$$

Thus

$$\begin{aligned} \Phi(f(1, y)) &= \Phi(a) + (f(1, y) - f(1, 0))\Phi'(\xi_y) \\ &\geqq \Phi(a) + \frac{3\delta}{4}\eta_1. \end{aligned}$$

Hence, since $f(0, y) \leqq a$, we have

$$\begin{aligned} \Delta(y) &= \Phi(f(1, y)) - \Phi(f(0, y)) \\ &\geqq \Phi(a) + \frac{3\delta}{4}\eta_1 - \Phi(a) \\ &= \frac{3\delta}{4}\eta_1. \end{aligned} \tag{4.20}$$

Since Φ is convex we obtain

$$\Phi(a) = \int_0^{a-\delta/4} \Phi'(t)dt + \int_{a-\delta/4}^{a} \Phi'(t)\,dt$$

$$\leqq \left(a - \frac{\delta}{4}\right)\varepsilon_1 + \eta_1 \frac{\delta}{4} \leqq \varepsilon_1 + \eta_1 \frac{\delta}{4}\,.$$

Now (4.15) and (4.16) ensure that $\varepsilon_1 \leqq \eta_1 \delta/2$ and therefore

$$\Phi(a) \leqq \frac{3\delta}{4}\eta_1\,. \tag{4.21}$$

This together with (4.20) confirms that

$$\Delta(y) \geqq \Phi(a)$$

when $f(0, y) \in \left[a - \frac{\delta}{4}, a\right]$.

When $f(0, y) > a$ one obtains for some $t_y \in [f(0, y), f(1, y)] \subseteq [a, 1]$,

$$\begin{aligned}\Delta(y) &= \Phi(f(1, y)) - \Phi(f(0, y)) \\ &= (f(1, y) - f(0, y))\Phi'(t_y) \\ &\geqq \delta\eta_1 \geqq \Phi(a)\end{aligned}$$

where we have used (4.21) in the last step.

It remains to show that

$$\Delta(y) := \Phi(f(1, y)) - \Phi(f(0, y)) \leqq \Phi(1) - \Phi(b)\,. \tag{4.22}$$

Since $a \leqq b < 1$ one has $a < \frac{b+1}{2}$. Considering first those y such that $f(1, y) \leqq \frac{b+1}{2}$, we obtain

$$\Delta(y) \leqq \Phi\left(\frac{b+1}{2}\right) \leqq \Phi(1) - \Phi(b) + \Phi\left(\frac{b+1}{2}\right) - \Phi(1) + \Phi(b) - \Phi(0)$$

and we have to make sure that

$$\Phi(b) - \phi(0) \leqq \Phi(1) - \Phi\left(\frac{b+1}{2}\right).$$

This is indeed the case if

$$\max_{0 \leqq t \leqq b} \Phi'(t)b \leqq \min_{b+1/2 \leqq t \leqq 1} \Phi'(t)\frac{1-b}{2}$$

which, in turn is guaranteed by (4.15) and (4.18).

Consider now the second possibility:

$$f(1, y) \geqq \frac{b+1}{2}\,.$$

We first estimate $f(0, y)$ from below. Setting

$$\eta_2 := \min\left\{ \Phi'(t) : \frac{b+1}{2} \leqq t \leqq 1 \right\}$$

and noting that

$$b - f(0, y) \leqq M(1 - y)\,,$$

$$1 - f(1, y) \geqq m(1 - y)\,,$$

we obtain

$$\Phi(1) - \Phi(f(1, y)) \geqq \eta_2 m(1 - y)$$

and

$$\Phi(b) - \Phi(f(0, y)) \leqq M(1 - y)\varepsilon_2$$

where

$$\varepsilon_2 := \max\{\Phi'(t) : 0 \leqq t \leqq b\}\,.$$

Hence

$$\begin{aligned}\Phi(f(1, y)) - \Phi(f(0, y)) &\leqq \Phi(1) - \Phi(b) + \Phi(f(1, y)) - \Phi(1)\\ &\quad + \Phi(b) - \Phi(f(0, y))\\ &\leqq M(1 - y)\varepsilon_2 + \Phi(1) - \Phi(b) - \eta_2 m(1 - y)\\ &\leqq \Phi(1) - \Phi(b)\end{aligned}$$

where we have used (4.15) and (4.18) in the last step. This proves that Φ satisfies (4.11).

Concerning (4.12), we will show first that

$$\Delta(x) = \Phi(f(x, 1)) - \Phi(f(x, 0)) \geqq \Phi(b)\,.$$

Considering first the case

$$f(x, 1) \geqq b + 2\mu \tag{4.23}$$

the convexity of $\Phi(t)$ yields

$$\begin{aligned}\Delta(x) &\geqq \Phi(b + 2\mu) - \Phi(b + \mu) + \Phi(b + \mu) - \Phi(b)\\ &\geqq \Phi(b) - \Phi(b - \mu) + \Phi(b + \mu) - \Phi(b)\,.\end{aligned}$$

Setting

$$\varepsilon_3 := \max\{\Phi'(t) : 0 \leqq t \leqq b - \mu\}$$

we obtain

$$\Phi(b - \mu) = \Phi(b - \mu) - \Phi(0) \leqq (b - \mu)\varepsilon_3 \leqq \varepsilon_3\,.$$

Hence (4.14), (4.15) and (4.17) yield, in view of (4.23),

$$\Phi(b - \mu) \leqq \mu\eta_3 \leqq \Phi(b + \mu) - \Phi(b)$$

where

$$\eta_3 := \min\left\{ \Phi'(t) : b \leqq t \leqq \frac{b+1}{2} \right\}.$$

This confirms that

$$\Delta(x) \geqq \Phi(b)$$

whenever (4.23) holds.

Suppose now $f(x, 1) \leqq b + 2\mu \leqq (b+1)/2$. In this case

$$mx \leqq f(x, 1) - b \leqq 2\mu$$

so that $x \leqq 2\mu/m$. Thus,

$$\begin{aligned}\Phi(f(x,1)) - \Phi(f(x,0)) &= \Phi(f(x,1)) - \Phi(b) - (\Phi(f(x,0)) - \Phi(0)) + \Phi(b)\\ &\geqq mx\eta_3 - xM\varepsilon_1 + \Phi(b)\,,\end{aligned}$$

where we have used (4.13) and the fact that $M \geqq 1$ in the last step. Now $f(x, 0) \leqq a - \frac{\delta}{4} \leqq b - \mu$ by (4.13) and (4.14). Hence, in view of (4.15) and (4.17), this confirms that

$$\Delta(x) \geqq \Phi(b)\,.$$

Concerning the upper bound, suppose first that $f(x, 1) \leqq b + 2\mu$ so that we may conclude, in view of (4.13), as above,

$$\begin{aligned}\Phi(f(x,1)) - \Phi(f(x,0)) &\leqq \Phi(1) - \Phi(a) + \Phi(a) - \Phi(f(x,0))\\ &\quad + \Phi(f(x,1)) - \Phi(1)\\ &\leqq \Phi(1) - \Phi(a) + \varepsilon_3(1-x)M - \eta_3 m(1-x)\end{aligned} \tag{4.24}$$

which, by (4.15) and (4.16), yields

$$\Delta(x) \leqq \Phi(1) - \Phi(a)\,.$$

Finally, when $f(x, 1) > b + 2\mu$, the last estimate in (4.24) may be replaced by

$$\Delta \leqq \Phi(1) - \Phi(a) + \varepsilon_2(1-x)M - \eta_2 m(1-x)\,.$$

Using (4.14), (4.15) and (4.18) provides again

$$\Delta(x) \leqq \Phi(1) - \Phi(a)$$

which completes the proof of Lemma 4.3 and also of Theorem 4.1. □

References

1. Utreras, F.I. (1987): Constrained Surface Construction. In: C.K. Chui, L.L. Schumaker, F.I. Utreras, (eds) Multivariate Approximation. Academic Press, pp. 233–254